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A Framework for the Control of Stable Aperiodic Walking in Underactuated Planar Bipeds

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***Abstract*— This paper presents a new definition of stable walking—that is not necessarily periodic—for a class of biped robots. The inspiration for the definition is the commonly-held notion of stable walking: the biped does not fall. To make the definition useful, an algorithm is given to verify if a given controller induces stable walking. Also given is a framework to synthesize controllers that induce stable walking. The results are illustrated on a model of a 5-link biped.**

I. Introduction

The most intuitive definition of stable biped walking is likely that "the biped does not fall." Although this definition is commonly held and intuitively pleasing, it is difficult to rigorously verify that a given controller induces walking that is stable in this sense. To help remedy this problem, this paper gives a rigorous definition for stable aperiodic walking in bipeds that have limited or no torque available at its ankles.

Other definitions of gait stability are the following:

For bipeds that walk flat-footed the most popular methods to ensure gait stability are to require that the projection of center of mass on the ground, the zero-moment point (ZMP), or the foot-rotation indicator (FRI) [1] lie within the support polygon. (These three concepts are closely related; for more details see [1]–[5].)

Recently, Pratt and Tedrake [6] proposed a velocity-based gait stability definition, which provides a sufficient condition for gait stability.

For periodic gaits, the Poincaré map is the most popular tool to assess gait stability. Use of a Poincaré map enables the stability of the orbit corresponding to a periodic gait to be determined by examination of an associated discrete-time system. Cheng and Lin [7] derived the linearization of the Poincaré map for the gait of a 5-link planar biped at the fixed point. The eigenvalues of the linearized return map were then used to design controllers that induced periodic gaits. In [8]–[12], the eigenvalues of the Poincaré maps for the gaits of several different bipeds were obtained numerically. In the work that provides the foundations for the developments of this paper, Grizzle et al. [13] and Westervelt et al. [14], [15], the controller design was used to reduce the dimension of the Poincaré map to one. In [14], the Poincaré map was shown to be linear thus allowing gait stability to be derived using simple stability metrics.

The proposed definition subsumes the common notion of gait stability: the robot does not fall. The new definition of gait stability makes it possible to prove rigorously that a gait is stable. The definition is *independent* of the controller used.

Using the proposed definition, a framework for the design of walking controllers is given. This framework extends the work of Westervelt et al. [14]. Unlike [14], which enabled the design of controllers that only induce stable *periodic* walking, the proposed framework allows the induced gaits to be aperiodic. Moreover, the gait associated with an individual controller of the framework is not necessarily asymptotically stable.

This paper is organized as follows. Section II describes the class of the biped model treated. Section III presents a series of definitions leading to the novel definition of stable biped walking. Section IV begins with a summary of results on the hybrid zero dynamics-based approach to the control of biped walking and then gives a control design approach to generate controllers that induce stable walking in the sense developed in Section III. In Section V the controller design approach is illustrated via simulation on a 5-link model. Conclusions are given in Section VI.

II. Hybrid Model of Biped Walking

The biped is assumed to be planar and consists of N rigid links connected by revolute joints. It is further assumed that the leg ends have point contact with the ground, and that there is no actuation between the stance leg tibia and the ground (see Figure 1). A step is composed of two phases: a swing phase, when only one leg end is in stationary contact with the ground; and an instantaneous double support phase, when both legs are in contact with the ground. The two phases result in a model for walking that is hybrid.

During the swing phase the robot is modeled as an N-link rigid-body system. The equations of motion are

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Bu, \tag{1}$$

where $q := (q_1; \ldots; q_N) \in \mathcal{Q}$ are the joint angles, $\mathcal{Q}$ is a simply-connected, open subset of $[0, 2\pi)^N$, $\dot{q} \in \mathbb{R}^N$, and $u \in \mathbb{R}^{N-1}$. Since the dimension of u is smaller than the dimension of q, the robot is underactuated. It is assumed that the coordinates are chosen to be all relative angles with only one absolute angle. The matrix $D(q)$ is the mass-inertia matrix, $C(q,\dot{q})$ is the matrix of centripetal and Coriolis terms, $G(q)$ is the gravity vector, and B is the input matrix.

Defining $x := (q; \dot{q})$, the model written in state space is

$$\dot{x} = \begin{bmatrix} \dot{q} \\ D^{-1}[-C\dot{q} - G] \end{bmatrix} + \begin{bmatrix} 0 \\ D^{-1}B \end{bmatrix} u \tag{2a}$$

$$=: f(x) + g(x)u \tag{2b}$$

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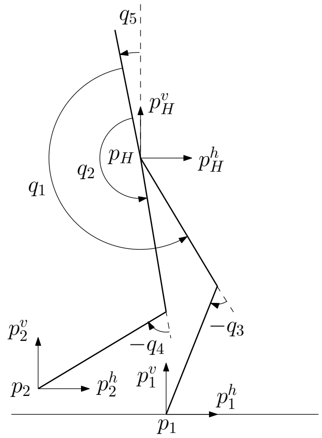

Fig. 1. A 5-link underactuated biped with the hip p_H, stance leg tip p_1, swing leg tip p_2, and joint angles indicated.

with state space $TQ := \{(q;\dot{q}) \mid q \in Q, \dot{q} \in \mathbb{R}^N\}$.

The double support phase is assumed to be instantaneous and modeled by a rigid impact (see [14, IH1–IH6] for a complete list of hypotheses concerning the impact.) The development of the impact model requires an $N+2$ DOF model of the biped. The state variables just after and just before impact are related by an algebraic map

$$x^+ = \Delta(x^-) \tag{3}$$

where x^+ is the state just after the impact, and x^- is the state just before the impact. The impact map is applied whenever the state enters the switching set $\mathcal{S}$ (at double support), where

$$\mathcal{S} := \{(q,\dot{q}) \in TQ \mid p_1^v(q) = 0,\ p_2^v(q) = 0,\ p_2^h(q) > 0\}. \tag{4}$$

The complete hybrid system may be written as

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)u & x^- \notin \mathcal{S} \\ x^+ = \Delta(x^-) & x^- \in \mathcal{S}. \end{cases} \tag{5}$$

III. A New Definition of Stable Bipedal Walking

This section gives a sequence of preliminary definitions leading to a precise definition of stable walking.

Let $\varphi(t, t_0, x_0, u_{[t_0,t)})$ be the solution of the swing phase dynamics at time $t \in \mathbb{R}$ with initial condition $x_0 \in TQ$ at $t_0 \in \mathbb{R}$ and control $u_{[t_0,t)} \in \mathcal{U}$, where $\mathcal{U}$ is the set of possible controllers. Then, the time of next impact may be defined as follows:

Definition 1 (Time of next impact)**.** *The time of next impact, $T_I : \mathbb{R} \times TQ \times \mathcal{U} \to \mathbb{R}$ is defined by*

$$T_I(t_0, x_0, u_{[t_0,t)}) := \inf\{t > t_0 \mid \varphi(t, t_0, x_0, u_{[t_0,t)}) \in \mathcal{S}\}. \tag{6}$$

When the robot does not complete a step, the time of next impact is infinite. In addition, implicit in the definition of the time of next impact is that $T_I(t_0, x_0, u_{[t_0,t)}) < \infty$ implies that $\varphi(\cdot)$ exists on $t \in [t_0, T_I(t_0, x_0, u_{[t_0,t)}))$.

Definition 2 (Feasible trajectory/feasible trajectory set)**.** *Given $(t_0, x_0^-) \in \mathbb{R} \times \mathcal{S}$ a feasible trajectory is a trajectory for which there exists $u_{[t_0,t)} \in \mathcal{U}$ such that $T_I(t_0, \Delta(x_0^-), u_{[t_0,t)}) < \infty$ and such that for all $t \in [t_0, T_I(t_0, \Delta(x_0^-), u_{[t_0,t)}))$ all constraints are satisfied. Such constraints include ground reaction force constraints, joint angles constraints, actuation limits, etc. All feasible trajectories associated with $(t_0, x_0^-) \in \mathbb{R} \times \mathcal{S}$ form the feasible trajectory set $\Phi(t_0, x_0^-)$, i.e.,*

$$\Phi(t_0, x_0^-) = \bigcup_{u_{[t_0,t)} \in \mathcal{U}} \varphi(\,\cdot\,, t_0, \Delta(x_0^-), u_{[t_0,t)}), \tag{7}$$

such that $T_I(t_0, \Delta(x_0^-), u_{[t_0,t)}) < \infty$.

Definition 3 (Proper switching set)**.** *The proper switching set $\mathcal{S}_P$ is a subset of the switching set $\mathcal{S}$, such that for all $x_0^- \in \mathcal{S}_P$ there exists $t_0 \in \mathbb{R}$ and $u_{[t_0,t)} \in \mathcal{U}$ such that $\varphi(\,\cdot\,, t_0, \Delta(x_0^-), u_{[t_0,t)}) \in \Phi(t_0, x_0^-)$.*

Definition 4 (Strictly proper switching set/strictly proper feasible trajectories/strictly proper feasible trajectory set)**.** *The strictly proper switching set $\mathcal{S}_{SP}$ is a subset of the proper switching set $\mathcal{S}_P$, with the property that for all $x_0^- \in \mathcal{S}_{SP}$ there exists $t_0 \in \mathbb{R}$ and $u_{[t_0,t)} \in \mathcal{U}$ such that $\varphi(\,\cdot\,, t_0, \Delta(x_0^-), u_{[t_0,t)}) \in \Phi(t_0, x_0^-)$ and $\varphi(T_I(t_0, \Delta(x_0^-), u_{[t_0,t)}),\, t_0, \Delta(x_0^-), u_{[t_0,t)}) \in \mathcal{S}_{SP}$. Such feasible trajectories are called strictly proper feasible trajectories, and all strictly proper feasible trajectories associated with $(t_0, x_0^-) \in \mathbb{R} \times \mathcal{S}_{SP}$ form the strictly proper feasible trajectory set $\Phi_{SP}(t_0, x_0^-) \subset \Phi(t_0, x_0^-)$.*

Note that $\mathcal{S}_{SP} \subset \mathcal{S}_P \subset \mathcal{S}$. In Definitions 3 and 4, for any given $(t_0, x_0^-) \in \mathbb{R} \times \mathcal{S}_P$ (resp. $\mathbb{R} \times \mathcal{S}_{SP}$), there may exist more than one feasible trajectory such that $\varphi(T_I(t_0, \Delta(x_0^-), u_{[t_0,t)}),\, t_0, \Delta(x_0^-), u_{[t_0,t)}) \in \mathcal{S}$ (resp. $\mathcal{S}_{SP}$). The non-uniqueness of the mapping is a result of the use of different controllers. Finally, note that the terminology "strictly proper switching set" is used instead of "basin of attraction" since the notion of a basin of attraction is usually associated with a periodic orbit or a fixed point, and not aperiodic orbits.

The previous definitions are concerned with only single steps. Building upon these definitions, walking and stable walking are defined next. Loosely speaking, a step consists of a swing phase and an impact event whereas walking consists of successive steps. This is made precise in the following.

Definition 5 (Step and walking)**.** *A step of the robot is the solution of (5) defined on the half-open interval $[t_0, T_I(t_0, \Delta(x_0^-), u_{[t_0,t)}))$ with $(t_0, x_0^-) \in \mathbb{R} \times \mathcal{S}$ and $u_{[t_0,t)} \in \mathcal{U}$ and $T_I(t_0, \Delta(x_0^-), u_{[t_0,t)})) < \infty$. Walking is defined as successive steps.*

Definition 6 (Stable walking)**.** *The biped walking is stable if $x_0^- := x_0(t_0^-) \in \mathcal{S}_{SP}$, and for any $k = 0, 1, 2, \ldots$, $\varphi_k(\,\cdot\,, t_k, \Delta_k(x_k^-), u_k) \in \Phi_{SP}(t_k, x_k^-)$ for all $t \in [t_k, t_{k+1})$, where $x_k^- := x(t_k^-)$, $t_{k+1} := T_I(t_k, \Delta(x_k^-), u_k)$ and $u_k := u_{[t_k,t_{k+1})}$.*

IV. Controller Design

In this section, results from [14] on the use of hybrid zero dynamics for the design of controllers that induce asymptotically stable gaits are reviewed. Then, a discussion of two conditions leading to gait instability follows. The development continues with the presentation of a hierarchical walking controller that acts by switching among a set of individual controllers. The conditions required for the existence of a switching policy that induces stable walking and sufficient conditions to find a policy are given. An approach is given for determining a set of individual controllers such that the conditions required for the existence of a switching policy that induces stable walking are satisfied.

A. Basic Facts and Gait Stability

Some results from [14] are now summarized. For the model (1), suppose that there exists a function $\theta : \mathcal{Q} \to \mathbb{R}$ that is monotonically increasing over the duration of a step. Define a parameterized set of holonomic constraints, $h_{d,\alpha}(\theta) : \mathbb{R} \to \mathbb{R}^{N-1}$ on the actuated coordinates. The $N-1$ constraint functions are chosen to be Bézier polynomials,

$$h_{di,\alpha^i} := \sum_{k=0}^{M} \alpha_k^i \frac{M!}{k!(M-k)!} s^k (1-s)^{M-k}, \qquad (8)$$

where $s := (\theta - \theta^+)/(\theta^- - \theta^+)$, $i = 1, \ldots, N-1$, $\alpha^i := (\alpha_0^i; \ldots; \alpha_M^i) \in \mathbb{R}^{M+1}$, and θ^+ and θ^- are the values of θ at the beginning and end of the swing phase.

Let $\alpha := \left[\alpha^1, \ldots, \alpha^{N-1}\right] \in \mathbb{R}^{(M+1)\times(N-1)}$ and define the output

$$y = h(q) := h_0(q) - h_{d,\alpha}(q) \qquad (9)$$

where $h_0(q) := (q_1; \ldots; q_{N-1})$. The set of parameters α is said to be *regular* if it satisfies output Hypothesis HH1–HH5 given in [14]. The regular parameter set implies that the associated decoupling matrix is invertible, there exists a two-dimensional zero dynamics during the swing phase, the associated zero dynamics manifold $\mathcal{Z}_\alpha := \{x \in T\mathcal{Q} \mid h(x) = 0, L_f h(x) = 0\}$ is rendered invariant by the feedback control $u^*(x) := -(L_g L_f h(x))^{-1} L_f^2 h(x)$, and, with appropriate initialization, the robot is able to complete at least one step. Let the feedback control Γ_α be any feedback control satisfying Hypothesis CH2–CH5 given in [14]. Then, $\mathcal{Z}_\alpha$ is invariant under Γ_α and is locally finite-time attractive. The controller Γ_α is termed an *individual controller* associated with $\mathcal{Z}_\alpha$. Furthermore, the hybrid zero dynamics exists if

$$\Delta(S \cap \mathcal{Z}_\alpha) \subset \mathcal{Z}_\alpha. \qquad (10)$$

Let (ξ_1, ξ_2) be coordinates for $\mathcal{Z}_\alpha$ where $\xi_1 := \theta$, $\xi_2 := \partial K/\partial \dot{q}_N|_{\mathcal{Z}_\alpha} = d_N(q)\dot{q}$ and where $K(q,\dot{q}) = \frac{1}{2}\dot{q}' D \dot{q}$ is the kinetic energy of the robot and d_N is the row of D corresponding to the absolute coordinate. In these coordinates, the hybrid zero dynamics takes the form

$$\begin{cases} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} k_1(\xi_1)\xi_2 \\ k_2(\xi_1) \end{bmatrix} & (\xi_1, \xi_2) \notin (S \cap \mathcal{Z}_\alpha) \\ \begin{bmatrix} \xi_1^+ \\ \xi_2^+ \end{bmatrix} = \begin{bmatrix} \theta^+ \\ \delta_{\text{zero}}^2 \xi_2^- \end{bmatrix} & (\xi_1, \xi_2) \in (S \cap \mathcal{Z}_\alpha) \end{cases} \qquad (11)$$

where δ_{zero}^2 is a constant which can be computed *a priori* [13].

Let $\zeta_2 = \frac{1}{2}\xi_2^2$. Since ξ_1 is monotonic over a step, ζ_2 may be integrated against ξ_1 over one step to obtain

$$\zeta_2(\xi_1) = \zeta_2^+ - V_{\text{zero}}(\xi_1), \qquad (12)$$

where $V_{\text{zero}}(\xi_1) = -\int_{\xi_1^+}^{\xi_1} k_2(\xi)/k_1(\xi) d\xi$. The quantity $V_{\text{zero}}(\xi_1)$ may be computed numerically. Two values of V_{zero} that are of particular interest are $V_{\text{zero}}(\xi_1^-)$ and

$$V_{\text{zero}}^{\max} := \max_{\theta^+ \leq \xi_1 \leq \theta^-} V_{\text{zero}}(\xi_1). \qquad (13)$$

Concerning the ground reaction forces on the stance leg, with the constraints perfectly imposed, the ground reaction forces may be calculated as

$$\begin{bmatrix} F_1^T(\xi_1, \zeta_2^-) \\ F_1^N(\xi_1, \zeta_2^-) \end{bmatrix} = \Lambda_1(\xi_1)\zeta_2^- + \Lambda_0(\xi_1), \qquad (14)$$

where F_1^T and F_1^N are the ground reaction forces on the tangential and normal directions, respectively, and Λ_0 and Λ_1 are smooth functions.

To ensure that the stance leg end does not leave the ground, the normal force F_1^N must always point upwards. In addition, the ratio of the tangential force and the normal force should be smaller than the friction coefficient to ensure the leg end will not slip on the ground. These two constraints result in a limit on the value of ζ_2^-. The limit, $\zeta_{2,|F_1^T/F_1^N|}^{\max}$, may be explicitly calculated as

$$\zeta_{2,F_1^N}^{\max} := \sup\left\{ \zeta_2^- > 0 \;\middle|\; \min_{\theta^+ \leq \xi_1 \leq \theta^-} \left|F_1^N(\xi_1, \zeta_2^-)\right| \geq 0 \right\} \qquad (15a)$$

$$\zeta_{2,|F_1^T/F_1^N|}^{\max} := \sup\left\{ 0 < \zeta_2^- < \zeta_{2,F_1^N}^{\max} \;\middle|\; \max_{\theta^+ \leq \xi_1 \leq \theta^-} \left| \frac{F_1^T(\xi_1, \zeta_2^-)}{F_1^N(\xi_1, \zeta_2^-)} \right| \leq \mu_s \right\}. \qquad (15b)$$

For the hybrid system, the restricted Poincaré map $\rho : S \cap \mathcal{Z}_\alpha \to S \cap \mathcal{Z}_\alpha$ has the form

$$\rho(\zeta_2^-) = \delta_{\text{zero}}^2 \zeta_2^- - V_{\text{zero}}(\xi_1^-) \qquad (16)$$

and its domain of definition is given by

$$\mathcal{S}_D := \left\{ \zeta_2^- > 0 \;\middle|\; \delta_{\text{zero}}^2 \zeta_2^- > V_{\text{zero}}^{\max}, \zeta_2^- \leq \zeta_{2,|F_1^T/F_1^N|}^{\max} \right\}. \qquad (17)$$

On the zero dynamics manifold $\mathcal{Z}_\alpha$, using [14, Eqns. (20) and (21)], maps can be defined between $(\xi_1; \zeta_2)$ and $(q; \dot{q})$ as follows:

$$q = \Pi_\alpha^{-1}(0, \xi_1) =: \Upsilon_{q,\alpha}(\xi_1), \qquad (18)$$

where $\Pi_\alpha(q) := (h; \theta(q))$ and

$$\dot{q} = \begin{bmatrix} \frac{\partial h}{\partial q} \\ d_N \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial h}{\partial q} \\ d_N \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\sqrt{2\zeta_2} \end{bmatrix} \qquad (19a)$$

$$=: \Upsilon_{\dot{q},\alpha}(\zeta_2). \qquad (19b)$$

Remark 1. *Since $\zeta_2 = \xi_2^2/2$, then $\xi_2 = \pm\sqrt{\zeta_2}$. The sign may be inferred when the walking direction is known. According* *to the coordinate system used here, and assuming that the biped walks forward, the minus sign should be chosen.*

1) Checking gait stability: There exists an exponentially stable periodic orbit of the hybrid zero dynamics if, and only if

$$\frac{\delta_{\text{zero}}^2}{1-\delta_{\text{zero}}^2} V_{\text{zero}}(\xi_1^-) + V_{\text{zero}}^{\max} < 0 \tag{20}$$

and

$$\delta_{\text{zero}}^2 < 1, \tag{21}$$

with the corresponding fixed point

$$\zeta_2^{*-} := -\frac{V_{\text{zero}}(\xi_1^-)}{1-\delta_{\text{zero}}^2} \in \mathcal{S}_D. \tag{22}$$

2) Conditions corresponding to gait instability: Assuming that the biped loses energy at every leg impact with the ground, i.e., $\delta_{\text{zero}}^2 < 1$, a fixed point of the return map exists. As a result, a gait will be unstable only if the gait's fixed point is outside the corresponding domain of definition, $\mathcal{S}_D$.

Since the domain of definition is one-dimensional, the fixed point may lie outside the domain by being on either side of the domain. Assume the biped is initialized at a point in the domain of definition of an individual controller whose fixed point is such that $\zeta_2^{*-} > \zeta_{2,|F_1^T/F_1^N|}^{\max}$. In this case, the biped's walking rate will increase, and eventually the stance leg end will slip due to violation of ground contact constraints. On the other hand, assume the biped is initialized at a point in the domain of definition of an individual controller whose fixed point is such that $\zeta_2^{*-} < V_{\text{zero}}^{\max}/\delta_{\text{zero}}^2$. In this case, the biped's walking rate will decrease, and eventually the biped will fail to finish a step.

3) Transition Gaits: Let α_i and α_j be two regular parameter sets of the output (9). Then, it is possible to construct a one step *transition controller* $\Gamma_{\alpha_{i\to j}}$. One valid choice for $\Gamma_{\alpha_{i\to j}}$ is given by (see [16])

$$\begin{aligned}
\alpha_{i\to j,0} &= \alpha_{i,0} \\
\alpha_{i\to j,1} &= \alpha_{i,0} - \frac{\theta_j^- - \theta_i^+}{\theta_i^- - \theta_i^+}(\alpha_{i,0} - \alpha_{i,1}) \\
\alpha_{i\to j,M-1} &= \alpha_{j,M} - \frac{\theta_j^- - \theta_i^+}{\theta_j^- - \theta_j^+}(\alpha_{j,0} - \alpha_{j,1}) \\
\alpha_{i\to j,M} &= \alpha_{j,0} \\
\theta_{i\to j}^+ &= \theta_i^+ \\
\theta_{i\to j}^- &= \theta_j^- \\
\alpha_{i\to j,m} &= \frac{1}{2}(\alpha_{i,m} + \alpha_{j,m}), \quad m = 2,\ldots,M-2.
\end{aligned} \tag{23}$$

B. Switching Controller Analysis

The proposed *walking controller* has a hierarchical structure with two layers: The lower layer consists of individual controllers that induce at least one step. The upper layer consists of a switching controller that, at the end of each step, chooses the next individual controller to be applied.

For convenience, the individual controllers of the lower layer are collected into a matrix:

$$\hat{\Gamma} := \begin{bmatrix} \Gamma_{\alpha_{1\to 1}} & \Gamma_{\alpha_{1\to 2}} & \cdots & \Gamma_{\alpha_{1\to n}} \\ \Gamma_{\alpha_{2\to 1}} & \Gamma_{\alpha_{2\to 2}} & \cdots & \Gamma_{\alpha_{2\to n}} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{\alpha_{n\to 1}} & \Gamma_{\alpha_{n\to 2}} & \cdots & \Gamma_{\alpha_{n\to n}} \end{bmatrix}. \tag{24}$$

The parameter set associated with each individual controller $\alpha_{i\to j}$ is assumed to be regular. Note that if the biped takes one step under the controllers on the main diagonal of $\hat{\Gamma}$, the final configuration will be the same as the initial configuration except with the legs swapped. These individual controllers are called *self-transition controllers*. Self-transition controllers that induce asymptotically stable gaits can be designed following the procedure summarized in Section IV-A. The swing phase zero dynamics manifold $\mathcal{Z}_{\alpha_{i\to j}}$ associated with the transition controller $\Gamma_{\alpha_{i\to j}}$ satisfies

$$\begin{aligned} \Delta(S\cap\mathcal{Z}_{\alpha_{i\to i}})\cap\mathcal{Z}_{\alpha_{i\to j}} &= \Delta(S\cap\mathcal{Z}_{\alpha_{i\to i}}) \\ \Delta(S\cap\mathcal{Z}_{\alpha_{i\to j}}) &= \Delta(S\cap\mathcal{Z}_{\alpha_{j\to j}}). \end{aligned} \tag{25}$$

That is, $\mathcal{Z}_{\alpha_{i\to j}}$ connects the zero dynamics manifolds $\mathcal{Z}_{\alpha_{i\to i}}$ and $\mathcal{Z}_{\alpha_{j\to j}}$ associated with the self-transition controllers $\Gamma_{\alpha_{i\to i}}$ and $\Gamma_{\alpha_{j\to j}}$. The requirement (25) on the transition controller zero dynamics manifold can be satisfied by choosing the transition controller's Bézier coefficients per (23), and this requirement implies that the configuration of the robot at the end of a step for controllers in the same column of $\hat{\Gamma}$ are the same, whereas the configuration of the robot at the beginning of a step are the same for controllers in the same row of $\hat{\Gamma}$.

Let $\rho_{i\to j}$ be defined as in (16) and $\mathcal{S}_{D,i\to j}$ be defined as in (17) denote the map and the domain of definition associated with the individual controller $\Gamma_{\alpha_{i\to j}}$. The image of $\mathcal{S}_{D,i\to j}$ under $\rho_{i\to j}$ is denoted by $\mathcal{S}_{I,i\to j} := \rho_{i\to j}(\mathcal{S}_{D,i\to j})$.

Based on the domain of definition given in Section IV-A, for each individual controller $\Gamma_{\alpha_{i\to j}}$, the corresponding domain of definition $\mathcal{S}_{D,i\to j}$ can be explicitly found as $(\zeta_{2,\min,i\to j}^-, \zeta_{2,\max,i\to j}^-)$ where

$$\zeta_{2,\min,i\to j}^- := \frac{1}{\delta_{\text{zero},i\to j}^2} V_{\text{zero},i\to j}^{\max} \tag{26}$$

$$\zeta_{2,\max,i\to j}^- := \zeta_{2,|F_1^T/F_1^N|,i\to j}^{\max}. \tag{27}$$

Define

$$\mathcal{S}_{D,i} := \bigcup_{j=1}^{n} \mathcal{S}_{D,i\to j}. \tag{28}$$

From each point in $\mathcal{S}_{D,i}$ there exists at least one individual controller that results in the robot taking a step.

Given the matrix $\hat{\Gamma}$ in (24), the following theorem gives sufficient conditions for the existence of a switching policy such that the biped in closed loop with the corresponding walking controller results in walking that is stable in the sense of Definition 6.

Theorem 1 (Switching policy existence)**.** *Assume that $\hat{\Gamma}$ in (24) is such that for all $i \in \{1,\ldots,n\}$, there exists a non-empty subset $\mathcal{S}_{D,i}^* \subseteq \mathcal{S}_{D,i}$ with the property that for all*

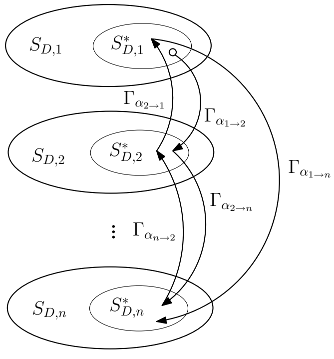

Fig. 2. Example transitions in the lower layer of a walking controller. The gait starts with the state in $(\Upsilon_{q,1}(\xi^-_{1,1});\ \Upsilon_{\dot q,1}(\mathcal{S}^*_{D,1}))$. Then, the individual controllers $\Gamma_{\alpha_{1\to2}}, \Gamma_{\alpha_{2\to n}}, \Gamma_{\alpha_{n\to2}}, \Gamma_{\alpha_{2\to1}}, \Gamma_{\alpha_{1\to n}}$ are applied.

$\zeta^-_{2,i} \in \mathcal{S}^*_{D,i}$, there exists $j \in \{1,\dots,n\}$ such that

$$\rho_{i\to j}(\zeta^-_{2,i}) \in \mathcal{S}^*_{D,j}. \tag{29}$$

Then, there exists a switching policy that induces stable walking.

Proof. The definition of $\mathcal{S}^*_{D,i}$, $i \in \{1,\dots,n\}$, means that for all $(\xi^-_1;\zeta^-_2) \in \bigcup_{i=1}^n \{\xi^-_{1,i}\} \times \mathcal{S}^*_{D,i}$ there exists at least one controller $\Gamma_{\alpha_{i\to j}}$ that will result in the biped taking one step and the state of the zero dynamics returning to the set $\bigcup_{i=1}^n \{\xi^-_{1,i}\} \times \mathcal{S}^*_{D,i}$. Hence, the set

$$\bigcup_{i=1,j=1}^{n,n} \left(\{\Upsilon_{q,i\to j}(\xi^-_{1,i})\} \times \Upsilon_{\dot q,i\to j}(\mathcal{S}^*_{D,i} \cap \mathcal{S}_{D,i\to j})\right) \subset \mathcal{S} \tag{30}$$

is a strictly proper switching set. A switching policy may be constructed by always taking the first controller $\Gamma_{\alpha_{i\to j}}$ that results in the biped's state returning to the set given in (30). Therefore, existence of a switching policy has been shown. □

Figure 2 illustrates transitions in the lower layer of a walking controller. From any point $\zeta^-_{2,i} \in \mathcal{S}^*_{D,i}$, there is at least one individual controller $\Gamma_{\alpha_{i\to j}}$ which can steer $\zeta^-_{2,i}$ into $\mathcal{S}^*_{D,j}$. The upper layer switching policy of the walking controller is responsible for making this selection.

For a given $\hat\Gamma$, once the existence of a switching policy has been established by Theorem 1, a sufficient condition for a switching policy to induce stable walking is given by the following:

Theorem 2 (A sufficient condition for the switching policy)**.** *Given $\hat\Gamma$ in (24) for which there exists a switching policy that induces stable walking (cf. Theorem 1), all switching policies satisfying the following are valid switching policies. At the end of step k under the individual controller $\Gamma_{\alpha_{i\to j}}$, the next individual controller $\Gamma_{\alpha_{j\to w}}$, where $w \in \{1,\dots,n\}$, must be such that*

$$\zeta^-_2(k+1) \in \mathcal{S}^*_{D,j} \cap \mathcal{S}_{D,j\to w}. \tag{31}$$

Proof. Selecting a controller $\Gamma_{\alpha_{j\to w}}$, $w \in \{1,\dots,n\}$, that satisfies (31) results in the biped taking one step and the state of the zero dynamics landing in $\{\xi^-_{1,w}\} \times \mathcal{S}^*_{D,w}$ from $\{\xi^-_{1,j}\} \times \mathcal{S}^*_{D,j}$. The images $\{\xi^-_{1,j}\} \times \mathcal{S}^*_{D,j}$ and $\{\xi^-_{1,w}\} \times \mathcal{S}^*_{D,w}$ in $T\mathcal{Q}$ are subsets of a strictly proper switching set as in (30). That is, the strictly proper switching set is invariant under the switching policy. Hence, stable walking is induced. □

TABLE I
PARAMETERS OF THE BIPED MODEL.

Link	Length) (m)	Mass (kg)	Moment of Inertia (kg·m^2)
Torso	0.2	13.56	0.0905
Femur	0.36	1.47	0.0238
Tibia	0.36	0.97	0.0184

Remark 2. *As an example of a switching policy, consider one that, at the end of each step, randomly selects among the available controllers satisfying (31). More generally, any freedom in the controller choice may be exploited to achieve additional goals, such as the biped stepping over an obstacle.*

C. An Applications of Theorem 1

The following application follows the proof of Theorem 1 to check the existence of $S^*_{D,i}$, $i \in \{1,\dots,n\}$:

Given n self-transition controllers $\Gamma_{\alpha_{i\to i}}$, $i = 1,\dots,n$, each with non-empty domain of definition, $\mathcal{S}_{D,i\to i}$, and whose parameter sets are regular, complete the following:

A1.1) Populate the matrix $\hat\Gamma$ with transition controllers $\Gamma_{\alpha_{i\to j}}$, for all $i,j = 1,\dots,n$, $i \neq j$ following the approach given in Section IV-A. Continue to Step A1.2) if every parameter set associated each transition controller is regular; otherwise, stop.

A1.2) Compute $\mathcal{S}_{D,i\to j}$ and $\mathcal{S}_{D,i}$, for $i,j = 1,\dots,n$. If for all $i \in \{1,\dots,n\}$, $\mathcal{S}^*_{D,i}$ is non-empty, then, by Theorem 1, there exists a switching policy that can result in stable biped walking; otherwise, no switching policy exists.

V. EXAMPLE

The following example illustrates the application of the procedure for controller synthesis given in Section IV-C that results in stable walking in the sense of Definition 6. The model used is of a 5-link biped that is in our laboratory. The model parameters are given in Table I. The friction coefficient between the ground and the robot's stance leg end is assumed to be $\mu_s = 0.6$. Sixth-degree Bézier polynomials were used to define $h_d(q)$, and θ was defined as $\theta := -q_5 - q_1 - q_3/2$.

The walking controller was synthesized following Section IV-C. Four self-transition controllers, $\Gamma_{\alpha_{i\to i}}$, $i = 1,\dots,4$, were designed by parameter optimization. Self-transition controllers $\Gamma_{\alpha_{1\to1}}$ and $\Gamma_{\alpha_{2\to2}}$ induce asymptotically stable gait at the rates of 0.36 m/s and 0.54 m/s, respectively. The gaits induced by self-transition controllers $\Gamma_{\alpha_{3\to3}}$ and $\Gamma_{\alpha_{4\to4}}$ are unstable because their fixed points are, respectively, above the upper boundary and below the lower boundary of their domain of definitions.

TABLE II

DOMAIN OF DEFINITION $\mathcal{S}_{D,i\to j}$ OF INDIVIDUAL CONTROLLER $\Gamma_{\alpha_{i\to j}}$.

i \ j	1	2	3	4
1	$(59, 171)$	$(59, 162.2)$	$(59, 84)$	*(65, 174)*
2	$(57, 213)$	$(57, 200)$	$(57, 100)$	*(65, 216)*
3	*(27, 141)*	$(26, 130)$	$(26, 106)$	*(40,143)*
4	$(61, 174)$	$(61, 165)$	$(61, 87)$	*(70, 147)*

TABLE III

IMAGE $\mathcal{S}_{I,i\to j}$ OF THE DOMAIN OF DEFINITION $\mathcal{S}_{D,i\to j}$.

i \ j	1	2	3	4
1	$(61, 135)$	$(66, 134)$	$(57, 74)$	$(63, 135)$
2	$(61, 162)$	$(66, 158)$	$(57, 85)$	$(64, 169)$
3	$(60, 147)$	$(64, 143)$	$(55, 116)$	$(68, 147)$
4	$(60, 135)$	$(65, 134)$	$(56, 74)$	$(65, 115)$

The matrix $\hat{\Gamma}$ was populated with transition controllers following Section IV-A.3. The domain of definition and its corresponding image for each individual controller are given in Tables II and III. In Table II, the entries in italics are subsets of the actual domain of definitions, and are chosen such that $\mathcal{S}^*_{D,i}$ will exist. It may be checked that $\mathcal{S}^*_{D,i} = \bigcup_{j=1}^{4} \mathcal{S}_{D,i\to j}$.

The switching policy is chosen such that the selection of individual controllers is done randomly as suggested in Remark 2. The resulting walking controller in closed-loop with the biped model was simulated for 20 steps. The individual controllers chosen are $\Gamma_{\alpha,0\to0}$, $\Gamma_{\alpha,0\to0}$, $\Gamma_{\alpha,0\to2}$, $\Gamma_{\alpha,2\to0}$, $\Gamma_{\alpha,0\to1}$, $\Gamma_{\alpha,1\to1}$, $\Gamma_{\alpha,1\to0}$, $\Gamma_{\alpha,0\to2}$, $\Gamma_{\alpha,2\to3}$, $\Gamma_{\alpha,3\to0}$, $\Gamma_{\alpha,0\to3}$, $\Gamma_{\alpha,3\to0}$, $\Gamma_{\alpha,0\to1}$, $\Gamma_{\alpha,1\to3}$, $\Gamma_{\alpha,3\to3}$, $\Gamma_{\alpha,3\to1}$, $\Gamma_{\alpha,1\to1}$, $\Gamma_{\alpha,1\to0}$, $\Gamma_{\alpha,0\to0}$, $\Gamma_{\alpha,0\to1}$. From Figure 3 it is clear that the biped's gait is aperiodic. During the walking, the nominal ground reaction force points upwards, and its value varies between 78 N and 194 N, the ratio of the tangential force and the nominal force is less than the friction coefficient, and the ground contact assumptions are satisfied.

VI. CONCLUSIONS

This paper presented a new definition of stable biped walking and gave a framework for the design of controllers that induce aperiodic walking that is stable in the sense of the given definition. The class of bipeds to which the results apply are in planar bipeds with point feet. The definition of stable walking given in this paper reflects the nature of the human bipedal walking, and is independent of the control strategy. The controller design framework was illustrated by simulation on a 5-link biped model.

VII. ACKNOWLEDGEMENTS

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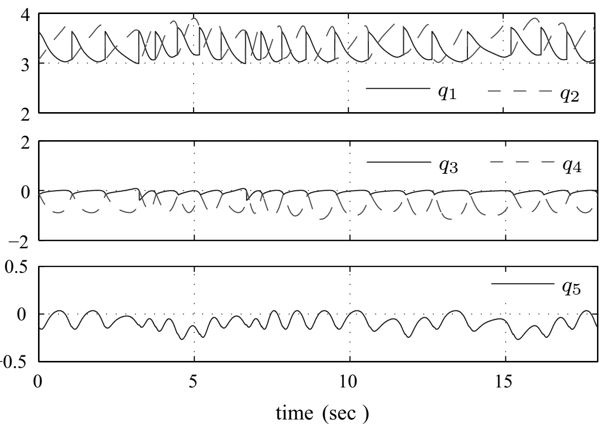

Fig. 3. Joint angle trajectories for twenty-steps of aperiodic walking. Angles q_1 and q_3 are the stance leg hip and knee angles, respectively; q_2 and q_4 are the swing leg hip and knee angles, respectively; and q_5 is the torso angle. The units are radians.

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