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Assembly Problem of Overconstrained and Clearance-free Parallel Manipulators

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***Abstract*— To avoid deteriorating the mechanism's performance, joint clearance can be eliminated by preloading the pairing elements of the joint. However, this paper proves rigorously that in the real world, the unavoidable assembly and manufacturing errors will cause overconstrained parallel manipulators to lose degree of freedoms, or even unable to be assembled if they are composed of purely clearance-free pairs(e.g., preloaded pairs). Introducing joint clearance is an essential and efficient way for the correct functioning and easy assembly of overconstrained parallel manipulators.**

I. INTRODUCTION

Introducing extra degree of freedoms(DoFs) to the pairing elements of joints, joint clearance deteriorates the mechanism's performance by generating pose(position and orientation) errors of the moving platform, impacts of the pairing elements, quicker wear of the pairs, etc ([1],[2]). Because of these negative effects, many designers try to eliminate joint clearance, or equivalently, the extra DoFs of the joints by preloading the pairing elements. However, parallel mechanisms with preloaded joints only is difficult to assemble, require very precise tolerances and consequently much higher manufacturing costs ([2]). More seriously, as we can show rigorously in the next section, the unavoidable assembly and manufacturing errors will cause *overconstrained* parallel manipulators to *lose DoFs or even unable to be assembled* if they are composed of purely clearance-free pairs(e.g., preloaded pairs). This is called the *assembly problem* of overconstrained parallel manipulators. We also show that introducing joint clearance is an essential and convenient way for the correct functioning and easy assembly of this kind of parallel manipulators.

Many literatures, e.g., [3] and [1], have pointed out that overconstrained parallel manipulators can not function well if all the pairs are clearance-free. The typical features of this kind of parallel manipulators in reality are the loss of mobility(DoFs), and the possibility of preventing assembly. However, to the author's knowledge, there lacks a rigorous and precise account for this common realistic phenomenon. This paper is probably the first time to attempt to verify this scenario using the precise mathematical tools. Using the tools from the Morse theory and differential topology, we differentiate the non-overconstrained parallel manipulators from the overconstrained ones, since the the former has a *transversal* subchains configuration spaces, while the latter does not. We prove that in the presence of inevitable assembly and manufacturing errors, parallel manipulators having clearance-free joints only in practice are all non-overconstrained, even if they are theoretically designed to be over-constrained ones. This implies that overconstrained parallel manipulators in practice will decrease DoFs or even can not be assembled, because originally designed dependent constraint forces are change to independent ones now by the assembly and manufacturing errors. Several examples are used to show the application of the proposed approach.

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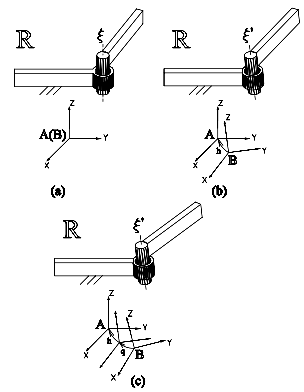

Fig. 1. Clearance-free Revolute Joint without and with Assembly and Manufacturing Error at the Home Configuration

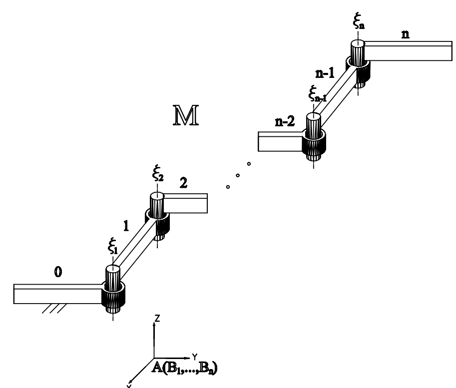

Fig. 2. A Subchain Composed by Clearance-free Revolute Joints without Assembly and Manufacturing Errors at the Home Configuration

II. ASSEMBLY PROBLEM OF OVERCONSTRAINED AND CLEARANCE-FREE PARALLEL MANIPULATORS

It is well-known that a revolute ($\mathcal{R}$) joint is composed of two pairing elements : a bearing and a shaft. For a *clearance-*

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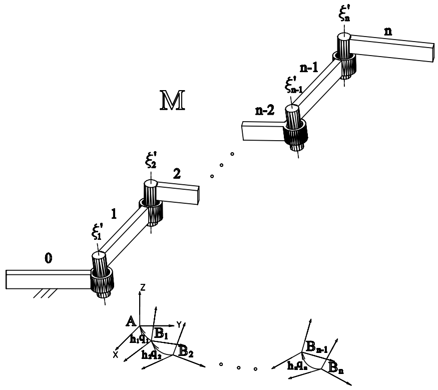

Fig. 3. A Subchain Composed by Clearance-free Revolute Joints with Assembly Errors at the Home Configuration

free $\mathcal{R}$ joint, the joint clearance between the pairing two elements has been eliminated, for example, by preloading the joint. Thus only one DoF rotations about an axis are allowed between the two pairing elements of a clearance-free $\mathcal{R}$ joint. After connecting two links to the pairing elements of a clearance-free $\mathcal{R}$ joint, an elementary clearance-free $\mathcal{R}$ *joint mechanism* is made and can be used to construct clearance-free serial and parallel mechanisms. In this section, first we study the error characteristic of an elementary clearance-free $\mathcal{R}$ joint mechanism. When two links are connected to a clearance-free $\mathcal{R}$ joint, assembly and manufacturing errors may occur at the adjacencies between the first link and the bearing and between the second link and the shaft. Hence the first objective is : in the presence of these assembly and manufacturing errors, how to derive the relative configuration space of the two links connected by a clearance-free $\mathcal{R}$ joint.

To describe the relative configurations of the two links connected by the $\mathcal{R}$ joint, we assume that the first link has no position and orientation error [1] and is fixed to the ground. Define an inertial frame A and a body frame B that are attached to the first and the second link respectively. Thus the relative configuration of the two links is given by the transformation matrix from the body frame B to the inertial frame A, which is an element of $SE(3)$(see [4] for more details). Without assembly and manufacturing errors, the bearing and the shaft of the $\mathcal{R}$ joint together with the second link will be at their nominal configurations. In this ideal case, if we suppose that the body frame B coincides with the inertial frame A at the home (or initial) configuration, and the rotation axis of the $\mathcal{R}$ joint has a twist coordinate $\xi \in \mathbb{R}^6$ when expressed in the frame A (and B), then the *ideal* relative configuration space of the two links, denoted by $C^I_{\mathcal{R}}$, is given by $C^I_{\mathcal{R}} = \{e^{\hat{\xi}\theta} | \theta \in (-\epsilon, \epsilon)\}$, where ϵ is the joint limit of the $\mathcal{R}$ joint. In the presence of the assembly and manufacturing errors, however, position errors may exist between the first link and the bearing and between the second link and the shaft. To precisely describe these position errors, first we assume that only the assembly and manufacturing error between the first link and the bearing exists and causes the $\mathcal{R}$ joint and the second link to deviate from their ideal configurations. In this case, supposing that the body frame B is displaced from the inertial frame A by an error transformation $h \in SE(3)$ at the home configuration (see Fig.1(b)), then the position error of the bearing with respect to the first link caused by the assembly and manufacturing error is uniquely characterized by h. Due to the position error of the bearing, the $\mathcal{R}$ joint will possess a changed rotational axis, with new twist coordinate ξ' when expressed in the inertial frame A. As the rotation axis is not changed as seen from the body frame, the twist representation of the $\mathcal{R}$ joint in the B frame of Fig.1(b) is still ξ. Hence immediately we get $\xi' = Ad_h\xi$. Second, assume that the adjacency between the second link the shaft also has the assembly and manufacturing error. Then the body frame B will further displace from the one in Fig.1(b) by an error transformation matrix q at the home configuration (see Fig.1(c)). Similarly, q describes the position error of the second link relative to the shaft caused by the assembly and manufacturing error too. As a consequence, the relative configuration space of the two links of the *actual* $\mathcal{R}$ joint mechanism, denoted by $C^a_{\mathcal{R}}$, is given by

$$\begin{aligned} C^a_{\mathcal{R}} &= \{e^{\hat{\xi}'\theta} \cdot h \cdot q | \theta \in (-\epsilon, \epsilon)\} \\ &= \{h \cdot e^{\hat{\xi}\theta} \cdot h^{-1} \cdot h \cdot q | \theta \in (-\epsilon, \epsilon)\} \\ &= \{h \cdot e^{\hat{\xi}\theta} \cdot q | \theta \in (-\epsilon, \epsilon)\} \\ &= h \cdot C^I_{\mathcal{R}} \cdot q \end{aligned} \quad (1)$$

For other lower pairs, e.g., prismatic (P) and helical (H) joint, we may also get $C^a_P = h \cdot C^I_P \cdot q$ and $C^a_H = h \cdot C^I_H \cdot q$ by a similar analysis. From the above analysis, one may observe that (h, q) together characterizes the assembly and manufacturing errors at two adjacencies within a clearance-free joint mechanism. Clearly, as the assembly and manufacturing errors are random when the two links are connected to the $\mathcal{R}$ (or P, H) joints, the values of h and q are stochastic for different $\mathcal{R}$ (or P, H) joint mechanisms with the same ideal joint motions.

Next, suppose that M is a serial subchain that is the cascading of n clearance-free $\mathcal{R}$ joint mechanisms (see Fig.2 and Fig.3). Denote the ith $\mathcal{R}$ joint mechanism by $\mathcal{R}_i$ afterwards, and label the two links of $\mathcal{R}_i$ by $i-1$ and i respectively. Clearly, link 0 is fixed with respect to the ground, and link n is called the *end-effector* of M. The second objective is : with assembly and manufacturing errors for all the joint mechanisms in place, how to derive the set of relative configurations between the end-effector and the base link (link 0). As we did before, we attach two frames B_{i-1} and B_i to the two links of $\mathcal{R}_i$, i.e., link $i-1$ and link i respectively. As B_0 is fixed with respect to the ground, it is also called the inertial frame and usually named by A. B_n, as attached to the end-effector, is also called the body frame of the subchain M and may be simply named by B. Thus the configuration space of M is the set of transformation matrices from B(or B_n) to A(or B_0). Fig.2 depicts the ideal situation of the subchain M (i.e., none of the

[1]For concision, position and orientation error will be simply called *position error* hereafter.

joint mechanisms has assembly and manufacturing error). In this ideal case, assuming that B_i coincides with B_{i-1} at the home configuration, and the rotation axis of $\mathcal{R}_i$ has a twist coordinate ξ_i when expressed in frame B_{i-1}, $i = 1, \cdots, n$, then the ideal relative configuration space of the two links of $\mathcal{R}_i$ is given by $C^I_{\mathcal{R}_i} = \{e^{\hat{\xi}_i\theta_i} | \theta_i \in (-\epsilon_i, \epsilon_i)\}$, where ϵ_i is the joint limit of $\mathcal{R}_i$. Consequently, the ideal configuration space of M, denoted C^I_M, is given by

$$C^I_M = C^I_{\mathcal{R}_1} \cdots C^I_{\mathcal{R}_n} = \{e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n} | \theta_i \in (-\epsilon_i, \epsilon_i)\} \ , \qquad (2)$$

In place of the assembly and manufacturing errors (see Fig.3), by our previous analysis, the actual relative configuration space of the two links of $\mathcal{R}_i$ should be $C^a_{\mathcal{R}_i} = \{e^{\hat{\xi}'_i\theta_i} \cdot h_i \cdot q_i | \theta_i \in (-\epsilon_i, \epsilon_i)\} = \{h_i \cdot e^{\hat{\xi}\theta} \cdot q_i | \theta \in (-\epsilon, \epsilon)\} = h_i \cdot C^I_{\mathcal{R}_i} \cdot q_i$, where h_i, q_i characterizes the assembly and manufacturing errors associated with $\mathcal{R}_i$, and ξ'_i is the twist coordinate of the actual rotation axis of $\mathcal{R}_i$ as seen from B_{i-1}. Therefore, the actual configuration space of M, C^a_M, is given by

$$\begin{aligned} C^a_M &= C^a_{\mathcal{R}_1} \cdots C^a_{\mathcal{R}_n} \\ &= \{h_1 \cdot e^{\hat{\xi}_1\theta_1} \cdot q_1 \cdots h_n \cdot e^{\hat{\xi}_n\theta_n} \cdot q_n | \theta_i \in (-\epsilon_i, \epsilon_i)\} \ . \end{aligned} \qquad (3)$$

Note that Eq.(2) and Eq.(3) can be derived in an alternative way. The initial configuration of M is the transformation from B_n to A at the home configuration. In the ideal situation, every joint has no assembly and manufacturing errors, hence all the frames B_i, $i = 1, \cdots, n$ will coincide with $B_0 = A$ at the home configuration (see Fig.2). Therefore, the ideal initial configuration of M is e. Furthermore, it is also clear that the twist coordinate of $\mathcal{R}_i$ in the inertial frame A is also ξ_i, $i = 1, \cdots, n$. Using the product of exponentials formula in [4], one may get that

$$C^I_M = \{e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_n\theta_n} | \theta_i \in (-\epsilon_i, \epsilon_i)\} \ .$$

In the presence of assembly and manufacturing errors, the actual initial configuration of M is $h_1 \cdot q_1 \cdots h_n \cdot q_n$ (see Fig.3), and the twist coordinate of $\mathcal{R}_i$ in the inertial frame A can be easily got as $\eta_1 = Ad_{h_1}\xi_1$, $\eta_i = Ad_{h_1 \cdot q_1 \cdots h_{i-1} \cdot q_{i-1} \cdot h_i}\xi_i$, $i = 2, \cdots, n$. Hence the actual configuration space of M is

$$\begin{aligned} C^a_M &= \{e^{\hat{\eta}_1\theta_1} \cdots e^{\hat{\eta}_n\theta_n} \cdot (h_1 \cdot q_1 \cdots h_n \cdot q_n) | \theta_i \in (-\epsilon_i, \epsilon_i)\} \\ &= \{h_1 \cdot e^{\hat{\xi}_1\theta_1} \cdot q_1 \cdots h_n \cdot e^{\hat{\xi}_n\theta_n} \cdot q_n | \theta_i \in (-\epsilon_i, \epsilon_i)\} \end{aligned}$$

which is the same as Eq.(3).

For simplicity, afterwards we will assume that $\xi_1, \cdots, \xi_n$ are linearly independent. As we remarked before, for all subchains that are the cascading of n clearance-free $\mathcal{R}$ joint mechanisms with the ideal twists $\xi_1, \cdots, \xi_n$, the matrices $h_1, q_1, \cdots, h_n, q_n$ are independent and random variables that may take any values in $SE(3)$. Hence we may define a map as follows :

$$\begin{aligned} F \ &: \ C^I_{\mathcal{R}_1} \times \cdots \times C^I_{\mathcal{R}_n} \times SE(3) \times \cdots \times SE(3) \to SE(3) \\ &: \ (g_1, \cdots, g_n, h_1, q_1, \cdots, h_n, q_n) \mapsto (h_1 g_1 q_1 \cdots h_n g_n q_n) \end{aligned}$$

where $C^I_{\mathcal{R}_i} = \{e^{\hat{\xi}_i\theta_i} | \theta_i \in (-\epsilon_i, \epsilon_i)\}$ is the ideal relative configuration space of the two links of $\mathcal{R}_i$, and $g_i = e^{\hat{\xi}_i\theta_i} \in C^I_{\mathcal{R}_i}$. As assembly and manufacturing errors in general are very small, h_i, q_i, $i = 1, \cdots, n$ in practice take values in a small neighborhood of e in $SE(3)$. Clearly, F is a smooth map. If we denote $g = (g_1, \cdots, g_n)$ and $s = (h_1, q_1, \cdots, h_n, q_n)$, then the restriction map of F by fixing $s = (h_1, q_1 \cdots, h_n, q_n)$

$$F_s : C^I_{\mathcal{R}_1} \times \cdots \times C^I_{\mathcal{R}_n} \to SE(3) : g \mapsto F(g, s)$$

maps $C^I_{\mathcal{R}_1} \times \cdots \times C^I_{\mathcal{R}_n}$ to the actual configuration space (C^a_M) of a particular subchain M affected by the assembly and manufacturing error s. Furthermore, it is easy to see that the other restriction map of F by fixing g

$$F_g : SE(3) \times \cdots \times SE(3) \to SE(3) : s \mapsto F(g, s)$$

is a submersion.

Proposition *1: Suppose that $F : X \times S \to Y$ is a smooth map of manifolds and for any fixed $x \in X$,the map $s \mapsto F(x, s)$ is a submersion $S \to Y$. Then for any submanifold Z of Y, the restriction map $F_s : X \to Y$ by fixing s is transversal to Z for almost every $s \in S$.*

Proof: See P.70 in [5].

Remark *1:* (i)For the subchain M, we may choose $X = C^I_{\mathcal{R}_1} \times \cdots \times C^I_{\mathcal{R}_n}$, $S = SE(3) \times \cdots \times SE(3)$ and $Y = SE(3)$. X and its image under F_s, C^a_M, are seen to be manifolds by shrinking $\epsilon_i, i = 1, \cdots, n$ if necessary. Thus Proposition 1 implies that C^a_M is transversal to any submanifold Z of $SE(3)$ for almost every assembly and manufacturing error $s = (h_1, q_1, \cdots, h_n, q_n)$; (ii)The same result may be obtained if the $\mathcal{R}$ joints are replaced by P or H joints.

Using the standard notation in [5], if X and Z are submanifolds of $SE(3)$, and X is transversal to Z, then we shall write $X \overline{\pitchfork} Z$. A parallel manipulator is the parallel connection of several serial subchains that share a common base ground and a common end-effector. By an abuse of notation, later on M will be used to denote a parallel manipulator. In this section, the parallel manipulator M under consideration is assumed to have k serial subchains $M_1, \cdots, M_k$, with each subchain consisting of clearance-free joint mechanisms only. Furthermore, all the subchains $M_1, \cdots, M_k$ share a common inertial frame A and a body frame B that are attached to the base ground and the end-effector respectively. Assume that the ideal initial configuration of the end-effector is e. Then under assembly and manufacturing errors, the actual configuration space of the end-effector, denoted C^a_M, is given by

$$C^a_M = C^a_{M_1} \cap C^a_{M_2} \cap \cdots \cap C^a_{M_k} \quad ,$$

where $C_{M_i}, i = 1, \cdots, k$ can be derived by Eq.(3), and the spatial velocity space of the end-effector at each $g \in C^a_M$, denoted $R_{g^{-1}*}T_g C^a_M$, is given by

$$R_{g^{-1}*}T_g C^a_M = R_{g^{-1}*}T_g C^a_{M_1} \cap \cdots \cap R_{g^{-1}*}T_g C^a_{M_k}$$

where $R_{g^{-1}*}T_g C^a_{M_i} \subset \mathbb{R}^6$ denotes the spatial velocity space of the ith subchain at g, namely, the subspace of $\mathbb{R}^6$ that is the span of all the twist coordinates of joints in M_i at g when expressed in the inertial frame A. Assume that $C^a_{M_1}, \cdots, C^a_{M_k}$ are all submanifolds by choosing small joint limits. Thus by Remark 1(i), it's clear that for any $i \neq j$, $1 \leq i, j \leq k$, we have $C^a_{M_i} \overline{\pitchfork} C^a_{M_j}$. If moreover $C^a_{M_i} \cap C^a_{M_j}$ is a submanifold, then for any other $l \leq k$, $l \neq i$ and $l \neq j$, we can also get by Remark 1(i) that $C^a_{M_l} \overline{\pitchfork} (C^a_{M_i} \cap C^a_{M_j})$.

Proposition *2: For two submanifolds X and Z of $SE(3)$, if $X \overline{\pitchfork} Z$, then $X \cap Z$ is again a submanifold of $SE(3)$. Furthermore, we have*

$$codim(X \cap Z) = codim(X) + codim(Z) \tag{4}$$

where for any submanifold Y of $SE(3)$, $codim(Y) = 6 - dim(Y)$.
Proof: See P.30 in [5].

For the parallel manipulator M, let $X = C^a_{M_i}$, $Z = C^a_{M_j}$. As $C^a_{M_i} \overline{\pitchfork} C^a_{M_j}$, Proposition 2 shows that $C^a_{M_i} \cap C^a_{M_j}$ is a submanifold. Let $Y = C^a_{M_l}$, then $dim(Y)$ is the dimension of the spatial velocity space of the lth subchain, i.e., the dimension of $R_{g^{-1}*} T_g C^a_{M_l}$ at any $g \in C^a_{M_l}$. Hence $codim(Y)$ is equal to the dimension of the constraint force space of the lth subchain at any $g \in C^a_{M_l}$, which is the annihilating subspace of $R_{g^{-1}*} T_g C^a_{M_l}$ in $\mathbb{R}^6$:

$$(R_{g^{-1}*} T_g C^a_{M_l})^\perp = \{f \in \mathbb{R}^6 \mid f \cdot \xi = 0, \forall \xi \in R_{g^{-1}*} T_g C^a_{M_l}\} .$$

It is clear that for any $g \in (C^a_{M_i} \cap C^a_{M_j})$, since

$$R_{g^{-1}*} T_g (C^a_{M_i} \cap C^a_{M_j}) = R_{g^{-1}*} T_g C^a_{M_i} \cap R_{g^{-1}*} T_g C^a_{M_j} \quad ,$$

we have

$$(R_{g^{-1}*} T_g (C^a_{M_i} \cap C^a_{M_j}))^\perp = (R_{g^{-1}*} T_g C^a_{M_i})^\perp + (R_{g^{-1}*} T_g C^a_{M_j})^\perp .$$

However, Eq.(4) of Proposition 2 further tells us that $\forall g \in C^a_{M_i} \cap C^a_{M_j}$,

$$\begin{aligned}(R_{g^{-1}*} T_g (C^a_{M_i} \cap C^a_{M_j}))^\perp = \;& (R_{g^{-1}*} T_g C^a_{M_i})^\perp \\ & \oplus (R_{g^{-1}*} T_g C^a_{M_j})^\perp \quad ,\end{aligned} \tag{5}$$

i.e., $(R_{g^{-1}*} T_g C^a_{M_i})^\perp \cap (R_{g^{-1}*} T_g C^a_{M_j})^\perp = \{(0, \cdots, 0)^T\}$. Since $C^a_{M_l} \overline{\pitchfork} (C^a_{M_i} \cap C^a_{M_j})$, by a similar analysis, it is easy to see that $C^a_{M_i} \cap C^a_{M_j} \cap C^a_{M_l}$ is also a submanifold, and $\forall g \in (C^a_{M_i} \cap C^a_{M_j} \cap C^a_{M_l})$,

$$\begin{aligned}(R_{g^{-1}*} T_g (C^a_{M_i} \cap C^a_{M_j} \cap C^a_{M_l}))^\perp = \;& (R_{g^{-1}*} T_g C^a_{M_i})^\perp \\ & \oplus (R_{g^{-1}*} T_g C^a_{M_j})^\perp \\ & \oplus (R_{g^{-1}*} T_g C^a_{M_l})^\perp .\end{aligned}$$

By deduction, one can get that C^a_M is a submanifold, and $\forall g \in C^a_M$,

$$(R_{g^{-1}*} T_g C^a_M)^\perp = (R_{g^{-1}*} T_g C^a_{M_1})^\perp \oplus \cdots \oplus (R_{g^{-1}*} T_g C^a_{M_k})^\perp ,$$

where $(R_{g^{-1}*} T_g C^a_M)^\perp$ is the constraint force space of the end-effector at g.

Finally, we remark that a necessary and sufficient condition for X to be transversal to Z is that for any $g \in X \cap Z$,

$$R_{g^{-1}*} T_g (X) + R_{g^{-1}*} T_g (Z) = \mathbb{R}^6 \ , \tag{6}$$

or equivalently

$$(R_{g^{-1}*} T_g (X \cap Z))^\perp = (R_{g^{-1}*} T_g X)^\perp \oplus (R_{g^{-1}*} T_g Z)^\perp \ , \tag{7}$$

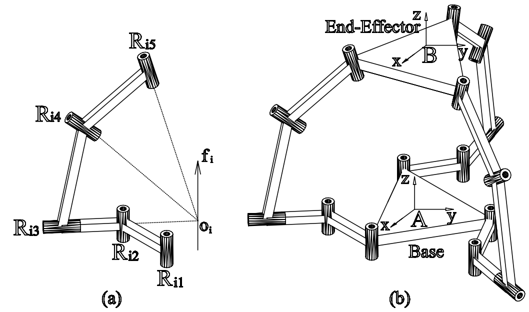

Fig. 4. A Parallel Manipulator With Theoretical 5-DoF

where $R_{g^{-1}*} T_g (X)$ is the spatial velocity space of X at g, and $(R_{g^{-1}*} T_g X)^\perp$ is the constraint force space of X at g. However, if $dim(X) + dim(Z) < 6$, then a special case of transversality(see P.30 in [5]) is that

$$X \overline{\pitchfork} Z \ \Leftrightarrow \ X \cap Z \text{ is empty} \tag{8}$$

Definition *1:* **Non-overconstrained and Overconstrained Parallel Manipulators**
Let M be a parallel manipulator that consists of a couple of serial subchains $M_1, \cdots, M_k$. Assume that joint limits are small such that the configuration spaces of all the subchains are submanifolds of $SE(3)$. At a configuration g of the end-effector of M, denote by $\Gamma_{M_i}(g)$ the constraint force space of M_i at g, $i = 1, \cdots, k$, and $\Gamma_M(g)$ the constraint force space of the end-effector at g. If for any configuration g of the end-effector of M, one can get

$$\Gamma_M(g) = \Gamma_{M_1}(g) \oplus \Gamma_{M_2}(g) \cdots \oplus \Gamma_{M_k}(g) \ , \tag{9}$$

then M is called a *non-overconstrained parallel manipulator*. Otherwise, M is called an *overconstrained parallel manipulator*.

Remark *2:* (i)Definition 1 assume that the parallel manipulator has small joint limit. For a general parallel manipulator with larger joint limits, to see whether it is non-overconstrained or not, one may reduce the joint limits and check the condition of Eq.(9) in the shrunk configuration space of the end-effector only; (ii) From Definition 1 and Proposition 2, it is easy to see that those whose configuration spaces of subchains are transversal submanifolds are non-overconstrained parallel manipulators, and vice versa.

By the previous analysis and Remark 2(ii), parallel manipulators having clearance-free joints only in practice are all non-overconstrained due to the assembly and manufacturing errors, even if they are theoretically designed to be over-constrained ones.

Example *1:* **A 5-DoF Overconstrained Parallel Manipulator** Consider a parallel manipulator M in Fig.4 with three symmetrically arranged subchains $M_1, \cdots, M_3$ ([6]). Theoretically, the ith subchain M_i consists of 5 revolute joint mechanisms $\mathcal{R}_{i1}, \cdots, \mathcal{R}_{i5}$, with the axis of $\mathcal{R}_{i1}$ and $\mathcal{R}_{i2}$ parallel to each other, and the axis of $\mathcal{R}_{i3}$, $\mathcal{R}_{i4}$ and $\mathcal{R}_{i5}$ converge to a point o_i (see Fig.4(a), where for concision,

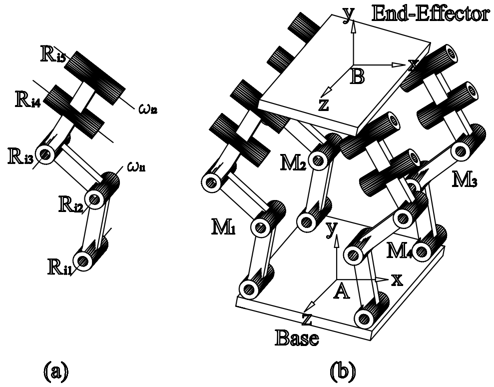

Fig. 5. A Parallel Manipulator With Theoretical 4-DoF

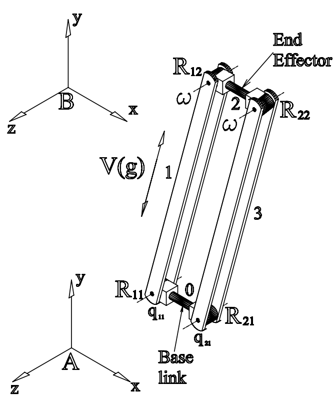

Fig. 6. A Parallelogram

the end-effector and the base are not depicted). At any configuration g of the end-effector, $\Gamma_{M_i}(g), i = 1, \cdots, 3$ is thus spanned by a pure force passing through o_i

$$\Gamma_{M_i}(g) = (R_{g^{-1}*}T_g C^I_{M_i})^\perp = \{(f_i, o_i \times f_i)\} \quad i = 1, \cdots, 3$$

where f_i is the direction vector parallel to the axis of $\mathcal{R}_{i1}$ and $\mathcal{R}_{i2}$. The three subchains are assembled such that $o_1 = o_2 = o_3$ and $f_1 = f_2 = f_3$. Then it is clear that $\Gamma_{M_1}(g) = \Gamma_{M_2}(g) = \Gamma_{M_3}(g)$ at any configuration g of the end-effector. Hence in theory, M is a 5-DoF overconstrained parallel manipulator by Definition 1. However, with assembly and manufacturing errors in place, the axis of $\mathcal{R}_{i1}$ and $\mathcal{R}_{i2}$ may not be parallel, and the axis of $\mathcal{R}_{i3}$, $\mathcal{R}_{i4}$ and $\mathcal{R}_{i5}$ may not be able to converge to a point. Thus by our previous analysis, $\Gamma_{M_i}(g) = (R_{g^{-1}*}T_g C^a_{M_i})^\perp$ and

$$\begin{aligned}\Gamma_M(g) &= (R_{g^{-1}*}T_g C^a_M)^\perp \\ &= (R_{g^{-1}*}T_g C^a_{M_1})^\perp \oplus \cdots \oplus (R_{g^{-1}*}T_g C^a_{M_3})^\perp \\ &= \Gamma_{M_1}(g) \oplus \cdots \oplus \Gamma_{M_3}(g)\end{aligned}$$

at any configuration g of the end-effector. Hence in the presence of assembly and manufacturing errors, M is a non-overconstrained parallel manipulator, with only $6 - 3 = 3$ DoF.

Example *2:* **A 4-DoF Overconstrained Parallel Manipulator** As another example, look at a parallel manipulator M in Fig.5. M composes of 4 identical subchains $M_1, \cdots, M_4$, and each subchain M_i has 5 revolute joint mechanisms $\mathcal{R}_{i1}, \cdots, \mathcal{R}_{i5}$. Without assembly and manufacturing errors, $\mathcal{R}_{i1}, \mathcal{R}_{i2}$, and $\mathcal{R}_{i3}$ have parallel rotation axis with direction $\omega_{i1} \in \mathbb{R}^3$, and $\mathcal{R}_{i4}, \mathcal{R}_{i5}$ also have parallel rotation axis, with direction $\omega_{i2} \neq \omega_{i1}$ (see Fig.5(a)). Thus at any configuration g of the end-effector, $\Gamma_{M_i}(g), i = 1, \cdots, 3$ is spanned by a pure torque

$$\Gamma_{M_i}(g) = (R_{g^{-1}*}T_g C^I_{M_i})^\perp = \{(0, \tau_i)\} \quad i = 1, \cdots, 3$$

where τ_i is the direction vector perpendicular to the plane spanned by ω_{i1} and ω_{i2}. As seen from Fig.5(b), ideally the four subchains are assembled in a way such that $\omega_{11} = \omega_{21}$, $\omega_{12} = \omega_{22}$, $\omega_{31} = \omega_{41}$, and $\omega_{32} = \omega_{42}$, which implies that $\tau_1 = \tau_2$, and $\tau_3 = \tau_4$. In other words, at any configuration g of the end-effector, $\Gamma_{M_1}(g) = \Gamma_{M_2}(g)$, $\Gamma_{M_3}(g) = \Gamma_{M_4}(g)$, and

$$\begin{aligned}\Gamma_M(g) &= (R_{g^{-1}*}T_g C^I_M)^\perp \\ &= \Gamma_{M_1}(g) + \Gamma_{M_2}(g) + \Gamma_{M_3}(g) + \Gamma_{M_4}(g) \\ &= \{(0, \tau_1), (0, \tau_3)\}\end{aligned}$$

Hence theoretically M is a 4-DoF overconstrained parallel manipulator. In place of assembly and manufacturing errors, however, the parallel conditions of the joints may not be able to be achieved. Furthermore, by our previous analysis, we can get

$$\begin{aligned}\Gamma_M(g) &= (R_{g^{-1}*}T_g C^a_M)^\perp \\ &= (R_{g^{-1}*}T_g C^a_{M_1})^\perp \oplus \cdots \oplus (R_{g^{-1}*}T_g C^a_{M_4})^\perp \\ &= \Gamma_{M_1}(g) \oplus \cdots \oplus \Gamma_{M_4}(g)\end{aligned}$$

at any configuration $g \in C^a_M$. That is, the manipulator in practice only has $6 - 4 = 2$ DoF due to the inevitable assembly and manufacturing errors.

Example *3:* **An Overconstrained Parallelogram** As an extreme example, we investigate a parallelogram in Fig.6, and show how seriously the assembly and manufacturing errors could ruin the parallel mechanism's performance. Denote the two subchains of a parallelogram by M_1 and M_2 respectively. Each subchain M_i consists of two revolute joint mechanisms $\mathcal{R}_{i1}$ and $\mathcal{R}_{i1}$. Ideally, the four revolute joints have parallel rotation axis, with direction $\omega \in \mathbb{R}^3$. Assume that at any configuration g of the end-effector of the ideal parallelogram, the vector of link 1 and 3 is $\mathbf{v}(g)$(see Fig.6). Then it is easy to get

$$\begin{aligned}\Gamma_{M_i}(g) &= (R_{g^{-1}*}T_g C^I_{M_i})^\perp \\ &= \{(\mathbf{v}(g), q_{i1} \times \mathbf{v}(g)), \qquad i = 1, 2 \\ &\quad (\omega, 0), (0, u), (0, v)\}\end{aligned}$$

where q_{i1} is a point on $\mathcal{R}_{i1}, i = 1, 2$, and u, v are two vectors such that $\{u, v, \omega\} = \mathbb{R}^3$. Note that $(\mathbf{v}(g), q_{11} \times \mathbf{v}(g))$ is a pure force along link 1, and $(\mathbf{v}(g), q_{11} \times \mathbf{v}(g))$ is a pure force along link 3. Thus it is clear that $\Gamma_{M_1}(g) \cap \Gamma_{M_2}(g) = \{(\omega, 0), (0, u), (0, v)\}$, and

$$\begin{aligned}\Gamma_M(g) &= (R_{g^{-1}*}T_g C^I_M)^\perp \\ &= \Gamma_{M_1}(g) + \Gamma_{M_2}(g) \\ &= \{(\mathbf{v}(g), q_{11} \times \mathbf{v}(g)), \\ &\quad (\mathbf{v}(g), q_{21} \times \mathbf{v}(g)), \\ &\quad (\omega, 0), (0, u), (0, v)\}\end{aligned}$$

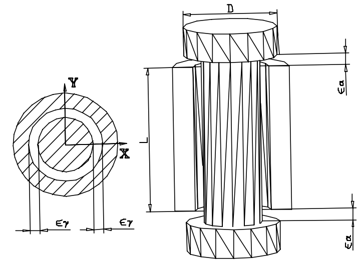

Fig. 7. Clearance-affected Revolute Joint

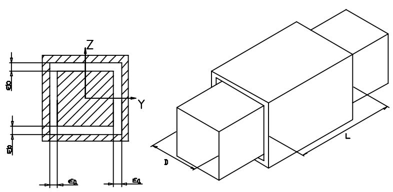

Fig. 8. Clearance-affected Prismatic Joint

at any $g \in C^I_M$. Hence an ideal parallelogram is an overconstrained parallel mechanism with 1-DoF. However, as we analyze before, $C^a_{M_1} \overline{\pitchfork} C^a_{M_2}$ in the presence of the assembly and manufacturing errors. As $dim(C^a_{M_1}) + dim(C^a_{M_2}) = 4 < 6$, by Eq.(8), $C^a_{M_1} \cap C^a_{M_2}$ is empty, i.e., such a mechanism is impossible to be assembled. This justifies that in the real world, a mobile parallelogram can not exist if it composes of purely clearance-free revolute joints.

Through these examples and our analysis, we have shown that overconstrained parallel manipulators can not be achieved in the real world if they are built solely by clearance-free lower pairs; in the presence of inevitable assembly and manufacturing errors, these mechanisms will lose DoFs or even can not move after they are assembled. This phenomenon is called the *assembly problem* of overconstrained parallel manipulators. However, nowadays many symmetrically arranged sub-6 DoF parallel manipulators are designed to be overconstrained, for example, the famous Delta robot. In particular, all 4 and 5-DoF parallel manipulators with symmetrical limbs, like those in Fig.4 and Fig.5, are in theory also overconstrained. Hence, one must overcome the assembly problem for overconstrained parallel manipulators before the application of them.

Using clearance-affected lower pairs is an alternative to solve this problem. A clearance-affected lower pair actually has 6-DoF and the rotation or translation axis of its nominal DoF motions is allowed to wobble in a bounded region determined by the clearance. For example, consider a journal bearing revolute pair with clearance(see Fig.7). Such a design is common for revolute pairs. From Fig.7, it is easy to see that instead of being only allowed to rotate about its nominal axis, the second element of the pair can also rotate about any two right axis perpendicular to the nominal one, and translate along any three independent directions in the space. These five extra DoF motions are bounded within a small region determined by the geometry and magnitude of clearance of the pairing elements. Similar analysis can be applied to a clearance-affected prismatic pair modelled by Fig.8. Thus, if a parallel manipulator is built all by such clearance-affected lower pairs, even though the assembly and manufacturing errors exist for every joint mechanism, the end-effector can still undergo desired motions through the self-adjustment of the rotation or translation axis of each lower pair to its desired position and orientation. The drawback is, in each ideal configuration of the parallel manipulator, there exists extra DoFs error motions for the end-effector due to the clearance and extra DoFs brought by the clearance-affected lower pairs. In our peer work [7], we have proposed an efficient method to evaluate these configuration(or pose) errors of the end-effector caused by the joint clearance.

III. Conclusion

In this paper, using differential topology, we make the definitions of non-overconstrained and overconstrained parallel manipulators according to the transversality condition of their subchains' configuration spaces. we prove that in the presence of inevitable assembly and manufacturing errors, parallel manipulators having clearance-free joints only in practice are all non-overconstrained, even if they are theoretically designed to be over-constrained ones. Thus overconstrained parallel mechanisms can not work with clearance-free lower pairs only, which is called the assembly problem for this type of parallel manipulators. We also have shown that introducing joint clearance is an essential and efficient way for the correct functioning and easy assembly of overconstrained parallel manipulators. Several examples are used to elucidate the application of the proposed approach.

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