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Static and Stiffness Analyses of a Class of Over-Constrained Parallel Manipulators with Legs of Type US and UP̲S

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Abstract—**This paper deals with a class of over-constrained parallel manipulators that can be obtained from architecturally singular parallel mechanisms with legs of type US (U and S stand for universal and spherical joints respectively) by the addition of legs of type UP̲S (P̲ stands for actuated prismatic pair). These manipulators can be internally preloaded in order to tune the system stiffness, to diminish system backlash and to control the state of tension of their structural elements. The paper, in particular, addresses the static and stiffness analyses of this class of manipulators. A symmetric stiffness matrix defined as the Hessian of the elastic potential energy of the system is used, as an alternative to the Cartesian stiffness matrix used in this field, to describe the stiffness characteristic of these manipulators. An over-constrained 2-dof parallel spherical wrist is analyzed as a case study.**

I. Introduction

ARCHITECTURALLY singular mechanisms [1]-[3] with n legs of type US (n-US-PMs for short) are unactuated closed-loop mechanisms that comprise a fixed base and a moving platform connected to each other through n kinematic chains (legs) each consisting of a binary link of type US (U and S are for universal and spherical joints, respectively) having constant length. These mechanisms are finitely mobile with mobility m greater than $(6 - n)$ (as the Grübler's formula would predict). They are over-constrained mechanisms that feature simple elements such as compression rods and lightweight cables. In practice, these mechanisms can be employed as internally preloaded, zero-backlash, stiff, strong and lightweight complex kinematic pairs to build over-constrained parallel manipulators with m degrees of freedom (dof).

A class of over-constrained parallel manipulators can indeed be obtained from n-US-PMs by the addition of m actuated legs of type UP̲S (P̲ stands for actuated prismatic P joint), each connected to the base and platform by means of a U joint and an S joint respectively, and placed so that its axis (the line passing through the centers of the U and S joints) does not belong to the linear variety of lines spanned by the axes of the other $(n + m - 1)$ legs. Over-constrained parallel manipulators of this kind potentially possess large (eventually tunable) stiffness, low inertia, high accuracy and large payload capacity.

An example of a manipulator belonging to this class is depicted in Fig. 1. The system comprises a Wren's mechanism [4], i.e. an n-US-PM with finite mobility $m = 1$ and platform with a helical motion, made by n legs (n = 6 in Fig. 1) of equal length and of type US whose U and S joint centers are arranged identically on equal circles in the base and in the platform, and by one leg of type UP̲S whose U and S joint centers are located on the centers of the aforementioned circles. This manipulator can be used to control the relative helical motion of base and platform about the axis of the leg of type UP̲S. The helical motion is provided by Wren's mechanism (a linear-to-angular displacement device [5]) that works as an unconventional helical pair that may enjoy high strength, low weight, high efficiency, absence of lubricants and low cost with respect to traditional transmissions based on gears or ball screws [5].

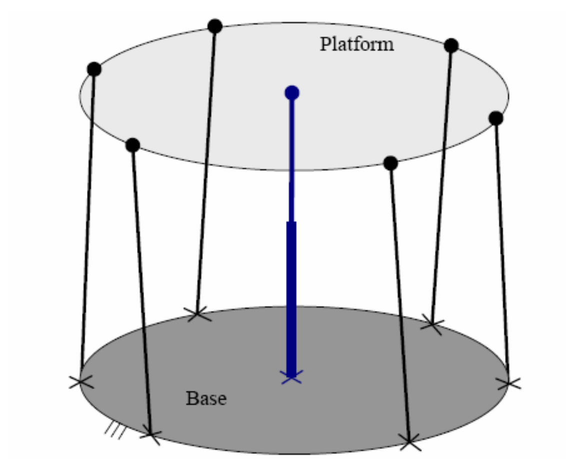

Fig. 1. Over-constrained parallel manipulator made by an architecturally singular 6-US-PM (Wren's mechanism) and by one actuated telescopic leg of type UP̲S.

Effective design, optimization and performance evaluation of this class of over-constrained parallel manipulators requires a framework to study their stiffness characteristic. Due to the slenderness of the legs with respect to the base and platform, such systems can be modeled as simple compliant couplings [6] (i.e. frictionless and massless mechanisms made by a rigid base and a rigid platform connected by translational springs by means of U and S joints) that are internally preloaded in their equilibrium position. In the literature [6]-[15], the stiffness characteristic of simple compliant couplings is always described by the

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Cartesian stiffness matrix, which defines how the components of a wrench acting on the platform change as the platform moves along basis twists. In particular, the Cartesian stiffness matrices of simple compliant couplings that stay near their unloaded configurations were discussed in [6]-[10] where these matrices were shown to be always symmetric. The Cartesian stiffness matrices of simple compliant couplings that stay near their (externally) loaded configurations were discussed in [11]-[15] where these matrices were shown to be usually asymmetric.

This paper discusses an approach for the study of the stiffness characteristic of a class of over-constrained parallel manipulators that are derived from architecturally singular mechanisms. In particular, the paper deals with the stiffness analysis of internally preloaded compliant couplings that stay near both their unloaded configurations and their (externally) loaded configurations. In this context, an alternative form of the stiffness matrix is presented, which is obtained from the elastic potential energy of simple compliant couplings via the second partial derivatives with respect to the independent generalized coordinates that uniquely describe the configuration of the system. It is shown that the alternative stiffness matrix has the property of being always symmetric even when the compliant couplings are near their loaded configurations. An application to a simple compliant coupling derived from a 2-dof spherical over-constrained parallel manipulator devised in [16] is provided which shows the potentials of the proposed approach and highlights the static and stiffness properties of the manipulator.

II. Internally Preloaded Compliant Couplings

In this section, internally preloaded compliant couplings are described by taking as a reference the system depicted in Fig. 2 that, in practice, can be used to model the stiffness characteristic of the over-constrained parallel manipulator shown in Fig. 1.

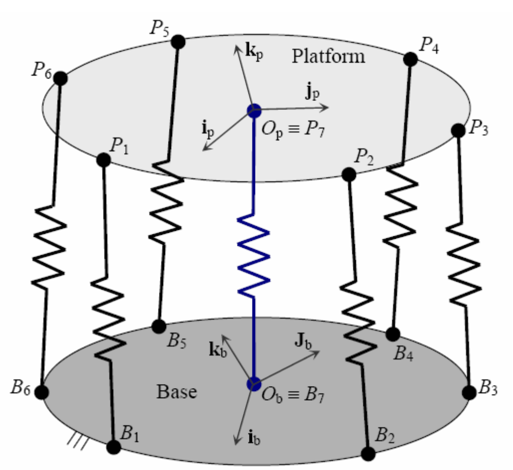

Fig. 2. Internally preloaded compliant coupling to be used as a model for the study of the stiffness characteristic of the over-constrained parallel manipulator depicted in Fig. 1.

Internally preloaded compliant couplings consist of two rigid bodies, a fixed base and a moving platform, which are connected by $(n + m)$ frictionless and massless translational springs acting in parallel (in Fig. 2 $n = 6$ and $m = 1$). Each translational spring is connected to the base and to the platform by a U joint and an S joint. It is worth mentioning that, for each translational spring, the location of the U and S joints can be interchanged. Thus, for clarity of representation, both U and S joints are depicted by the same symbol (a dot) in Fig. 2. By comparison with the over-constrained parallel manipulator shown in Fig. 1, it can be understood that the n springs model the structural compliance of the legs of type US while the m springs model the structural and servo (i.e. related to the loop gain of the control system) compliance of the $\underline{\text{P}}$ actuators of the legs of type U$\underline{\text{P}}$S.

For every (both regular and singular) configuration of the system, i.e. for every relative location of base and platform, the axes of the n translational springs form a linear variety of lines with rank $(6 - m)$. For every (regular) configuration of the system the axes of the m translational springs do not belong to the linear variety of lines spanned by the others $(n + m - 1)$ so that the axes of the $(n + m)$ translational springs form a linear variety of lines with rank 6. The linear dependency of the lines of action of the n translational springs allows the system to be internally preloaded. Indeed, by properly changing the stiffness and the free lengths of the n legs it is possible to tune the state of tension of the n springs without affecting the state of equilibrium (configuration and external loads) of the overall system. Internal preload is important since it makes it possible to define the state of tension and consequently the type of elements (compression rods or lightweight cables) that compose the system so as to make it stiffer and more lightweight, to reduce system backlash and tune the stiffness characteristic of the system to make it stable and more or less compliant.

III. Notation

In this section, the symbols and the definitions that characterize the geometry and the configuration of preloaded compliant couplings are introduced.

For each h-th translational spring, $B_{(h)}$ and $P_{(h)}$ are the attachment points, i.e. the centers of the U and S joints connected to the base and to the platform respectively. Each translational spring is linear and characterized by its actual length $l_{(h)} = \left\| \overline{B_{(h)}P_{(h)}} \right\|$, free length $l_{(h)0}$ and stiffness $\kappa_{(h)}$. The length $l_{(h)}$ varies and depends on the relative location of the base and platform, while $l_{(h)0}$ and $\kappa_{(h)}$ are constant and known. The ratio between the free and the actual length of the h-th spring is defined by $\delta_{(h)}$, i.e. $\delta_{(h)} = l_{(h)0} / l_{(h)}$.

Description of the relative location of base and platform requires the definition of the two coordinate systems $S_b \equiv \{O_b;\ \mathbf{i}_b,\ \mathbf{j}_b,\ \mathbf{k}_b\}$, with origin O_b, and $S_p \equiv \{O_p;\ \mathbf{i}_p,\ \mathbf{j}_p,\ \mathbf{k}_p\}$, with origin O_p, which are embedded in the base and in the platform respectively. Here, $\mathbf{i}_b$, $\mathbf{j}_b$, $\mathbf{k}_b$ are the unit vectors of the axes X, Y, Z of the system S_b and $\mathbf{i}_p$, $\mathbf{j}_p$, $\mathbf{k}_p$ are the unit vectors of the axes x, y, z of the system S_p.

The geometry of compliant couplings is defined by the arrays of coordinates $\boldsymbol{P}_{(h)} = [x_{(h)}, y_{(h)}, z_{(h)}]^T$ of the attachment point $P_{(h)}$ with respect to S_p and by the arrays of coordinates $\boldsymbol{B}_{(h)} = [X_{(h)}, Y_{(h)}, Z_{(h)}]^T$ of the attachment points $B_{(h)}$ with respect to S_b.

The relative location of base and platform is characterized by the three displacements, q_1, q_2, q_3 of point O_p with respect to the X-axis, Y-axis and Z-axis, respectively, and by three successive rotations defined by the angles q_4, q_5 and q_6 performed in the following sequence: first q_6 about the z-axis, then q_5 about the y-axis and finally q_4 about the x-axis. In compact form, the configuration of a compliant coupling is uniquely identified by the generalized coordinate vector $\boldsymbol{q}$ of the independent parameters q_i

$$\boldsymbol{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \end{bmatrix}^T . \tag{1}$$

IV. Elastic Potential Energy and Static Equilibrium

Compliant couplings are conservative systems whose associated elastic potential energy Ψ is given by

$$\Psi = \frac{1}{2}\sum_{h=1}^{n+m} \kappa_{(h)} \left[l_{(h)} - l_{(h)0} \right]^2 , \tag{2}$$

which is a function of the generalized coordinate vector $\boldsymbol{q}$ through the relation

$$l_{(h)} = \sqrt{\boldsymbol{P}_{(h)}^T \boldsymbol{P}_{(h)} + \boldsymbol{U}_{(h)}^T \left(\boldsymbol{U}_{(h)} + 2\mathbf{R}\boldsymbol{P}_{(h)} \right)} , \tag{3}$$

where

$$\boldsymbol{U}_{(h)} = \boldsymbol{O}_p - \boldsymbol{B}_{(h)} , \tag{4}$$

$$\boldsymbol{O}_p = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^T , \tag{5}$$

and $\mathbf{R}$ is the 3×3 rotation matrix for the transformation of vector components from S_p to S_b. According to the generalized coordinates defined in Section III, $\mathbf{R}$ reads as

$$\mathbf{R} = \begin{bmatrix} c_5 c_6 & s_5 c_6 s_4 - s_6 c_4 & s_5 c_6 c_4 + s_6 s_4 \\ c_5 s_6 & s_5 s_6 s_4 + c_6 c_4 & s_5 s_6 c_4 - c_6 s_4 \\ -s_5 & c_5 s_4 & c_5 c_4 \end{bmatrix} . \tag{6}$$

where $c_i = \cos(q_i)$ and $s_i = \sin(q_i)$, for $i = 4, 5, 6$.

Based on the principles of statics, a compliant coupling is in a configuration of static equilibrium when

$$Q_i = \frac{\partial \Psi}{\partial q_i} = \sum_{h=1}^{n+m} \kappa_{(h)} \left(1 - \delta_{(h)}\right) \left(\boldsymbol{U}_{(h)} + \mathbf{R}\boldsymbol{P}_{(h)} \right)_i , \; i = 1, 2, 3 \tag{7.1}$$

$$Q_i = \frac{\partial \Psi}{\partial q_i} = \sum_{h=1}^{n+m} \kappa_{(h)} \left(1 - \delta_h\right) \boldsymbol{U}_{(h)}^T \frac{\partial \mathbf{R}}{\partial q_i} \boldsymbol{P}_{(h)} , \; i = 4, 5, 6 \tag{7.2}$$

where the Q_i are the generalized forces related to the system of external forces and moments (all but the elastic ones) that act on the compliant coupling. Note that, in Eq. (7.1) and in the following ones as well, the notation $(\boldsymbol{v})_i$ is used to indicate the i-th component of a vector $\boldsymbol{v}$ with respect to S_b.

As for the physical interpretation, the generalized forces Q_1, Q_2 and Q_3 are the components of the total elastic force $\boldsymbol{F}^E$

$$\boldsymbol{F}^E = \sum_{h=1}^{n+m} \kappa_{(h)} \left(1 - \delta_{(h)}\right) \overline{B_{(h)} P_{(h)}} , \tag{8}$$

along the unit vectors $\mathbf{i}_b$, $\mathbf{j}_b$ and $\mathbf{k}_b$ respectively, while the generalized forces Q_4, Q_5 and Q_6 are the components of the total elastic moment $\boldsymbol{M}_{O_p}^E$ (with respect to O_p) of the elastic forces

$$\boldsymbol{M}_{O_p}^E = \sum_{h=1}^{n+m} \kappa_{(h)} \left(1 - \delta_{(h)}\right) \overline{O_p B_{(h)}} \times \overline{O_p P_{(h)}} , \tag{9}$$

along the unit vectors $\mathbf{i}_p$, $\mathbf{j}'$ and $\mathbf{k}_b$, where $\mathbf{j}'$ is the unit vector of the moving axis about which the rotation by the angle q_5 is performed (the components of $\mathbf{j}'$ in S_b are $[-s_3, c_3, 0]^T$).

In compact notation, Eqs. (7) can be rewritten as

$$\boldsymbol{Q} = \mathbf{A}\boldsymbol{f} , \tag{10}$$

where

$$\boldsymbol{Q} = \begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \end{bmatrix}^T , \tag{11}$$

is the 6x1 vector of the generalized forces,

$$\boldsymbol{f} = \left[\kappa_{(1)} \left(l_{(1)} - l_{(1)0} \right) \;\; \ldots \;\; \kappa_{(n+m)} \left(l_{(n+m)} - l_{(n+m)0} \right) \right]^T , \tag{12}$$

is the $(n + m) \times 1$ vector of the elastic forces exerted by the translational springs, and

$$\mathbf{A} = \begin{bmatrix} \left(\boldsymbol{U}_{(1)} + \mathbf{R}\boldsymbol{P}_{(1)} \right) / l_{(1)} & \ldots & \left(\boldsymbol{U}_{(n+m)} + \mathbf{R}\boldsymbol{P}_{(n+m)} \right) / l_{(n+m)} \\ \dfrac{1}{l_{(1)}} \boldsymbol{U}_{(1)}^T \dfrac{\partial \mathbf{R}}{\partial q_i} \boldsymbol{P}_{(1)} & \ldots & \dfrac{1}{l_{(n+m)}} \boldsymbol{U}_{(n+m)}^T \dfrac{\partial \mathbf{R}}{\partial q_i} \boldsymbol{P}_{(n+m)} \end{bmatrix} , \tag{13}$$

is the $6 \times (n + m)$ matrix that maps the spring forces $\boldsymbol{f}$ into the generalized forces $\boldsymbol{Q}$. In regular configurations, since the axes of the $(n + m)$ springs form a linear variety of lines with rank 6, matrix **A** has rank equal to 6.

The generalized force $\boldsymbol{Q}$ is a function of the relative position of the base and platform and is representative of the external load that is required to keep the system in equilibrium in the configuration $\boldsymbol{q}$. Indeed, for a given compliant coupling with known $\kappa_{(h)}$ and $l_{(h)0}$, for $h = 1, \ldots, (n + m)$, the generalized coordinate vector $\boldsymbol{q}$ defines the actual length $l_{(h)}$ and thus it defines the vector $\boldsymbol{f}$ that determines via Eq. (10) the external load $\boldsymbol{Q}$ required for equilibrium.

If a $\boldsymbol{q}$ exists for which $\boldsymbol{f} = 0$ (that is $\delta_{(h)} = 1$, for every h), then $\boldsymbol{Q}$ vanishes, which means that no external load is required to maintain the system at $\boldsymbol{q}$. If a $\boldsymbol{q}$ exists for which $\boldsymbol{f} \neq 0$ ($\delta_{(h)} \neq 1$, for some h) and $\boldsymbol{Q} = 0$, then the system is internally preloaded but no external load is required to maintain the system at $\boldsymbol{q}$.

V. External and Internal Preload

Compliant couplings, which are in equilibrium in the configuration $\boldsymbol{q}$ under the external load $\boldsymbol{Q}$, are said to be preloaded when some components of the elastic force vector $\boldsymbol{f}$ are non zero. The study of the preloading of a compliant coupling amounts to finding all the elastic force vectors $\boldsymbol{f}$ that satisfy Eq. (10) for known $\boldsymbol{q}$ and $\boldsymbol{Q}$. Since the matrix **A** has dimension $6 \times (n + m)$, i.e. it has more columns than rows, it cannot be directly inverted. As customary in linear algebra, solution of the problem can be obtained by resorting to both the notion of weighed pseudoinverse of **A**, i.e. the $(n + m) \times 6$ matrix $\mathbf{A}_{\mathrm{W}}^{+}$ defined as

$$\mathbf{A}_{\mathrm{W}}^{+} = \mathbf{W}\mathbf{A}^{T}\left(\mathbf{A}\mathbf{W}\mathbf{A}^{T}\right)^{-1}, \tag{14}$$

where the $(n + m) \times (n + m)$ weighting matrix **W** is chosen here as

$$\mathbf{W} = \mathrm{diag}\left(\kappa_{(1)}, \ldots, \kappa_{(n+m)}\right), \tag{15}$$

and the notion of nullspace of **A**, i.e. the vector space defined by the $(n + m - 6)$ linearly independent vectors $\boldsymbol{n}_i$ such that

$$\mathbf{A}\boldsymbol{n}_i = 0 . \tag{16}$$

Note that, the vectors $\boldsymbol{n}_i$ can be obtained by either the QR decomposition or the singular value decomposition of **A**.

As a matter of fact, the expressions

$$\boldsymbol{f} = \mathbf{A}_{\mathrm{W}}^{+}\boldsymbol{Q} + \sum_{i=1}^{n+m-6} a_i \boldsymbol{n}_i \text{ , for } \forall a_i \tag{17}$$

are solutions of Eq. (10). Equations (17) collect the $\infty^{(n+m-6)}$ states of preloading that guarantee the equilibrium of the compliant coupling in the configuration $\boldsymbol{q}$ under the external load $\boldsymbol{Q}$. The term $\mathbf{A}_{\mathrm{W}}^{+}\boldsymbol{Q}$ represents the (external) preload that is necessary in order to balance the external load $\boldsymbol{Q}$. Note that, among all the possible choices, the solution $\boldsymbol{f} = \mathbf{A}_{\mathrm{W}}^{+}\boldsymbol{Q}$ minimizes the elastic potential energy Ψ, given in Eq. (2), which is stored in the compliant coupling in the state of equilibrium defined by the configuration $\boldsymbol{q}$ and by the external load $\boldsymbol{Q}$. The term $\sum_{i=1}^{n+m-6} a_i \boldsymbol{n}_i$ represents the internal preload that can be given (as desired) to the compliant coupling in order to simplify the type of elements of the system (compression rods and lightweight cables), to stabilize the system, to reduce system backlash and to tune the system stiffness (for details refer to section VI).

Note that once a desired preload $\boldsymbol{f}$ is chosen, it can be given to the system by choosing the following free lengths $l_{(h)0}$

$$l_{(h)0} = l_{(h)} - \kappa_{(h)}^{-1} f_{(h)}, \quad h = 1, \ldots, (n + m). \tag{18}$$

VI. Stiffness Matrix

Stiffness is a geometric mapping that transforms variations of the generalized coordinates of the mechanism into incremental changes of the generalized forces acting on it. As such, the stiffness of compliant couplings can be described by the stiffness matrix **K** whose components K_{ij} (also called coefficients of influence) read as

$$K_{ij} = \frac{\partial Q_i}{\partial q_j} = \frac{\partial^2 \Psi}{\partial q_i \partial q_j}, \quad i, j = 1, \ldots, 6. \tag{19}$$

Of course, matrix **K** is 6×6 and is symmetric (because of the Schwartz theorem).

Use of Eqs. (7) in Eq. (19) leads to the following expressions

$$K_{ij} = \sum_{h=1}^{n+m} \kappa_{(h)} \left\{ \left(1 - \delta_{(h)}\right)\delta_{ij} + \frac{\delta_{(h)}}{l_{(h)}^2} \begin{bmatrix} \left(\boldsymbol{U}_{(h)} + \mathbf{R}\boldsymbol{P}_{(h)}\right)_i \cdot \\ \cdot \left(\boldsymbol{U}_{(h)} + \mathbf{R}\boldsymbol{P}_{(h)}\right)_j \end{bmatrix} \right\}, \tag{20}$$

$i, j = 1, 2, 3$

$$K_{ij} = \sum_{h=1}^{n+m} \kappa_{(h)} \left\{ \boldsymbol{U}_{(h)}^T \begin{bmatrix} \left(1 - \delta_{(h)}\right) \dfrac{\partial^2 \mathbf{R}}{\partial q_i \partial q_j} + \\ + \dfrac{\delta_{(h)}}{l_{(h)}^2} \dfrac{\partial \mathbf{R}}{\partial q_i} \boldsymbol{P}_{(h)} \boldsymbol{U}_{(h)}^T \dfrac{\partial \mathbf{R}}{\partial q_j} \end{bmatrix} \boldsymbol{P}_{(h)} \right\}, \tag{21}$$

$i, j = 4, 5, 6$

$$K_{ij} = \sum_{h=1}^{n+m} \kappa_{(h)} \left\{ \begin{bmatrix} \left(1 - \delta_{(h)}\right) \left(\dfrac{\partial \mathbf{R}}{\partial q_j} \boldsymbol{P}_{(h)} \right)_i + \\ + \dfrac{\delta_{(h)}}{l_{(h)}^2} \left(\boldsymbol{U}_{(h)} + \mathbf{R}\boldsymbol{P}_{(h)}\right)_i \boldsymbol{U}_{(h)}^T \dfrac{\partial \mathbf{R}}{\partial q_j} \boldsymbol{P}_{(h)} \end{bmatrix} \right\}, \tag{22}$$

$i = 1, 2, 3$ and $j = 4, 5, 6$

which are functions of the generalized coordinate vector $\boldsymbol{q}$, i.e. of the relative position of base and platform. Note that the Kronecker delta δ_{ij} has been used in Eq. (20). As for the terms that contribute to the coefficients of the stiffness matrix, the terms multiplied by $(1 - \delta_{(h)})$ represent the contributions provided by the internal and by the external preloads, while the terms multiplied by $\delta_{(h)}$ represent the contributions provided by the system geometry.

Equations (20)-(22) show that, for a given internally preloaded compliant coupling that is in equilibrium in the configuration $\boldsymbol{q}$ under the external load $\boldsymbol{Q}$, the coefficients of the matrix $\mathbf{K}$, and thus the stiffness characteristic of the system, can be tuned by proper choices of the free lengths $l_{(h)0}$ that lie within the space defined by Eq. (18). In particular, choosing the $l_{(h)0}$ that render the matrix $\mathbf{K}$ positive definite is the way to make the system stable; choosing the $l_{(h)0}$ that guarantee that certain coefficients K_{ij} are above or below a given threshold is the way to make the system selectively compliant in certain directions.

VII. Example: Analysis of an Over-Constrained 2-dof Spherical Parallel Manipulator

In this section, the framework proposed for the stiffness analysis of internally preloaded compliant couplings is used to investigate the preloading and the stiffness characteristic of an over-constrained 2-dof spherical parallel manipulator. The architecture of the system and the states of equilibrium about which the analyses are performed have been kept as simple as possible in order to obtain results that are meaningful yet fit within the size of the paper.

A. Manipulator Description

A class of over-constrained 2-dof spherical parallel manipulators that can be obtained from three families of architecturally singular spherical 5-US-PMs (n = 5, m = 2) by the addition of 2 actuated legs of type UPS were described in [16].

One of these manipulators is depicted in Fig. 3. It features a fixed base and a moving platform, five legs of type US (P_1B_1, P_2B_2, P_3B_3, P_4B_4 and P_5B_5) and two legs of type UPS (P_6B_6 and P_7B_7) with actuated prismatic pairs. The locations of the U and S joints in the base and in the platform, and the lengths of the five legs of type US are chosen so that for every configuration of the US-PM their axes belong to a degenerate congruence with rank 4, i.e. the variety of lines that lie in the plane defined by the unit vectors $\mathbf{k}_b$ and $\mathbf{i}_p$ or pass through point O_p (the intersection of $\mathbf{k}_b$ and $\mathbf{i}_p$). The center B_6 of the joint that connects leg P_6B_6 to the base is located on the axis $\mathbf{k}_b$, while the center P_7 of the joint that connects leg P_7B_7 to the platform is located on the axis $\mathbf{i}_p$. The manipulator has two decoupled degrees of freedom that are represented by a rotation of the platform about $\mathbf{i}_p$ that is controlled by P_6B_6 only, and of a rotation of the platform about $\mathbf{k}_b$ that is controlled by P_7B_7 only.

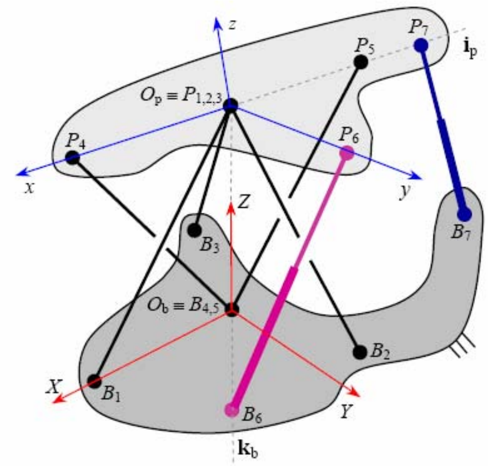

Fig. 3. 2-dof spherical over-constrained parallel manipulator made by a spherical architecturally singular 5-US-PM [16] and by two actuated telescopic legs of type UPS.

By properly choosing the stiffness and the free lengths of the five legs of type US (P_iB_i, i = 1, …, 5), the manipulator can be internally preloaded to increase the mechanism stiffness-to-weight and strength-to-weight ratios, to reduce system backlash, to tune the system stiffness and to allow the system to be built through simpler elements such as compression rods and lightweight cables (e.g. legs P_4B_4 and P_5B_5 can be compression rods while legs P_1B_1, P_2B_2 and P_3B_3 can be lightweight cables, or vice versa). The manipulator offers the potential for stiff, lightweight yet robust systems and is suited for high demanding applications, for instance in aerospace and automotive fields, such as pointing systems for mirrors and antennas, steering systems of vehicles and joints for biomimetic robots.

The manipulator considered in this example has the following geometry: the parameters $X_{(4,5,6)}$, $Y_{(1,4,5,6)}$, $Z_{(1,2,3,4,5)}$, $x_{(1,2,3,6)}$, $y_{(1,2,3,4,5,7)}$ and $z_{(1,2,3,4,5,6,7)}$ are zero, while $X_{(1)} = a$, $X_{(2)} = a\cos(2\pi/3)$, $X_{(3)} = a\cos(-2\pi/3)$, $X_{(7)} = -d$, $Y_{(2)} = a\sin(2\pi/3)$, $Y_{(3)} = a\sin(-2\pi/3)$, $Y_{(7)} = -d$, $Z_{(6)} = (a - b)$, $Z_{(7)} = a$, $x_{(4)} = a$, $x_{(5)} = -a$, $x_{(7)} = -d$, $y_{(6)} = b$. The lengths of the legs of type US are $l_{(1,2,3,4,5)} = a\sqrt{2}$. The vectors $\mathbf{k}_b$ and $\mathbf{i}_p$ are orthogonal.

The configuration of the manipulator is defined by the generalized coordinate vector $\boldsymbol{q}$ = [0, 0, a, q_4, 0, q_6]^T, for every q_4 and q_6 (i.e. the degrees of freedom of the system).

B. Preloading and Stiffness Analysis

The stiffness characteristic of the manipulator described in the previous subsection is studied by resorting to the internally preloaded compliant coupling (i.e. the model) that is obtained from the system depicted in Fig. 3 by replacing the links $P_{(h)}B_{(h)}$, h = 1, …, 7, with translational springs. In particular, the links of the legs P_1B_1, P_2B_2 and P_3B_3 are replaced by springs with equal stiffness κ and equal free length $l_{(1)0}$, the links of the legs P_4B_4 and P_5B_5 are replaced by springs with equal stiffness κ and equal free length $l_{(4)0}$, and the telescopic legs P_6B_6 and P_7B_7 are replaced by springs with stiffness $\kappa_{(6)}$ and $\kappa_{(7)}$ and free lengths $l_{(6)0}$ and $l_{(7)0}$ respectively. In practice, κ models the structural rigidity of the manipulator legs of type US, while $\kappa_{(6)}$ and $\kappa_{(7)}$ model the total (structural and servo) rigidity of the telescopic legs. The free lengths $l_{(1)0}$ and $l_{(4)0}$ are independent of the manipulator configuration $\boldsymbol{q}$ and are chosen at the outset, while $l_{(6)0}$ and $l_{(7)0}$ depend on the manipulator configuration, i.e. on q_4 and q_6, and are controlled by the servos.

The internal preloading and the stiffness studies were performed according to the guidelines reported in Sections IV, V and VI. For brevity, only the results that are related to the ∞^2 states of unloaded equilibrium (no external loads, i.e. $\boldsymbol{Q} = 0$) in the configurations $\boldsymbol{q} = [0, 0, \text{a}, q_4, 0, q_6]^T$, for every q_4 and q_6, are reported in the following.

As for the internal preloading, use of Eqs. (7) shows that for the ∞^2 states of unloaded equilibrium the free lengths must comply with

$$l_{(6)0} = l_{(6)} = b\sqrt{2\left(1+\text{s}_4\right)}\ , \tag{23}$$

$$l_{(7)0} = l_{(7)0} = d\sqrt{3-2\text{c}_6-2\text{s}_6}\ , \tag{24}$$

$$l_{(1)0} = \frac{5\sqrt{2}}{3}a - \frac{2}{3}l_{(4)0}\ . \tag{25}$$

That is, the actuated legs P_6B_6 and P_7B_7 can be externally preloaded only. The internal preload acts on legs P_1B_1, P_2B_2, P_3B_3, P_4B_4 and P_5B_5 and can be governed by one independent parameter, either the free length $l_{(1)0}$ or $l_{(4)0}$. No internal preload exists if $l_{(1)0} = l_{(4)0} = a\sqrt{2}$. Legs P_1B_1, P_2B_2, P_3B_3, and legs P_4B_4, P_5B_5 can be made, respectively, by lightweight cables and by compression rods when $l_{(4)0} > a\sqrt{2}$ (vice versa if $l_{(4)0} < a\sqrt{2}$).

As for the stiffness, use of equations (20)-(25) shows that for all the ∞^2 states of unloaded equilibrium the manipulator has 10 (among 36) coefficients of influence that are always zero. The expressions of the vanishing and non-vanishing coefficients are given in the Appendix. Equations (A.1)-(A.17) show that the coefficients of influence are functions of the parameters q_4 and q_6, i.e. functions of the configuration of the system. These expressions also show that the coefficients K_{11}, K_{12}, K_{15}, K_{22}, K_{25} and K_{55} are linear functions of $l_{(4)0}$ and, therefore, can be properly tuned by internally preloading the system. For example, a large value of $l_{(4)0}$ ($l_{(4)0} > a\sqrt{2}$) can be used to increase the absolute values of K_{15}, K_{25} and K_{55} so that the system become more stiff against external torques that act on the platform about the axis $\mathbf{k}_\text{b}$ and $\mathbf{i}_\text{p}$ (which act in the plane defined by vectors $\mathbf{k}_\text{b}$ and $\mathbf{i}_\text{p}$).

VIII. Conclusion

Architecturally singular parallel mechanisms with legs of type US (U and S stand for universal and spherical joints respectively) can be employed as complex kinematic couplings to build reduced-backlash, stiff, strong and lightweight over-constrained parallel manipulators. These mechanisms have redundant elements (legs of type US) which on one hand act in parallel to sustain the loads applied to the system and on the other make it possible to internally preload the mechanism in order to compensate for backlash in the kinematic pairs and to modify the overall stiffness of the system.

This paper discussed an approach for the study of the static and stiffness analyses of these manipulators. In particular, a symmetric stiffness matrix defined as the Hessian of the elastic potential energy of the system has been proposed, as an alternative to the Cartesian stiffness matrix systematically used in this field, to investigate the stiffness characteristic of these over-constrained parallel manipulators.

Application to a 2-dof spherical over-constrained parallel manipulator has been reported which highlights the effective potentials of the proposed approach and shows the stiffness performances of the manipulator.

Appendix

In the following, the expressions of the influence coefficients of the over-constrained 2-dof spherical parallel manipulator (described in Section VII) that is in equilibrium in one of the ∞^2 configurations defined by the generalized coordinates $\boldsymbol{q} = [0, 0, \text{a}, q_4, 0, q_6]^T$, for every q_4 and q_6, under no external load, i.e. $\boldsymbol{Q} = 0$, are provided:

$$K_{35} = K_{53} = K_{36} = K_{63} = K_{45} = K_{54} = 0\ , \tag{A.1}$$

$$K_{46} = K_{64} = K_{56} = K_{65} = 0\ , \tag{A.2}$$

$$K_{11} = \left\{ \begin{array}{r} \kappa\left[\dfrac{5}{4} + \dfrac{\left(2\text{c}_6^2-1\right)}{2\sqrt{2}}\dfrac{l_{(4)0}}{a}\right] + \kappa_{(6)}\dfrac{\text{c}_4^2\text{s}_6^2}{2\left(1+\text{s}_4\right)} + \\ +\kappa_{(7)}\dfrac{\left(1-\text{c}_6\right)^2}{\left(3-2\text{c}_6-2\text{s}_6\right)} \end{array} \right\}, \tag{A.2}$$

$$K_{12} = K_{21} = \left\{ \begin{array}{r} \kappa\dfrac{\text{s}_6\text{c}_6}{\sqrt{2}}\dfrac{l_{(4)0}}{a} - \kappa_{(6)}\dfrac{\text{c}_4^2\,\text{c}_6\,\text{s}_6}{2\left(1+\text{s}_4\right)} + \\ +\kappa_{(7)}\dfrac{\left(1-\text{s}_6\right)\left(1-\text{c}_6\right)}{\left(3-2\text{c}_6-2\text{s}_6\right)} \end{array} \right\}, \tag{A.3}$$

$$K_{13} = K_{31} = -\kappa_{(6)}\frac{\text{c}_4\text{s}_6}{2}\ , \tag{A.4}$$

$$K_{14} = K_{41} = -\kappa_{(6)} \frac{c_4^2 s_6}{2\left(1+s_4\right)} b, \tag{A.5}$$

$$K_{15} = K_{51} = -\kappa \frac{c_6}{\sqrt{2}} l_{(4)0}, \tag{A.6}$$

$$K_{16} = K_{61} = \kappa_{(7)} \frac{\left(s_6 - c_6\right)\left(1 - c_6\right)}{\left(3 - 2c_6 - 2s_6\right)} d, \tag{A.7}$$

$$K_{22} = \left\{ \begin{array}{r} \kappa \left[\dfrac{5}{4} + \dfrac{\left(2s_6^2 - 1\right)}{2\sqrt{2}} \dfrac{l_{(4)0}}{a} \right] + \kappa_{(6)} \dfrac{c_4^2 c_6^2}{2\left(1+s_4\right)} + \\ + \kappa_{(7)} \dfrac{\left(1 - s_6\right)^2}{\left(3 - 2c_6 - 2s_6\right)} \end{array} \right\}, \tag{A.8}$$

$$K_{23} = K_{32} = \kappa_{(6)} \frac{c_4 c_6}{2}, \tag{A.9}$$

$$K_{24} = K_{42} = \kappa_{(6)} \frac{c_4^2 c_6}{2\left(1+s_4\right)} b, \tag{A.10}$$

$$K_{25} = K_{52} = -\kappa \frac{s_6}{\sqrt{2}} l_{(4)0}, \tag{A.11}$$

$$K_{26} = K_{62} = \kappa_{(7)} \frac{\left(s_6 - c_6\right)\left(1 - s_6\right)}{\left(3 - 2c_6 - 2s_6\right)} d, \tag{A.12}$$

$$K_{33} = \kappa \frac{1}{2} + \kappa_{(6)} \frac{\left(1+s_4\right)}{2}, \tag{A.13}$$

$$K_{34} = K_{43} = \kappa_{(6)} \frac{c_4}{2} b, \tag{A.14}$$

$$K_{44} = \kappa_{(6)} \frac{c_4^2}{2\left(1+s_4\right)} b^2, \tag{A.15}$$

$$K_{55} = \kappa \frac{a}{\sqrt{2}} l_{(4)0}, \tag{A.16}$$

$$K_{66} = \kappa_{(7)} \frac{\left(s_6 - c_6\right)^2}{\left(3 - 2c_6 - 2s_6\right)} d^2. \tag{A.17}$$

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