Mixed Initiative Control of Autonomous Vehicles

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Abstract— While there is extensive work on motion planning and control for navigation tasks with guarantees, there is no systematic way for human operators to modify the resulting plans without losing the guarantees. In this paper we propose a systematic way of composing behaviors resulting from human inputs with behaviors derived from navigation functions. The proposed controller is based on a new class of navigation function based controllers that posses weak Input-to-State stability properties. The resulting system has analytically guaranteed safety and convergence properties. The feasibility of the proposed methodology is demonstrated through simulation examples and hardware experiments.

I. INTRODUCTION

Usually when developing automated systems, generally one only considers the development of safe, fully autonomous systems. However, human intuition is often indispensable and a systematic way of incorporating it while guaranteeing the safety and liveness properties of the system is required.

Our approach is based on the concept of navigation functions [3]. The reason for choosing the class of navigation functions is due to their favorable analytic properties that can be exploited for developing the sought controllers. Navigation functions are a special category of cost functions, that are free of local minima. The only minimum they possess is at the target configuration. They provide for guaranteed a.e.[1] convergence through a Lyapunov-LaSalle type of argument. Moreover by their design, they provide for guaranteed safety properties in terms of avoidance control [4]. Our navigation function based controllers inherit all of the analytic properties of the navigation functions and moreover they allow for interconnections with other systems.

Regarding the composition of human inputs with navigation tasks, no general approach exists. Results available in the literature [6] provide for only local guarantees for the navigation task. We need a general framework, that will be able to provide global guarantees for safety and convergence.

A composition framework that would be relevant to our application is the Input-to-State Stability (ISS) framework [10] under which one can interconnect the input of an ISS system with the output of a globally asymptotically stable (GAS) system and get a GAS system, as shown in figure 1.

Due to the topological properties imposed on navigation functions, a navigation function based controller cannot

[1]a.e.: almost everywhere i.e. everywhere up to a set of measure zero of initial conditions
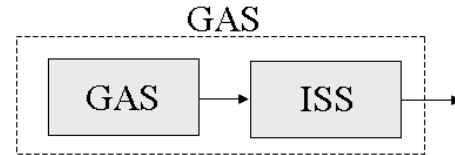

Fig. 1 : *The Input to State Stability (ISS) framework*

directly accept an ISS characterization. However, in [5] we show that there exist navigation function based controllers that accept a weaker form of input to state stability characterization, namely the almost ISS (aISS) characterization [1]. Our basic approach is to use an aISS navigation function based controller, similar to the ones proposed in [5], with its inputs being interconnected to the output of a human interface system.

The rest of the paper is organized as follows: Section II presents the necessary notions for the following analysis while section III analyzes the proposed control framework. Section IV describes how the proposed control framework can be utilized in the construction of mixed initiative controllers, while in section V we present experimental results from the application of our methodology. The paper concludes with section VI

II. PRELIMINARIES

A. The aISS framework

As we have mentioned in the introduction, the compositional framework that motivated our approach was the ISS framework. We hereby briefly state some definitions and results from the ISS literature.

Definition 1: [10] Consider a system of the form $\dot{x} = f(x, u)$ evolving in finite dimensional spaces $\mathbb{R}^n$ with inputs $u \in \mathbb{R}^m$ that are measurable essentially locally bounded. The map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz and satisfies $f(0,0) = 0$. The system is input to state stable (ISS) if

$$\|x(t)\| \le \beta(\|x(0)\|, t) + \gamma(\|u\|_\infty) \qquad (1)$$

for some $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ and for all $t \ge 0$.

Unfortunately almost GAS systems cannot be characterized by the above $\beta + \gamma$ type of estimate [1]. An equivalent approach in terms of asymptotic gains is more suitable for such systems and gives rise to an almost global definition of input to state stability:

Definition 2: [1] A system $\dot{x} = f(x,u)$ evolving on a smooth manifold M with $f : M \times \mathbb{U} \to TM$ a locally Lipschitz manifold map satisfying $f(x,u) \in T_x M$ for all $x \in M$ and all $u \in \mathbb{U} \subset \mathbb{R}^m$ is almost ISS (aISS) with respect to an invariant compact set $\mathcal{A} \subset M$, if $\mathcal{A}$ is locally asymptotically stable and

$$\forall u, \forall \ a.\ a.\ \xi \in M, \ \limsup_{t \to +\infty} |x(t,\xi,u)|_{\mathcal{A}} \leq \gamma(\|u\|_\infty) \quad (2)$$

where $\gamma \in \mathcal{K}$ and $|\cdot|_{\mathcal{A}}$ denotes the standard point to set distance.

While ISS properties are propagated through cascade interconnections and under small gain conditions for feedback interconnections, propagation of aISS properties through cascades and feedback interconnections is currently an open research topic. A weaker result was established by [1] regarding the interconnection of an almost GAS with an aISS system stating that the resulting system is almost GAS:

Theorem 1: [1] Consider the cascaded system

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(y) \end{aligned} \quad (3)$$

with state $z = \begin{bmatrix} x^T, y^T \end{bmatrix}^T \in M \times N$ where N a smooth manifold. Assume that f and g satisfy $f(0_M, 0_N) = 0$ and $g(0_N) = 0$ for some points $0_M \in M$ and $0_N \in N$. Let the x-subsystem be almost ISS with respect to the equilibrium 0_M and the input y and the y-subsystem be almost GAS at 0_N. Then the interconnection (3) is almost GAS at $0_{M \times N} := 0_M \times 0_N$

B. Navigation Function Based Controllers

Navigation Functions (NFs) are real valued maps, realized through cost functions, the negated gradient field of which is attractive towards the goal configuration and repulsive with respect to obstacles. Considering a trivial system described kinematically as $\dot{q} = u$ the basic idea behind navigation functions is to use a control law of the form $u = -\nabla \varphi(q)$ where $\varphi(q)$ is a navigation function, to drive the system to its destination (figure 2).
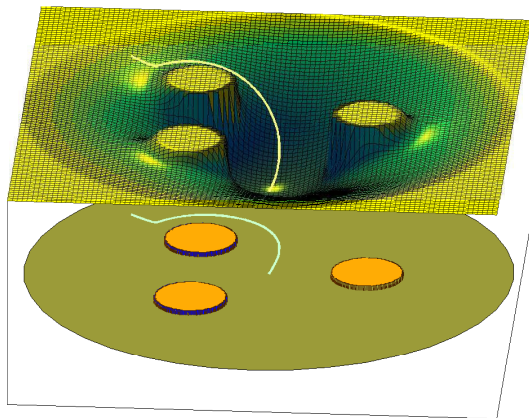

Fig. 2 : *Navigation Function with three obstacles and the resulting gradient following path*

It has been shown (Koditschek and Rimon [3]) that strict global navigation (i.e. with a globally attracting equilibrium

state) is not possible and a smooth vector field on any sphere world, which has a unique attractor, must have at least as many saddles as obstacles. It has been shown [3] that navigation properties are invariant under diffeomorphisms hence any world that can be diffeomorphically transformed to a sphere world can accept a navigation function [9], [8], [11].

III. CONTROLLER DESIGN

We will be primarily concerned with systems that are trivially described by first order kinematic models. Without loss of generality the treatment is performed on the sphere model world where the destination configuration is considered to be the origin. The following result will be useful in transferring our results from simple systems to more complicated ones:

Proposition 1: Consider the systems:

$$\dot{x} = f(x) + r(x)u(t) \quad (4)$$

and

$$\dot{y} = k(y,t)f(y) + k(y,t)r(y)u(t) \quad (5)$$

where the functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $r : \mathbb{R}^n \to \mathbb{R}$, $u : \mathbb{R}^+ \to \mathbb{R}^n$, $k : \mathbb{R}^n \times \mathbb{R}_0^+ \to \mathbb{R}$ are Lipschitz continuous in x (or y respectively) and measurable in t. Moreover we require $k(y,t) \geq \varepsilon > 0$ for some fixed ε. Then the trajectories of system (5) are the same with the trajectories of system (4) up to a time parametrization.

Proof: See Appendix A ∎

Based on the above result we can state the following stability equivalence between two aISS systems:

Proposition 2: If there exist $a, \varepsilon > 0$ such that $a \geq k(y,t) \geq \varepsilon$ then the aISS property of system (4) implies the aISS property of system (5) and vice versa.

Proof: See Appendix B ∎

Motivated by the results in [5], let us define the following:

$$S_0 : \quad \dot{x} = -K\nabla\varphi(x) + r(x)u(t) \quad (6)$$

where $r : \mathbb{R}^n \to \mathbb{R}_0^+$ is a positive, Lipschitz function and K is a positive scalar. Function $u : \mathbb{R}^+ \to \mathbb{U}$ is the input to the system and φ a navigation function.

Let $\mathcal{F} \subset E^n$ be a compact Riemannian manifold denoting the system's workspace. We have the following result regarding the safety of the system:

Proposition 3: (Safety) The trajectories of system S_0 for any measurable u satisfy: $x(t, x_0, u(t)) \in \mathcal{F}$ for all $t \geq 0$, $x_0 \in \mathcal{F}$ as long as $r(\partial \mathcal{F}) = 0$.

Proof: See Appendix C ∎

Regarding the characterization of the stability of system S_0, we have the following:

Proposition 4: With r upper bounded, system S_0 is aISS with respect to the origin.

Proof: See Appendix D ∎

IV. MIXED INITIATIVE CONTROLLER

A. Concept

In composing a human input with a navigation function based controller, we assume that the human user can use an interface to provide direction and velocity commands to the system. We can assume that the output of the human -

interface system is globally asymptotically stable, since at some point the human user will stop providing inputs to the system. By using a navigation function based system, we seek to interconnect it with a human interface system by means of of Theorem 1 as shown in figure 3. This interconnection will then have global stability and safety properties.
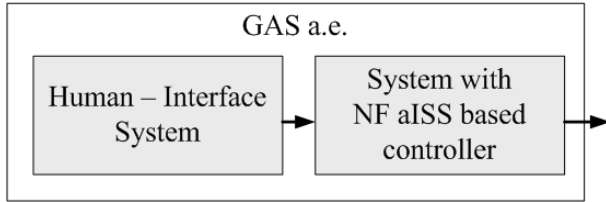

Fig. 3 : *Interfacing Human inputs with navigation function based controllers*

B. Feedback Linearization Controller

Consider the differential drive robot schematically depicted in figure 4. Its kinematics are described by the following model:

$$\begin{array}{rcl} \dot{x} & = & \frac{r}{2}(w_l + w_r)\cos(\theta) \\ \dot{y} & = & \frac{r}{2}(w_l + w_r)\sin(\theta) \\ \dot{\theta} & = & \frac{r}{L}(w_r - w_l) \end{array} \quad (7)$$
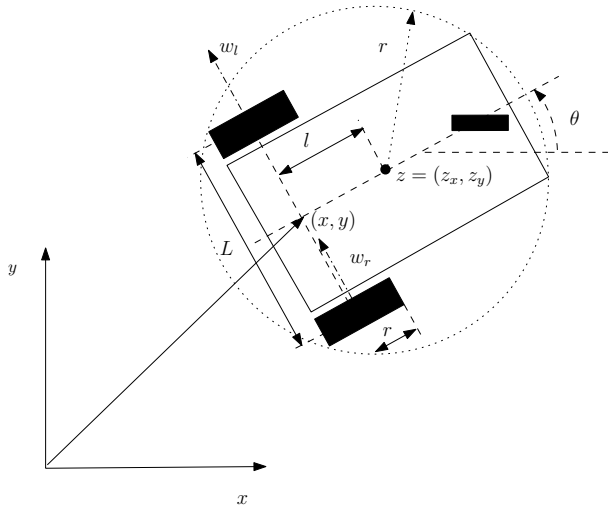

Fig. 4 : *Differential drive robot*

Using the transformations: $u = \frac{r}{2}(w_l + w_r)$ and $w = \frac{r}{L}(w_r - w_l)$, the differential drive model is transformed to the equivalent representation of the non-holonomic unicycle model:

$$\begin{array}{rcl} \dot{x} & = & u\cos(\theta) \\ \dot{y} & = & u\sin(\theta) \\ \dot{\theta} & = & \omega \end{array} \quad (8)$$

In order to be able to apply navigation function based techniques to the non-holonomic system (8) we can use a feedback linearization approach to drive a point z of the robot located at a distance l from the wheel axis central point. Consider the transformation:

$$\left[\begin{array}{c} z_x \\ z_y \end{array} \right] = \left[\begin{array}{c} x \\ y \end{array} \right] + \left[\begin{array}{c} \cos(\theta) \\ \sin(\theta) \end{array} \right] \cdot l \quad (9)$$

where point $z = (z_x, z_y)$ is located in distance l from the wheel rotation axis as shown in figure 4. Taking the time derivative of (9) and substituting (8) we get:

$$\dot{z} = A(\theta) \cdot \left[\begin{array}{c} u \\ w \end{array} \right] \quad (10)$$

where $A(\theta) = \left[\begin{array}{cc} \cos(\theta) & -l\sin(\theta) \\ \sin(\theta) & l\cos(\theta) \end{array} \right]$. Matrix $A(\theta)$ is invertible as long as $l > 0$. The inverse is given by $A(\theta)^{-1} = \left[\begin{array}{cc} \cos(\theta) & \sin(\theta) \\ -\frac{\sin(\theta)}{l} & \frac{\cos(\theta)}{l} \end{array} \right]$ so equation (10) can be written as

$$\left[\begin{array}{c} u \\ w \end{array} \right] = A(\theta)^{-1} \cdot \dot{z} \quad (11)$$

The importance of equation (11) is that we have now a way to transform any vector field to appropriate linear and angular velocities of the system (8) and effectively drive point z of the robot to any location of the free workspace. In this approach we are not interested in achieving a specific orientation of our system but just to ensure that the angular velocity goes to zero when the position of our system converges. This transformation has this property since due to the non-degeneracy of matrix $A(\theta)$, when $\dot{z} = 0$ then $u = w = 0$.

Since for the applications we are considering the achievable linear and angular velocities of the system are bounded, we need to also consider the velocity constraints in our controller design. Assume that

$$|u| \leq u_{\max}, \quad |w| \leq w_{\max} \quad (12)$$

Define the matrix $M = \left[\begin{array}{cc} \frac{1}{u_{\max}} & 0 \\ 0 & \frac{1}{w_{\max}} \end{array} \right]$ and define the expression $\mu(x, \theta) = \max\left(\left[1 \vdots \left| MA(\theta)^{-1} \cdot x \right|^T \right] \right)$ which returns the maximum element of the augmented vector $\left[1 \vdots \left| MA(\theta)^{-1} \cdot x \right|^T \right]$ where $x \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$.

Setting $\dot{z} = v\mu(v, \theta)^{-1}$ in (11), we get:

$$\left[\begin{array}{c} u \\ w \end{array} \right] = A(\theta)^{-1} \cdot \left(v\mu(v, \theta)^{-1} \right) \quad (13)$$

Regarding the velocity constraints satisfaction of system (8) we can state the following:

Lemma 1: System (8) with its input defined by the transformation (13), satisfies the velocity constraints (12) for any input vector field v

Proof: See Appendix E. ∎

C. Navigation Function Based Controller

Now that we have a way of translating the velocity of point z of the robot to appropriate linear and angular velocities, we want to design an appropriate vector field to drive the robot. This vector field will be created based on the navigation functions framework. Assume that we have a workspace with radius r_w and a set obstacles in the workspace with radii r_{O_i} with $i \in \{1, \ldots, N_O\}$ where N_O the number of obstacles. However the navigation functions framework deals with point robot. This constraint can be overcome in our case, by modifying the workspace and obstacle radii so as they can accommodate the volume of the robot and the robot will appear as a point in this modified workspace. This is achieved by setting $r'_w = r_w - r$ for the workspace radius and $r'_{O_i} = r_{O_i} + r$ for the i'th obstacle's radius. The center of the robot is considered to be the z point (see figure 4) we used in the feedback linearization and its radius r the maximum distance of z from the robot's boundary.

With this in mind we construct a navigation function φ on the modified workspace $\mathcal{F}$. Then by Proposition 4 the vector field

$$v = -K\nabla\varphi(z) + r(z)u(t) \tag{14}$$

provides an aISS controller for the system $\dot{z} = v$ for an appropriate choice of $r(z)$. The significance of $r(z)$ is that is has to be zero on the workspace boundary and close to the unit away from it in order to allow for meaningful inputs from the user. In order to increase the robustness of our system to noise, we have selected the following smooth function:

$$r(z) = \frac{\upsilon\left(\beta(z) - r_s\right)}{\upsilon\left(\beta(z) - r_s\right) + \upsilon\left(\varepsilon + r_s - \beta(z)\right)}$$

where $\varepsilon > 0$, $r_s \geq 0$, $\beta(x)$ is the obstacle function for the navigation function φ (see [3] for the construction), that denotes the distance of the robot from the workspace boundary and $\upsilon(t) \triangleq \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases}$. Figure 5 shows a plot of the function. r_s is the safety margin of $b(z)$ below which $r(z)$ is identically zero and $r(z)$ is identically 1 for $\beta(z) > r_s + \varepsilon$.
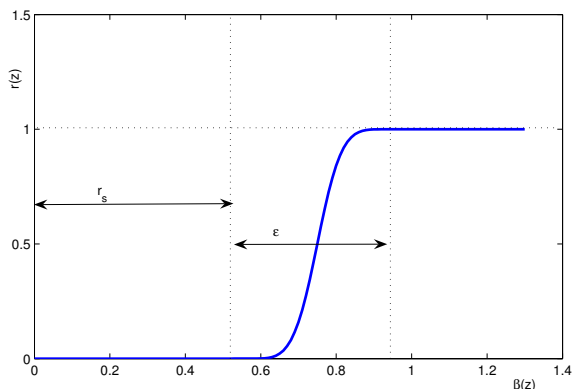
We can immediately observe that $0 \leq r(z) \leq 1$ and its Lipschitz continuity is implied from it's smoothness property.

However the vector field provided by $\dot{z} = v$ does not satisfy the constraints (12). This can be fixed by setting

$$\dot{z} = v\mu(v,\theta)^{-1} \tag{15}$$

with v as defined in (14).

In view of Proposition 2 we can state the following:

Proposition 5: System (15) is aISS

Proof: See Appendix F ∎

Corollary 1: System 15 is safe, i.e. its trajectories for any measurable u satisfy: $z(t, z_0, u(t)) \in \mathcal{F}$ for all $t \geq 0$, $z_0 \in \mathcal{F}$

Proof: See Appendix G ∎

V. EXPERIMENT

A. Setup

The experimental setup consists of a custom made autonomous differential drive mobile robot (see figure 6) developed in the GRASP lab. The software architecture is based on the Player framework [2]. The localization system is vision based and uses 4 overhead cameras to extract the posture of the robot and the obstacles in the workspace using fiducial marks. These measurements are then fused with odometry measurements from the robot wheel encoders using an extended Kalman filter with outlier rejection. The client code for the controller was developed in C++. For the user interface we used a commercially available joystick. Due to the relatively low speeds achievable by the mobile robot and due to the nature of the motors and the motor control system, the kinematic description of the robot proved to be adequate for our purposes without the need to consider the dynamics of the robot.

B. Results

To verify the feasibility of proposed scheme, we have set up an experiment with the Merl robotic platform in a workspace with three obstacles as shown in figure 6.
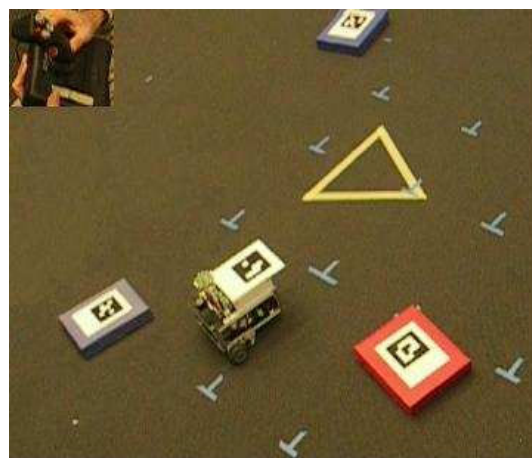

Fig. 6 : *Snapshot: Human interaction in a navigation scenario in an obstacle cluttered environment*

Fig. 5 : *Plot of the chosen r(x) function*

The robot was initially placed at it's destination configuration, when the user started applying an input to the system. Under the influence of the user input and the navigation vector field, the robot initially headed towards the obstacles, avoiding collisions. The user continued providing input to the system until the robot was placed behind an obstacle, where the user stopped providing any input to the system. The system, under the influence of the navigation field was guided to it's destination configuration and was stabilized there. Figure 7 depicts the trajectory of the robot in the obstacle cluttered environment. Notice that human interaction with the planner introduced paths that are not homotopically equivalent to the paths that are normally produced by the flows of the navigation function.
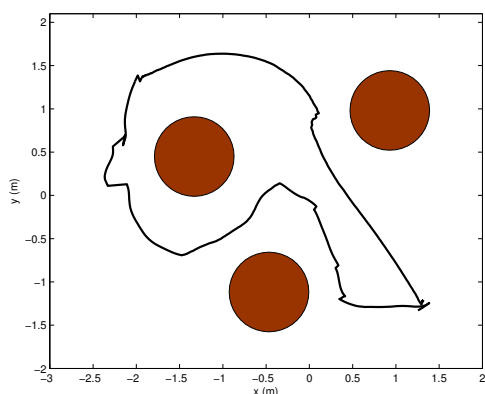

Fig. 7 : *Trajectory of the robot*

In figure 8 we can see the evolution of the $z - z_{goal}$ error variables and the evolution of the robot's angle. As we can see the robot after the user input vanishes, converges to its destination configuration.
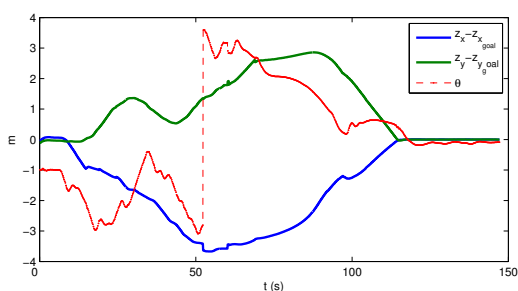

Fig. 8 : *X-Y error and robot orientation*

Figure 9 depicts the levels of input in each direction provided by the used during the evolution of the system.

Figure 10 depicts the linear and angular velocity commands applied to the robot. As we can see the controls are
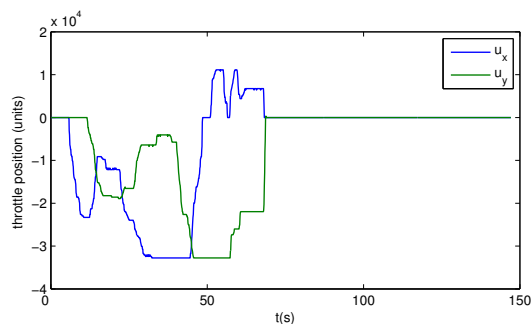

Fig. 9 : *User input*
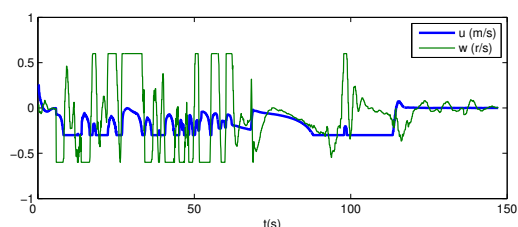
bounded and they saturate at their maximum values.

Fig. 10 : *Control signal*

VI. CONCLUSIONS

We have proposed a novel technique of composing human inputs with navigation function based controllers. The technique is based on the new concept of almost Input to State Stability property of a class of navigation function based controllers. The proposed methodology provides theoretically guaranteed properties of safety and convergence, while allowing the human operator to deviate from the planned path. Experimental results from the implementation of our algorithms confirmed the feasibility of the proposed methodology.

Our future research will include extending the aISS NF framework to several novel classes of navigation functions including decentralized navigation systems for robotic swarms.

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APPENDIX

A. Proof of Proposition 1

Proof: Let $y(y_0, t)$ be a solution to (5) for the initial condition y_0. Then at any time instant the solution $y(y_0, t)$ satisfies: $\dot{y}(y_0, t) = k(y(y_0, t), t)f(y(y_0, t)) + k(y(y_0, t), t)r(y(y_0, t))u(t)$. Setting $k_m(t) = k(y(y_0, t), t)$ we can write the equation as

$$\dot{y}(y_0, t) = k_m(t)f(y(y_0, t)) + k_m(t)r(y(y_0, t))u(t) \quad \text{(A-1)}$$

Let us now choose a different time parametrization to traverse the solution: $y(y_0, \tau(t))$. Taking the time derivative we get

$$\frac{d\left(y(y_0, \tau(t))\right)}{dt} = \frac{dy(y_0, \tau(t))}{d\tau}\dot{\tau}(t) \quad \text{(A-2)}$$

Combining equations (A-2) and (A-1) we get:

$$\dot{y}(y_0, \tau)\dot{\tau}(t) = k_m(\tau)f(y(y_0, \tau)) + k_m(\tau)r(y(y_0, \tau))u(\tau) \quad \text{(A-3)}$$

Setting $\tau(t) = \int_0^t k(\sigma)d\sigma$ and since $k > 0$, equation (A-3) becomes:

$$\dot{y}(y_0, \tau) = f(y(y_0, \tau)) + r(y(y_0, \tau))u(\tau) \quad \text{(A-4)}$$

Note that since $k(\sigma) \geq \varepsilon > 0$ it implies that $\int_0^\infty k(\sigma)d\sigma = \infty$ hence we do not have pathological conditions where system (5) stops along some non-stationary point of the trajectory of (4). Now since this holds for any feasible initial conditions, we can omit the initial condition from (A-4) and get:

$$\dot{y}(\tau) = f(y(\tau)) + r(y(\tau))u_2(\tau) \quad \text{(A-5)}$$

which is equivalent to equation (4). ∎

B. Proof of Proposition 2

Proof: Since the solutions of the two systems are equivalent as shown in Proposition 1 and since $k(y, t) > 0$ we will only have forward time progression for both systems across their solution. Since asymptotic stability describes the limit behavior as $t \to \infty$ the asymptotic stability of the one system implies the asymptotic stability of the other system. Now assume that system (4) is aISS, i.e. in addition to the asymptotic stability the following property of Definition 2 holds:

$$\forall u, \forall \, a.\, a. \, \xi \in M, \, \limsup_{t \to +\infty} |x(t, \xi, u)|_{\mathcal{A}} \leq \gamma(\|u\|_\infty)$$

since we are examining the limit behavior of the two systems and their trajectories are equivalent we can write for system (5) that:

$$\forall u, \forall \, a.\, a. \, \xi \in M, \, \limsup_{t \to +\infty} |y(t, \xi, u)|_{\mathcal{A}} \leq \gamma(a\|u\|_\infty)$$

By setting $\gamma_a(\|u\|_\infty) = \gamma(a\|u\|_\infty))$ we note that $\gamma_a(\cdot) \in \mathcal{K}$ since $\gamma(\cdot) \in \mathcal{K}$. Hence all the requirements of definition 2 are satisfied and system (5) is aISS. The opposite direction is proved in a similar way by selecting $\gamma_\varepsilon(\|u\|_\infty) = \gamma\left(\frac{1}{\varepsilon}\|u\|_\infty\right))$ and noting that $\gamma_\varepsilon(\cdot) \in \mathcal{K}$. ∎

C. Proof of Proposition 3

Proof: The navigation function φ is by definition [3] uniformly maximal over the workspace boundary, i.e. $\varphi(q) = 1$ for $q \in \partial\mathcal{F}$. This implies that $-\nabla\varphi$ is transverse across the workspace boundary and points inwards. Taking the inner product of $-\nabla\varphi$ with $\dot{x}$ we have that $\dot{x}^T(-\nabla\varphi) = k(x, t)K\|\nabla\varphi\|^2 - k(x, t)r(x)u_1(t)^T\nabla\varphi$ But for $x \in \partial\mathcal{F}$ we have that $r(x) = 0$ hence we get $\dot{x}^T(-\nabla\varphi) = k(x, t)K\|\nabla\varphi\|^2 > 0$ since $\nabla\varphi$ and $k(x, t)$ are non-zero across the workspace boundary and the workspace is positive invariant. ∎

D. Proof of Proposition 4

Proof: A similar proof can be found in [5] and is based on the concept of combining primal and dual Lyapunov techniques [7]. Please refer to [12] for an extended version of this paper with a detailed proof. ∎

E. Proof of Proposition 1

Proof: Assume that $\begin{bmatrix} u_0 \\ w_0 \end{bmatrix} = A(\theta)^{-1}v$. If $|u_0\| \leq u_{\max}$ and $|w_0| \leq w_{\max}$ then $\mu(v, \theta) = 1$ and $u = u_0$ and $w = w_0$ and the constraints are satisfied. If $|u_0| > u_{\max}$ or $|w_0| > w_{\max}$, let $m = \max(|u_0|/u_{\max}, |w_0|/w_{\max})$. Then $\mu(v, \theta) = m$ and $u = u_0/m < u_{\max}$ and $w = w_0/m < w_{\max}$ and the constraints are satisfied. ∎

F. Proof of Proposition 5

Proof: Our selection of the $k(z, t) = \mu(z, t)^{-1}$ (we write $\mu = \mu(z, t)$ with slight abuse of notation since $v = v(z, t)$) function satisfies the Lipschitz continuity requirements in z due to the smoothness of $\varphi(z)$ and $r(z)$. It is by construction bounded above by 1 and due to smoothness and compactness arguments for the considered workspace and due to the assumption that u is measurable, $\mu(z, t)^{-1}$ is also below bounded by $a = \frac{\min(u_{\max}, w_{\max})}{K\max\limits_{z \in \mathcal{F}}\|\nabla\varphi\| + \max\limits_{t}u(t)} > 0$. Since by Proposition 4 system $\dot{z} = v$ is aISS, invoking Proposition 2 and noting that all its requirements are satisfied, we have the result. ∎

G. Proof of Proposition 1

Proof: System $\dot{z} = v$ is safe since it satisfies all the requirements of Proposition 3. By Proposition 1 system (15) has the same trajectories up to a time parametrization, hence it also satisfies $z(t, z_0, u(t)) \in \mathcal{F}$ for all $t \geq 0$, $z_0 \in \mathcal{F}$ ∎