

Mobile Robot Localization based on a Polynomial Approach

C. Manes, A. Martinelli, F. Martinelli and P. Palumbo

Abstract—This paper introduces a new analytical algorithm to perform the localization of a mobile robot using odometry and laser readings. Based on a polynomial approach, the proposed algorithm provides the optimal affine filter in a class of suitably defined estimators. The performance of the algorithm has been evaluated through simulations. The comparison with the standard Extended Kalman Filter shows that the proposed filter provides good estimates also in critical situations where the system nonlinearities cause a bad behavior of the EKF.

I. INTRODUCTION

In most cases, autonomous mobile robots are required to know precisely their position and orientation in order to successfully perform their mission. This is usually achieved by fusing proprioceptive data (gathered by sensors monitoring the motion of the vehicle, like encoders) with exteroceptive data (e.g. [1], [6], [7], [18]). One of the most common methods adopted to perform this fusion is the Extended Kalman Filter (EKF, e.g. see [7]).

Apart from very few cases, both the dynamics of a mobile robot and the link between the data gathered by the robot sensors and the robot configuration are nonlinear functions. As a result, the EKF is not an optimal filter. It introduces an approximation by linearizing these functions around the current estimated state. In many cases, this approximation can lead to divergence. This can happen when the sensor data are delivered at a very low frequency with respect to the robot speed and/or the data are not precise enough.

In order to avoid the problems resulting from the system nonlinearities, usually numerical methods to approximate the posterior density function for the state are adopted. Many numerical approaches to the localization problem are based on the *Markov Localization* (e.g. [2], [10], [14], [20], [21]). Another very successful numerical approach in this framework is the *Monte Carlo Localization* in [22], which is based on particle filters [9], [15], [19]. Other approaches for robot localization include [12], the Unscented Kalman Filter [13] and set membership methods (see e.g. [8], [5]).

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The methodology presented in this paper, unlike the approaches cited above, is an analytical contribution with well defined optimal properties. It is based on a polynomial approach, which provided a great deal of results in the last decade in the framework of filtering linear [3], bilinear [4] or nonlinear [11] systems, both in the Gaussian or non-Gaussian case. The first step in the filter construction is to pre-process the measurements available from the robot in order to write the new output equations as a second-order polynomial transformation of a suitably defined extended state. Then a generation model of the pre-processed data is achieved in the form of a bilinear system (linear drift and multiplicative noise), to which the optimal linear filter is finally applied [4]. It is important to stress that the filter construction is not based on the linear approximation of the system but on the exact system equations. The computations in the paper are quite messy but the idea behind them is rather simple: introduce a transformation in the description of the system in order to obtain a form for which the optimal linear filter can be applied. The simulations show the effectiveness of the proposed approach under several parameter settings, and the improvements w.r.t. the standard EKF.

II. SYSTEM NOTATION AND PROBLEM FORMULATION

Consider a robot moving in a 2D environment with coordinates (x_t, y_t) and orientation θ_t . Assuming a discrete time unicycle differential model, the dynamics is given by:

$$x_{t+1} = x_t + \delta\rho_t \cos(\theta_t), \quad (1)$$

$$y_{t+1} = y_t + \delta\rho_t \sin(\theta_t), \quad (2)$$

$$\theta_{t+1} = \theta_t + \delta\theta_t, \quad (3)$$

where $t = 0, 1, \dots$; $\delta\rho_t$ and $\delta\theta_t$ are the robot shift and rotation during the sample time. Let $\delta\rho_t^e$ and $\delta\theta_t^e$ denote the noisy encoder readings. Under the assumption of synchronous drive, the synchronous odometry error model can be adopted [17], so that $\{\delta\rho_t\}$ and $\{\delta\theta_t\}$ can be modeled as independent sequences of independent Gaussian random variables generated as follows:

$$\delta\rho_t = \delta\rho_t^e + \sqrt{\delta\rho_t^e} \nu_{\rho,t}, \quad \text{with } \nu_{\rho,t} = \mathcal{N}(0, K_\rho), \quad (4)$$

$$\delta\theta_t = \delta\theta_t^e + \sqrt{\delta\rho_t^e} \nu_{\theta,t}, \quad \text{with } \nu_{\theta,t} = \mathcal{N}(0, K_\theta). \quad (5)$$

In practise, it is assumed that the odometry is perfectly calibrated (mean value equal to the reading) and the variances increase linearly with the traveled distance (as in the

diffusion motion). Equations (1-3) become:

$$x_{t+1} = x_t + \delta\rho_t^e \cos(\theta_t) + \sqrt{\delta\rho_t^e} \nu_{\rho,t} \cos(\theta_t), \quad (6)$$

$$y_{t+1} = y_t + \delta\rho_t^e \sin(\theta_t) + \sqrt{\delta\rho_t^e} \nu_{\rho,t} \sin(\theta_t), \quad (7)$$

$$\theta_{t+1} = \theta_t + \delta\theta_t^e + \sqrt{\delta\rho_t^e} \nu_{\theta,t}. \quad (8)$$

The environment is perfectly known and is represented by line segments. The robot is equipped with a laser range finder providing the distance at m directions, each with angle θ_i w.r.t. the robot orientation θ_t . The equation of the environment line segment sensed in the laser direction i ($i = 1, 2, \dots, m$) at time t will be denoted by $y = m_{it}x + q_{it}$ (in the following it is assumed $m_{it} < \infty$: the extension to the general case is straightforward). By denoting $\rho_{i,t}$ the corresponding laser range finder reading, we have:

$$\rho_{i,t} = \sqrt{(x_t - x_{p_{it}})^2 + (y_t - y_{p_{it}})^2} + n_{i,t}, \quad (9)$$

where $(x_{p_{it}}, y_{p_{it}})$ is the intersection of the i -th laser beam with the line $y = m_{it}x + q_{it}$. $\{n_{i,t}\}$ are independent sequences of zero-mean independent Gaussian variables, independent of $\{\nu_{\rho,t}\}$, $\{\nu_{\theta,t}\}$, with variance $\zeta_{i,2}$ (in the following $\zeta_{i,j} = \mathbb{E}[n_{i,t}^j]$ indicates the j -th order moment).

The aim of the paper is to estimate both the position (x_t, y_t) and the orientation θ_t starting from the odometry and the readings $\rho_{i,t}$, $i = 1, \dots, m$.

III. A BILINEAR MODEL FOR THE ROBOT

The first step is to pre-process the available measurements in order to write a new output equation, as a polynomial transformation of a suitably defined extended state.

A. The output equations

Consider the measurement equations (9) and substitute the values of the pair $(x_{p_{it}}, y_{p_{it}})$. Then, after computations:

$$\rho_{i,t} - n_{i,t} = \frac{|y_t - m_{it}x_t - q_{it}|}{d}, \quad (10)$$

where

$$d = |(\sin(\theta_i) - m_{it} \cos(\theta_i)) \cos(\theta_t) + (\cos(\theta_i) + m_{it} \sin(\theta_i)) \sin(\theta_t)|.$$

Let $c_t = \cos(\theta_t)$ and $s_t = \sin(\theta_t)$ and define an extended state $X(t) = (x_t, y_t, c_t, s_t)^T$. Taking the squares of both terms in (10), it is possible to obtain:

$$0 = C_{i,0}(t) + C_{i,1}(t)X(t) + C_{i,2}(t)X^{[2]}(t) + D_i(t)X^{[2]}(t)N_{o,i}(t), \quad (11)$$

where $N_{o,i}(t) = n_{i,t}^2 - \zeta_{i,2} - 2\rho_{i,t}n_{i,t}$ and the matrices $C_{i,0}$, $C_{i,1}$, $C_{i,2}$, D_i are given in (12-15) and (16-21). The superscript in square brackets in (11) denotes the Kronecker power of a vector (see [4] and references therein). By taking into account the trigonometric identity:

$$\cos^2(\theta_t) + \sin^2(\theta_t) = 1 \iff X_3^2(t) + X_4^2(t) = 1, \quad (22)$$

and defining $Y(t) = (Y_0(t), Y_1(t), \dots, Y_m(t))^T$, with $Y_0(t) \equiv 1$, $Y_i(t) \equiv 0$, $i = 1, \dots, m$, equations (11) and (22) may be written in the following more compact form:

$$Y(t) = C_1(t)X(t) + C_2(t)X^{[2]}(t) + C_0(t) + H(t)X^{[2]}(t), \quad (23)$$

with $X(t) \in \mathbb{R}^n$, $n = 4$, $Y(t) \in \mathbb{R}^{m+1}$ and $C_0(t) = [0, C_{1,0}(t), \dots, C_{m,0}(t)]^T$, $C_1(t) = [O_{n \times 1}, C_{1,1}^T(t), \dots, C_{m,1}^T(t)]^T$, $C_2(t) = [C_{0,2}^T, C_{1,2}^T(t), \dots, C_{m,2}^T(t)]^T$, $H(t) = [O_{n^2 \times 1}, D_1^T(t)N_{o,1}(t), \dots, D_m^T(t)N_{o,m}(t)]^T$, with $C_{0,2} = [O_{1 \times 10} \ 1 \ O_{1 \times 4} \ 1]$ and $O_{r \times c}$ is a generic $r \times c$ zero matrix. Notice that $H(t)$ is a stochastic matrix.

B. The extended state dynamics

In order to obtain a generation model for (23), the recursive equations of $X(t)$ are derived. Notice that $\cos(\sqrt{\delta\rho_t^e} \nu_{\theta,t})$ and $\sin(\sqrt{\delta\rho_t^e} \nu_{\theta,t})$ are both non-Gaussian random variables with the following mean values:

$$\bar{\xi}_{c1,t} := \mathbb{E}[\cos(\sqrt{\delta\rho_t^e} \nu_{\theta,t})] = e^{-\frac{\delta\rho_t^e K_\theta}{2}}, \quad (24)$$

$$\mathbb{E}[\sin(\sqrt{\delta\rho_t^e} \nu_{\theta,t})] = 0. \quad (25)$$

To have zero-mean noises, let $N_c(t) = \cos(\sqrt{\delta\rho_t^e} \nu_{\theta,t}) - \bar{\xi}_{c1,t}$ and $N_s(t) = \sin(\sqrt{\delta\rho_t^e} \nu_{\theta,t})$. With this positions, according to (6-8), the extended state dynamics obeys the following bilinear laws:

$$X_1(t+1) = X_1(t) + \delta\rho_t^e X_3(t) + \sqrt{\delta\rho_t^e} \nu_{\rho,t} X_3(t), \quad (26)$$

$$X_2(t+1) = X_2(t) + \delta\rho_t^e X_4(t) + \sqrt{\delta\rho_t^e} \nu_{\rho,t} X_4(t), \quad (27)$$

$$X_3(t+1) = (X_3(t) \cos(\delta\theta_t^e) - X_4(t) \sin(\delta\theta_t^e)) \bar{\xi}_{c1,t} + (X_3(t) \cos(\delta\theta_t^e) - X_4(t) \sin(\delta\theta_t^e)) N_c(t) - (X_4(t) \cos(\delta\theta_t^e) + X_3(t) \sin(\delta\theta_t^e)) N_s(t), \quad (28)$$

$$X_4(t+1) = (X_4(t) \cos(\delta\theta_t^e) + X_3(t) \sin(\delta\theta_t^e)) \bar{\xi}_{c1,t} + (X_4(t) \cos(\delta\theta_t^e) + X_3(t) \sin(\delta\theta_t^e)) N_c(t) + (X_3(t) \cos(\delta\theta_t^e) - X_4(t) \sin(\delta\theta_t^e)) N_s(t). \quad (29)$$

The moments of $N_c(t)$ and $N_s(t)$ will be denoted by $\xi_{ci,t} = \mathbb{E}[N_c^i(t)]$, $\xi_{si,t} = \mathbb{E}[N_s^i(t)]$, $\xi_{ci,sj,t} = \mathbb{E}[N_c^i(t)N_s^j(t)]$. As it will be clearer in the sequel, they will be required to be finite and available up to order 4. According to their construction, it readily comes that $\xi_{c1,t} = \xi_{s1,t} = \xi_{s3,t} = \xi_{ci,sj,t} = 0$, $\forall t \geq 0$, $\forall i = 1, \dots, 4$, $\forall j = 1, 3$.

In summary, by using the further position $N(t) = \sqrt{\delta\rho_t^e} \nu_{\rho,t}$ we have, from (26-29):

$$X(t+1) = A(t)X(t) + S_1(t)X(t), \quad (30)$$

with $S_1(t) = BN(t) + B_1(t)N_c(t) + B_2(t)N_s(t)$ and:

$$A(t) = \begin{bmatrix} I_2 & \delta\rho_t^e I_2 \\ O_2 & \bar{\xi}_{c1,t} R(\delta\theta_t^e) \end{bmatrix}, \quad B = \begin{bmatrix} O_2 & I_2 \\ O_2 & O_2 \end{bmatrix},$$

$$B_1(t) = \begin{bmatrix} O_2 & O_2 \\ O_2 & R(\delta\theta_t^e) \end{bmatrix}, \quad B_2(t) = \begin{bmatrix} O_2 & O_2 \\ O_2 & R(\delta\theta_t^e + \frac{\pi}{2}) \end{bmatrix},$$

where O_k is a square matrix of 0's of order k , I_k is the identity matrix of order k and $R(\phi) =$

$$C_{i,2}(t) = [-m_{it}^2 \quad m_{it} \quad 0 \quad 0 \quad m_{it} \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \quad C_{i,2}(t)|_{11} \quad C_{i,2}(t)|_{12} \quad 0 \quad 0 \quad C_{i,2}(t)|_{15} \quad C_{i,2}(t)|_{16}], \quad (12)$$

$$C_{i,1}(t) = [-2m_{it}q_{it} \quad 2q_{it} \quad 0 \quad 0], \quad (13)$$

$$C_{i,0}(t) = -q_{it}^2, \quad (14)$$

$$D_i(t) = [O_{1 \times 10} \quad D_i(t)|_{11} \quad D_i(t)|_{12} \quad 0 \quad 0 \quad D_i(t)|_{15} \quad D_i(t)|_{16}], \quad (15)$$

$$C_{i,2}(t)|_{11} = (\rho_{i,t}^2 + \zeta_{i,2})(\sin(\theta_i) - m_{it} \cos(\theta_i))^2, \quad (16)$$

$$C_{i,2}(t)|_{12} = C_{i,2}(t)|_{15} = (\rho_{i,t}^2 + \zeta_{i,2})(\sin(\theta_i) - m_{it} \cos(\theta_i))(\cos(\theta_i) + m_{it} \sin(\theta_i)), \quad (17)$$

$$C_{i,2}(t)|_{16} = (\rho_{i,t}^2 + \zeta_{i,2})(\cos(\theta_i) + m_{it} \sin(\theta_i))^2, \quad (18)$$

$$D_i(t)|_{11} = (\sin(\theta_i) - m_{it} \cos(\theta_i))^2, \quad (19)$$

$$D_i(t)|_{12} = D_i(t)|_{15} = (\sin(\theta_i) - m_{it} \cos(\theta_i))(\cos(\theta_i) + m_{it} \sin(\theta_i)), \quad (20)$$

$$D_i(t)|_{16} = (\cos(\theta_i) + m_{it} \sin(\theta_i))^2. \quad (21)$$

$[\cos(\phi), -\sin(\phi); \sin(\phi), \cos(\phi)]$ is a rotation matrix. Note that the first term in (30) is a linear drift; the other is a multiplicative noise. In the sequel, the moments $\mathbb{E}[N^i(t)]$ of the Gaussian variable N will be indicated by $\eta_{i,t}$ and will be needed up to degree 4 (remind that $\eta_{1,t} \equiv \eta_{3,t} \equiv 0$).

C. A bilinear generation model

The output equation (23) can be seen both as a quadratic transformation of the extended state, and as a linear transformation of an *augmented state* $\mathcal{X}(t) \in \mathbb{R}^{n+n^2}$, whose components are the first and second order Kronecker powers of the extended state $\mathcal{X}(t) = (\mathcal{X}_1^T(t) \quad \mathcal{X}_2^T(t))^T$, $\mathcal{X}_1(t) = X(t)$, $\mathcal{X}_2(t) = X^{[2]}(t)$. Then:

$$Y(t) = \mathcal{C}(t)\mathcal{X}(t) + \mathcal{V}(t) + \mathcal{N}_o(t), \quad (31)$$

with $\mathcal{C}(t) = [C_1(t) \quad C_2(t)]$; $\mathcal{V}(t) = C_0(t)$ a deterministic drift, and $\mathcal{N}_o(t) = H(t)\mathcal{X}_2(t)$ a multiplicative noise.

In order to obtain a bilinear (w.r.t. \mathcal{X}) generation model of the output equation in (31), the Kronecker square of the extended state $X(t)$ is needed. Using the binomial expansion of a Kronecker power (see [4] for more details):

$$X^{[2]}(t+1) = A_{22}(t)X^{[2]}(t) + S_2(t)X^{[2]}(t),$$

where:

$$A_{22}(t) = A^{[2]}(t) + B^{[2]}\eta_{2,t} + B_1^{[2]}(t)\xi_{c2,t} + B_2^{[2]}(t)\xi_{s2,t},$$

and $S_2(t)$ is the stochastic matrix given in (32), where \otimes denotes the Kronecker product among matrices. In summary:

$$\mathcal{X}(t+1) = \mathcal{A}(t)\mathcal{X}(t) + \mathcal{N}_s(t), \quad (33)$$

with:

$$\mathcal{A}(t) = \begin{bmatrix} A(t) & O_{n \times n^2} \\ O_{n^2 \times n} & A_{22}(t) \end{bmatrix};$$

$\mathcal{N}_s(t)$ is the following multiplicative noise:

$$\mathcal{N}_s(t) = \mathcal{S}(t)\mathcal{X}(t) = \begin{bmatrix} \mathcal{N}_{s1}(t) \\ \mathcal{N}_{s2}(t) \end{bmatrix} = \begin{bmatrix} S_1(t)\mathcal{X}_1(t) \\ S_2(t)\mathcal{X}_2(t) \end{bmatrix},$$

with:

$$\mathcal{S}(t) = \begin{bmatrix} S_1(t) & O_{n \times n^2} \\ O_{n^2 \times n} & S_2(t) \end{bmatrix}.$$

In order to compute the covariance matrices of $\mathcal{N}_s(t)$ and $\mathcal{N}_o(t)$, the definition of the *stack* operator is needed. The stack of a matrix A is the vector in $\mathbb{R}^{r \cdot c}$ that piles up all the columns of matrix A , and is denoted $\text{st}(A)$. The inverse operation is denoted $\text{st}_{r,c}^{-1}(\cdot)$, and transforms a vector of size $r \cdot c$ in an $r \times c$ matrix.

According to the multiplicative features of the noises, the computation of their covariance matrices requires the knowledge of the mean values of the Kronecker powers of the extended state up to degree 4. By denoting $Z_i(t) = \mathbb{E}[X^{[i]}(t)]$, $i = 1, \dots, 4$, after some Kronecker machineries:

$$\Psi_{s_{11}}(t) = \mathbb{E}[\mathcal{N}_{s1}(t)\mathcal{N}_{s1}^T(t)] = \text{st}_{n,n}^{-1}(\mathbb{E}[S_1^{[2]}(t)]Z_2(t)), \quad (34)$$

$$\begin{aligned} \Psi_{s_{12}}(t) &= \mathbb{E}[\mathcal{N}_{s1}(t)\mathcal{N}_{s2}^T(t)] \\ &= \text{st}_{n,n^2}^{-1}(\mathbb{E}[S_2(t) \otimes S_1(t)]Z_3(t)), \end{aligned} \quad (35)$$

$$\Psi_{s_{22}}(t) = \mathbb{E}[\mathcal{N}_{s2}(t)\mathcal{N}_{s2}^T(t)] = \text{st}_{n^2,n^2}^{-1}(\mathbb{E}[S_2^{[2]}(t)]Z_4(t)), \quad (36)$$

with:

$$\mathbb{E}[S_1^{[2]}(t)] = B^{[2]}\eta_{2,t} + B_1^{[2]}(t)\xi_{c2,t} + B_2^{[2]}(t)\xi_{s2,t} \quad (37)$$

The expression of $\mathbb{E}[S_2(t) \otimes S_1(t)]$ and $\mathbb{E}[S_2^{[2]}(t)]$, omitted here for space reasons, can be found in [16]. As far as the output noise covariance matrix is concerned, denote e_i , $i = 0, 1, \dots, m$ the natural basis of \mathbb{R}^{m+1} . Then, $H(t) = \sum_{i=1}^m e_i D_i(t) N_{o,i}(t)$, and, observing that $\{N_{o,i}, N_{o,j}\}$ are uncorrelated for $i \neq j$:

$$\begin{aligned} \Psi_o(t) &= \mathbb{E}[\mathcal{N}_o(t)\mathcal{N}_o^T(t)] = \mathbb{E}[H(t)\mathcal{X}_2(t)\mathcal{X}_2^T(t)H^T(t)] = \\ &= \sum_{i=1}^m e_i D_i(t) \cdot \text{st}_{n^2,n^2}^{-1}(Z_4(t)) D_i^T(t) e_i^T \mathbb{E}[N_{o,i}^2(t)], \end{aligned} \quad (38)$$

where $\mathbb{E}[N_{o,i}^2(t)] = \zeta_{i,4} - \zeta_{i,2}^2 + 4\rho_{i,t}^2 \zeta_{i,2}$.

$$S_2(t) = B^{[2]}(N^2(t) - \eta_{2,t}) + B_1^{[2]}(t)(N_c^2(t) - \xi_{c2,t}) + B_2^{[2]}(t)(N_s^2(t) - \xi_{s2,t}) + M_1^2(n) \left\{ (A(t) \otimes B)N(t) + (A(t) \otimes B_1(t))N_c(t) \right. \\ \left. + (A(t) \otimes B_2(t))N_s(t) + (B \otimes B_1(t))N(t)N_c(t) + (B \otimes B_2(t))N(t)N_s(t) + (B_1(t) \otimes B_2(t))N_c(t)N_s(t) \right\} \quad (32)$$

IV. THE FILTERING ALGORITHM

The position of the robot is given by the first two components of the augmented state, whose bilinear generation model, endowed with the output equation, is given by (33) and (31). The pair (x_t, y_t) is, then, estimated by means of the first two components of the augmented state estimate $\hat{\mathcal{X}}(t)$. As it is well known, the optimal choice for $\hat{\mathcal{X}}(t)$ would be the conditional expectation w.r.t. all the Borel transformations of the measurements, whose computation in general cannot be obtained through algorithms of finite dimension. Nevertheless, from an applicative point of view, it is useful to look for finite-dimensional approximations of the optimal filter. In the present paper an implementable recursive filter is proposed, providing the optimal estimate w.r.t. the Hilbert space of all the affine (first-order polynomial) transformations of the output $Y(\tau)$, $\tau = 0, \dots, t$ and performed as the projection of $\mathcal{X}(t)$ onto such a space [4]. As a consequence of the linear relationship between the pair (x_t, y_t) and $\mathcal{X}(t)$, it comes that the position estimate (\hat{x}_t, \hat{y}_t) is the optimal linear estimate w.r.t. the output Y . The orientation angle estimate is provided, by means of the third and fourth components of $\hat{\mathcal{X}}(t)$: $\hat{\theta}_t = \text{atan2}(\hat{s}_t, \hat{c}_t)$. According to the multiplicative feature of the noises, the computation of the covariance matrices of the noises $\mathcal{N}_s(t)$, $\mathcal{N}_o(t)$ requires the knowledge of the mean values of the Kronecker powers of the extended state up to degree 4 (as already reported in (34-36) and (38)). By defining $Z(t) = [Z_1^T(t) Z_2^T(t) Z_3^T(t) Z_4^T(t)]^T$, it is:

$$Z(t+1) = \mathcal{A}_Z(t)Z(t), \quad (39)$$

where $\mathcal{A}_Z(t)$ is a block-diagonal matrix with diagonal blocks: $A(t)$, $A_{22}(t)$, $A_{33}(t) = A_{22}(t) \otimes A(t) + \mathbb{E}[S_2(t) \otimes S_1(t)]$ and $A_{44}(t) = A_{22}^{[2]}(t) + \mathbb{E}[S_2^{[2]}(t)]$. The initial condition $Z(0)$ has to be taken from the *a priori* knowledge concerning the initial state of the system. Assume x_0, y_0, θ_0 are independent Gaussian random variables, with mean value $\bar{x}_0, \bar{y}_0, \bar{\theta}_0$ and variance $\sigma_{x_0}^2, \sigma_{y_0}^2, \sigma_{\theta_0}^2$, respectively. From these statistics, by a direct computation and using the moments $\xi_{ci,sj,t}$, it is possible to derive $Z(0)$ and the covariance matrix $P_P(0) = \text{Cov}(\mathcal{X}(0))$ of the initial extended state vector. The filter algorithm is below reported:

I) Set $t = -1$ and compute the initial conditions:

$$\hat{\mathcal{X}}(0|-1) = \begin{bmatrix} Z_1(0) \\ Z_2(0) \end{bmatrix}, \quad P_P(0) = \text{Cov}(\mathcal{X}(0));$$

II) compute the output prediction:

$$\hat{Y}(t+1|t) = \mathcal{C}(t+1)\hat{\mathcal{X}}(t+1|t) + \mathcal{V}(t+1);$$

III) compute the output noise covariance matrix $\Psi_o(t+1)$ by means of (38);

IV) compute the Kalman gain and error covariance:

$$K(t+1) = P_P(t+1)\mathcal{C}^T(t+1) \cdot$$

$$\left(\mathcal{C}(t+1)P_P(t+1)\mathcal{C}^T(t+1) + \Psi_o(t+1) \right)^\dagger,$$

$$P(t+1) = \left(I_{n+n^2} - K(t+1)\mathcal{C}(t+1) \right) P_P(t+1);$$

where \dagger denotes the Moore-Penrose pseudo-inverse;

V) compute the estimate $\hat{\mathcal{X}}(t+1)$:

$$\hat{\mathcal{X}}(t+1) = \hat{\mathcal{X}}(t+1|t) + K(t+1)(Y(t+1) - \hat{Y}(t+1|t)),$$

$$\hat{X}(t+1) = [I_n \quad O_{n \times n^2}] \hat{\mathcal{X}}(t+1),$$

VI) increment the counter: $t \mapsto t+1$;

VII) compute the prediction: $\hat{\mathcal{X}}(t+1|t) = \mathcal{A}(t)\hat{\mathcal{X}}(t)$;

VIII) compute the state noise covariance matrix $\Psi_s(t)$ by means of (34-36);

IX) compute the error covariance of the one-step prediction:

$$P_P(t+1) = \mathcal{A}(t)P(t)\mathcal{A}^T(t) + \Psi_s(t);$$

X) compute $Z(t)$ by means of (39) and go to Step **II**.

Remark. For consistency with all the developments made in the paper, the proposed algorithm has been here presented in a form that is not computationally optimized, in that the Kronecker powers contain redundant components (if $x \in \mathbb{R}^n$ then $x^{[i]} \in \mathbb{R}^{n^i}$, but only $\tilde{n}_i = \binom{n+i-1}{i}$ monomials are independent). Such redundancies can be avoided through the definition of *reduced Kronecker powers*, containing the independent components of ordinary Kronecker powers (see [3]). More in detail, denoting with $x^{(i)} \in \mathbb{R}^{n^i}$ the reduced i -th Kronecker power of x , it is always possible to define a selection matrix $T_i(n) \in \mathbb{R}^{n^i \times n^i}$, of 0's and 1's, such that:

$$x^{(i)} = T_i(n)x^{[i]} \quad (40)$$

(note that the choice of matrix $T_i(n)$ is not univocal). Similarly, the ordinary Kronecker power $x^{[i]}$ is recovered from the reduced power $x^{(i)}$ through multiplication with a suitable matrix $\tilde{T}_i(n) \in \mathbb{R}^{n^i \times \tilde{n}_i}$. Straightforward but tedious substitutions in the above algorithm provide a filter with a reduced computational burden, and this last should be considered for efficient implementations.

V. SIMULATION RESULTS

In this section the performances of our algorithm are compared with those of a standard EKF. A circular trajectory with radius 1.5 m in a rectangular room is considered (see Fig. 1). The robot moves along the circle in the counterclockwise direction at a constant speed starting from the asterisk. The results reported in this section refer to simulations performed with $K_\rho = 0.01\text{m}$, $K_\theta = 0.02\text{rad}^2/\text{m}$ and $m = 16$ laser

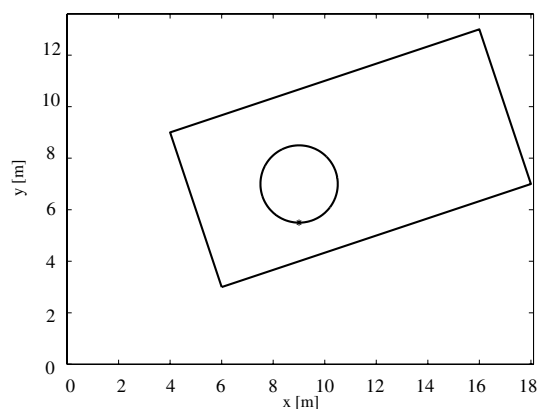


Fig. 1. The environment and the actual robot trajectory

directions. The motion is performed in $T = 1000$ steps, hence $\delta\rho_t = 9.4 \cdot 10^{-3}$ m and $\delta\theta_t = 6.3 \cdot 10^{-3}$ rad. We assume that the odometry readings are available at each motion step, while the laser readings are available every N_s steps, with $N_s \geq 1$. We have considered $N_s = 1$ in the simulations of Fig. 2 while $N_s = 50$ in Figs. 3, 4 and 5. Figs. 2 and 3 display the robot trajectory estimated through the EKF, the proposed method and the odometry together with the actual trajectory. With $N_s = 1$ both the proposed approach and the EKF provide a very good estimate (Fig. 2). However, with $N_s = 50$ the proposed filter performs nearly exact corrections unlike the EKF, whose linear approximations become too rough when the estimation errors are too large (Fig. 3). Figs. 4 and 5 report respectively the position and the orientation errors as a function of the traveled distance when applying the EKF, the proposed method and the odometry for $N_s = 50$. Finally, in Figs. 6 and 7 the effect of an increasing N_s on the EKF and the proposed method is illustrated. In Fig. 6 each point represents the square root of the mean square position error $(\frac{1}{T} \sum_t [(x_t - \hat{x}_t)^2 + (y_t - \hat{y}_t)^2])^{1/2}$ and is obtained as the average of 50 different simulations (with different noise realizations). Fig. 7 reports the same for the orientation. It is important to stress that for values of $N_s > 50$, in some simulations the EKF estimate has diverged, while our approach has maintained a bounded error. The performance of the proposed method has been tested under several maneuvers, with many different noise parameters, both in the odometry and in the laser readings, and similar results have been obtained.

VI. CONCLUSIONS

In this paper we introduced and discussed a new analytical solution to the localization problem. In contrast with other previous approaches able to deal with the system non linearities (Markov Localization, Monte Carlo Localization), our algorithm is not a numerical solution. In particular, by introducing a new set of variables for the dynamics and for the readings and suitably exploiting some results of the polynomial filtering approach [3], [4], [11] a bilinear description of the system is derived and the best affine

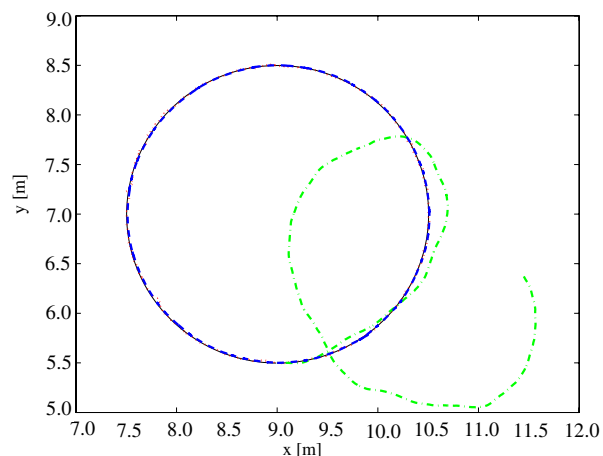


Fig. 2. Real (black solid) and estimated trajectories with $N_s = 1$. Odometry: green dot-dashed; EKF: blue dashed; proposed filter: red dotted

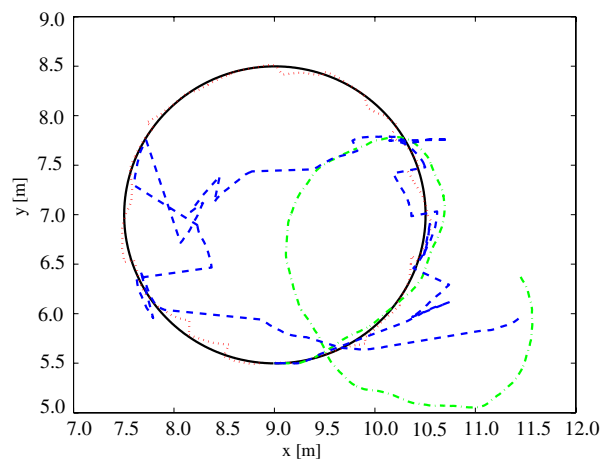


Fig. 3. Real (black solid) and estimated trajectories with $N_s = 50$. Odometry: green dot-dashed; EKF: blue dashed; proposed filter: red dotted

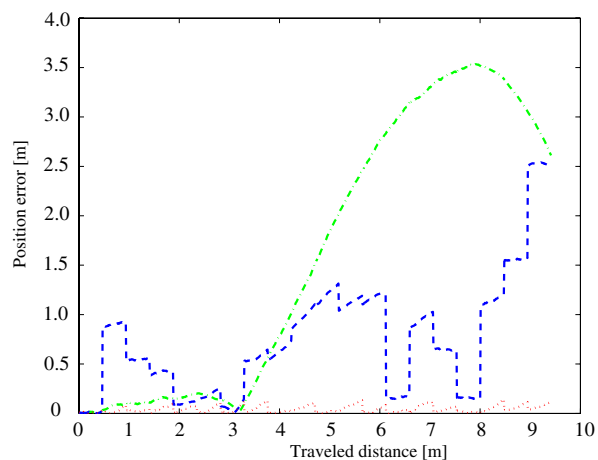


Fig. 4. Position error with $N_s = 50$ vs traveled distance. Odometry: green dot-dashed; EKF: blue dashed; proposed filter: red dotted

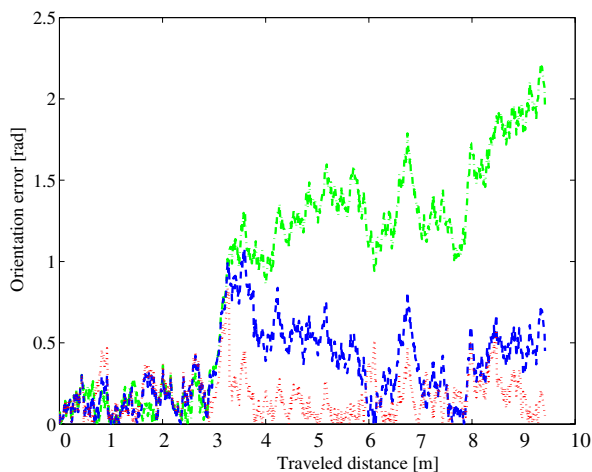


Fig. 5. Orientation error with $N_s = 50$ vs traveled distance. Odometry: green dot-dashed; EKF: blue dashed; proposed filter: red dotted

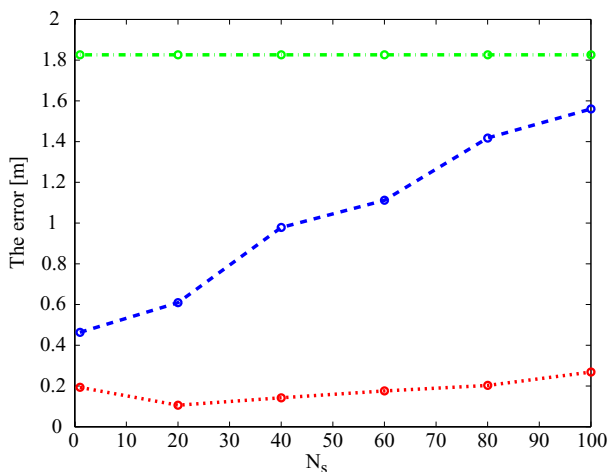


Fig. 6. Square root of the mean square position error vs N_s . Odometry: green dot-dashed; EKF: blue dashed; proposed filter: red dotted

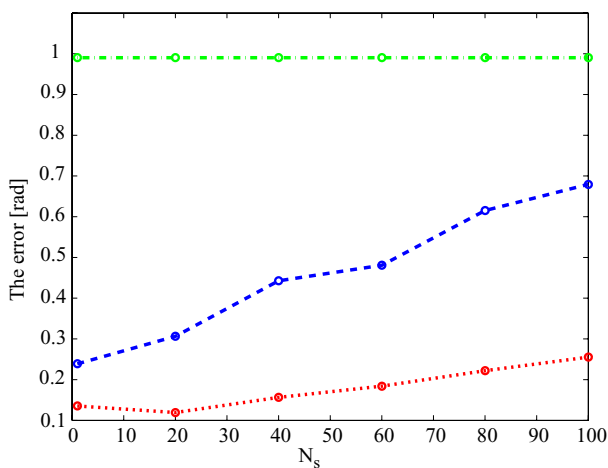


Fig. 7. Square root of the mean square orientation error vs N_s . Odometry: green dot-dashed; EKF: blue dashed; proposed filter: red dotted

estimator of the robot configuration is obtained.

In order to evaluate the performance of our algorithm we carried out a comparison with the standard EKF. The simulation results clearly show that our algorithm outperforms the standard EKF especially in critical situations (low sensor precision, low data frequency).

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