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A Control Lyapunov Approach for Feedback Control of Cable-Suspended Robots

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***Abstract*— This paper considers a feedback control technique for cable suspended robots under input constraints, using Control Lyapunov Functions (CLF). The motivation for this work is to develop an explicit feedback control law for cable robots to asymptotically stabilize it to a goal point with positive input constraints. The main contributions of this paper are as follows: (i) proposal for a CLF candidate for a cable robot, (ii) a CLF based positive controllers for multiple inputs. An example of a three degrees-of-freedom cable suspended robot is presented to illustrate the proposed methods.**

***Index Terms*— Cable Suspended Robot, Control Lyapunov Functions, Input Constraints.**

I. INTRODUCTION

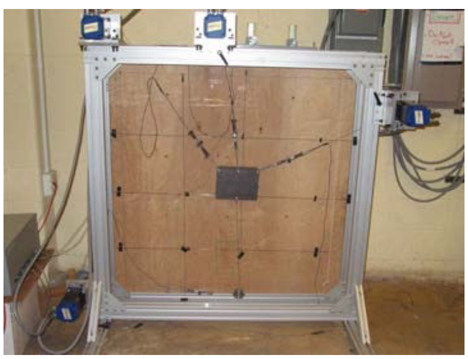

Fig. 1. A camera image of a cable-suspended robot designed and built in our laboratory.

Some research has been previously conducted to guarantee positive tension in the cables while the end-effector is moving. The idea of redundancy was utilized in cable system control ([1],[2]). A force distribution method was proposed to avoid slackness and excessive tension in cables [3]. Furthermore, the dynamic workspace for specific directions of motion and accelerations was studied ([4],[5]). Another proposed approach was to design a reference governor that restricts the reference signal to avoid cable slackness ([6]-[8]).

In this work, we present a *constructive nonlinear control strategy for a broad class of cable robots with input constraints*. The motivation for studying this type of controller stems from the observation that control laws in previous works were not explicit but were determined computationally by solving a minimization problem at each instant of time. This requires a high degree of computational load.

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In previous works, it was shown that if a control Lyapunov function (CLF) can be determined for a nonlinear system, the CLF and the system equations can be used to find explicit control laws that can render the system asymptotically stable ([9], [10]). These were called universal formula because they depend only on the CLF and the system equations and not on the particular structure of these equations. The reference deals with SISO systems and is unable to guarantee performance in certain regions of the state space [11].

Motivated by these considerations, in this paper, two control designs are proposed using CLF, such that asymptotic stability and positivity of multiple inputs are guaranteed for cable robots. *In this paper, first, Sontag's work regarding positive control is extended to a general class of nonlinear system with multiple inputs. To cope with the shortcoming of the CLF control law in regions of the state space, an assistive controller is implemented. The transition rules between these control laws are discussed to achieve asymptotical stability.*

The salient feature of the nonlinear CLF control design is its capability to systematically construct both the positive control and the stability. Hence, this study provides new insights into CLF-based nonlinear control of systems with multiple inputs and has the potential to be a useful tool in the design and analysis of constrained nonlinear system.

The rest of this paper is organized as follows: In Section II, the dynamic model of the cable system is described. A promising CLF for cable robots with multiple input is presented in Section III. Section IV show a method to design the CLF based controllers for MIMO systems. We provide an example of a cable-suspended system to demonstrate the proposed control technique in Section IV.

II. SYSTEM DYNAMIC MODEL

Our model of a planar cable robot consists of a moving platform (MP) that is connected by n cables to an inertially fixed platform shown in Fig. 2. A cable i is connected to MP shown in Fig. 2. An inertial reference frame $F_0(\hat{X}\hat{Y})$ is located at 0 and a moving reference frame $F_M(\hat{x}\hat{y})$ is located on MP at its center of mass M. The orientation of MP is specified by θ_e. The origin of F_M is given by a vector from 0 to M with x_e and y_e as its components. The ith cable orientation in the frame F_M is denoted by α_i.

A. Cable Kinematics and Statics

The position vector of point a_i in the frame F_M is written as

$$\left[\begin{array}{cc} b_i\mathrm{c}\alpha_i & b_i\mathrm{s}\alpha_i \end{array}\right]^T, \tag{1}$$

where c and s stand for cos and sin, respectively and b_i is the distance between points M and a_i. The transformation matrix of frame F_M with respect to frame F_0 can be written as

$$^0T_M = \begin{bmatrix} \mathtt{c}\theta_e & -\mathtt{s}\theta_e & x_e \\ \mathtt{s}\theta_e & \mathtt{c}\theta_e & y_e \\ 0 & 0 & 1 \end{bmatrix}. \quad (2)$$

Therefore, the position vector of points a_i with respect to F_0 is

$$\begin{bmatrix} ^0r_i \\ 1 \end{bmatrix} = ^0T_M \begin{bmatrix} ^Mr_i \\ 1 \end{bmatrix}, i = 1 \cdots n. \quad (3)$$

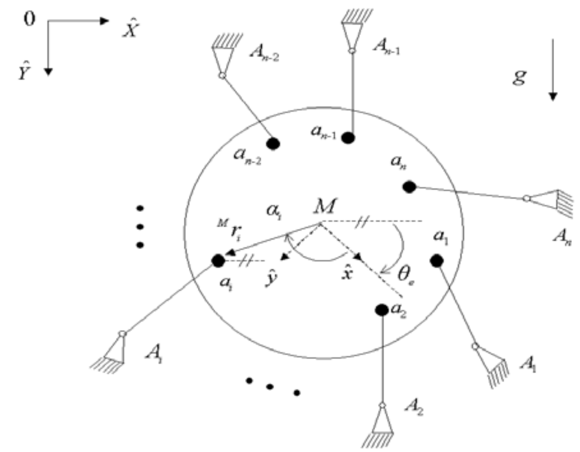

Fig. 2. A sketch of the cable system along with geometric parameters for the robot with n cables.

Upon substitution of ^0T_M from Eq. (2) into Eq. (3), one obtains

$$^0r_i = \begin{bmatrix} x_e + b_i\mathtt{c}\theta_e\mathtt{c}\alpha_i - b_i\mathtt{s}\theta_e\mathtt{s}\alpha_i \\ y_e + b_i\mathtt{s}\theta_e\mathtt{c}\alpha_i + b_i\mathtt{c}\theta_e\mathtt{s}\alpha_i \end{bmatrix}, i = 1 \cdots n. \quad (4)$$

Moreover, the position vector of suspension point A_i of cable i with respect to reference point 0 is written as

$$^0p_i = \begin{bmatrix} d_i \\ h_i \end{bmatrix}, i = 1 \cdots n. \quad (5)$$

Hence, the vector $\overrightarrow{a_iA_i}$ for cable i

$$l_i = {}^0p_i - {}^0r_i = \begin{bmatrix} l_{ix} \\ l_{iy} \end{bmatrix} = \begin{bmatrix} d_i - x_e - b_i\mathtt{c}\theta_e\mathtt{c}\alpha_i + b_i\mathtt{s}\theta_e\mathtt{s}\alpha_i \\ h_i - y_e - b_i\mathtt{s}\theta_e\mathtt{c}\alpha_i - b_i\mathtt{c}\theta_e\mathtt{s}\alpha_i \end{bmatrix} \quad i = 1 \cdots n. \quad (6)$$

The static equilibrium equation of MP can be used to obtain the forces in the cables.

$$\begin{aligned} \sum F_x = 0 & \qquad \sum_1^n T_i\mathtt{c}\theta_i = 0 \\ \sum F_y = 0 & \quad \Rightarrow \quad \sum_1^n T_i\mathtt{s}\theta_i + mg = 0 \\ \sum M_z = 0 & \qquad \sum_1^n T_i s_i = 0 \end{aligned} \quad (7)$$

where

$$\mathtt{c}\theta_i = \frac{l_{ix}}{\| l_i \|}, \mathtt{s}\theta_i = \frac{l_{iy}}{\| l_i \|}, i = 1 \cdots n$$

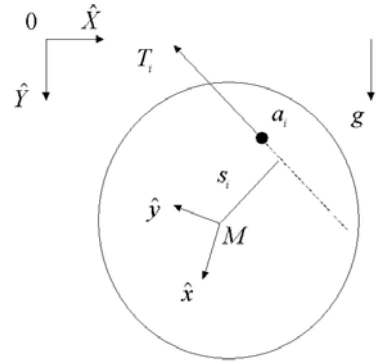

Fig. 3. A sketch of parameter s_i, which is the normal distance between M and the i-th cable.

and s_i is the normal distance between M and the cable axis i and can be expressed using Fig. 3 as $s_i = b_i \cdot \mathtt{s}(\theta_e + \alpha_i - \theta_i)$. Eqs. (7) can be written in matrix form as

$$A(\mathbf{x})\mathbf{u} = F \quad (8)$$

where

$$A(\mathbf{x}) = \begin{bmatrix} \frac{l_{1x}}{\|l_1\|} & \cdots & \frac{l_{nx}}{\|l_n\|} \\ \frac{l_{1y}}{\|l_1\|} & \cdots & \frac{l_{ny}}{\|l_n\|} \\ s_1 & \cdots & s_n \end{bmatrix} \quad (9)$$

$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$$
$$F = \begin{bmatrix} F_x & F_y & M_z \end{bmatrix}. \quad (10)$$

B. System Dynamics

During motion,

$$F = \begin{bmatrix} m\ddot{x}_e \\ m(\ddot{y}_e - g) \\ I_z\ddot{\theta}_e \end{bmatrix}, \quad (11)$$

where m is the mass and I_z is the moment of inertia of the end-effector about its center of mass along $\hat{Z}$. The equations of motion can be written alternatively in the following general form

$$D\ddot{\mathbf{x}} + \mathcal{G} = A(\mathbf{x})\mathbf{u} \quad (12)$$

where D is the inertia matrix for the system and $\mathcal{G}$ is the vector of gravity terms. Their expressions are

$$D = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_Z \end{bmatrix}, \mathcal{G} = \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix}$$

and $\mathbf{x} = [x_e, y_e, \theta_e]^T$. The above dynamic model is valid only for $u_i \geq 0$, i.e., the cables are in tension. A positive tension implies that the cable is pulling the attachment point of the end-effector.

III. CLF APPROACH FOR A MULTI-CABLE ROBOT

The state-space form of Eq. (12) is represented as follows

$$\frac{d}{dt}\underbrace{\begin{Bmatrix}\mathbf{x}\\ \dot{\mathbf{x}}\end{Bmatrix}}_{\underline{\mathbf{x}}} = \underbrace{\begin{Bmatrix}\dot{\mathbf{x}}\\ -D^{-1}\mathcal{G}\end{Bmatrix}}_{\mathbf{f}(\underline{\mathbf{x}})} + \underbrace{\begin{Bmatrix}O_{3\times 3}\\ D^{-1}A(\mathbf{x})\end{Bmatrix}}_{\mathbf{G}(\underline{\mathbf{x}})}\mathbf{u}, \quad (13)$$

where $\underline{\mathbf{x}}(t) \in R^6$ denotes the vector of state variables, $\mathbf{u}(t) = [u_1(t)\cdots u_n(t)]^T$ denotes the vector of manipulated inputs taking values in the nonempty compact subset $\mathcal{U} := \{\mathbf{u} \in R^n : \mathbf{u} \geq 0\}$. All entries of the vector $\mathbf{f}$ and the matrix $\mathbf{G}(\mathbf{x}) = [\mathbf{g}_1\cdots\mathbf{g}_n]^T$ are smooth functions. Note that $\mathbf{g}_i$ is the i-th column of the matrix $G(\mathbf{x})$.

Definition 1: A positive definite radially unbounded function V is called a CLF for the system $\dot{\underline{\mathbf{x}}} = \overline{\mathbf{f}}(\underline{\mathbf{x}}, \mathbf{u})$ if for each $\underline{\mathbf{x}} \neq 0$, there exists $\mathbf{u}$ such that

$$\dot{V} = V_{\underline{\mathbf{x}}}\overline{\mathbf{f}}(\underline{\mathbf{x}}, \mathbf{u}) < 0. \quad (14)$$

If $\overline{\mathbf{f}}(\underline{\mathbf{x}}, \mathbf{u}) = \mathbf{f}(\underline{\mathbf{x}}) + \mathbf{G}(\underline{\mathbf{x}})\mathbf{u}$, V is a CLF if and only if $V_{\underline{\mathbf{x}}}\mathbf{f}(\underline{\mathbf{x}}) < 0$ for all $\underline{\mathbf{x}} \neq 0$ such that $||V_{\underline{\mathbf{x}}}\mathbf{G}(\underline{\mathbf{x}})|| = 0$.

The importance of this concept is that, once a CLF is chosen, an explicit stabilizing control law can be selected [14]. The existence of a CLF implies that there exists a control law such that the CLF is a Lyapunov function for the closed-loop system. Hence, the CLF can be viewed as a candidate Lyapunov function, where a stabilizing control law has not yet been specified.

A parameterized CLF candidate can be selected as

$$\begin{aligned} V = {} & \tfrac{1}{2}\Big[\big(\dot{x}_e + \lambda_x(x_e - x_d)\big)^2 + \eta_x(x_e - x_d)^2\Big] \\ & + \tfrac{1}{2}\Big[\big(\dot{y}_e + \lambda_y(y_e - y_d)\big)^2 + \eta_y(y_e - y_d)^2\Big] \\ & + \tfrac{1}{2}\Big[\big(\dot{\theta}_e + \lambda_\theta(\theta_e - \theta_d)\big)^2 + \eta_\theta(\theta_e - \theta_d)^2\Big], \end{aligned} \quad (15)$$

with

$$V_{\underline{\mathbf{x}}} = \begin{bmatrix} \frac{\partial V}{\partial x_e} & \frac{\partial V}{\partial y_e} & \frac{\partial V}{\partial \theta_e} & \frac{\partial V}{\partial \dot{x}_e} & \frac{\partial V}{\partial \dot{y}_e} & \frac{\partial V}{\partial \dot{\theta}_e} \end{bmatrix}. \quad (16)$$

$$\begin{aligned} V_{\underline{\mathbf{x}}}\mathbf{f}(\underline{\mathbf{x}}) = {} & \big(\lambda_x\dot{x}_e + \lambda_x^2(x_e - x_d) + \eta_x(x_e - x_d)\big)\dot{x}_e \\ & + \big(\lambda_y\dot{y}_e + \lambda_y^2(y_e - y_d) + \eta_y(y_e - y_d)\big)\dot{y}_e \\ & + \big(\lambda_z\dot{\theta}_e + \lambda_\theta^2(\theta_e - \theta_d) + \eta_\theta(\theta_e - \theta_d)\big)\dot{z}_e \\ & + \big(\dot{x}_e + \lambda_x(x_e - x_d)\big)\mathbf{f}_{L1}(\underline{\mathbf{x}}) \\ & + \big(\dot{y}_e + \lambda_y(y_e - y_d)\big)\mathbf{f}_{L2}(\underline{\mathbf{x}}) \\ & + \big(\dot{\theta}_e + \lambda_\theta(\theta_e - z_d)\big)\mathbf{f}_{L3}(\underline{\mathbf{x}}) \end{aligned} \quad (17)$$

$$V_{\underline{\mathbf{x}}}\mathbf{g}(\underline{\mathbf{x}}) = \begin{pmatrix} \dot{x}_e + \lambda_x(x_e - x_d) \\ \dot{y}_e + \lambda_y(y_e - y_d) \\ \dot{\theta}_e + \lambda_\theta(\theta_e - \theta_d) \end{pmatrix}^T \begin{pmatrix} | & & | \\ \mathbf{g}_{L1}(\mathbf{x}) & \cdots & \mathbf{g}_{Ln}(\mathbf{x}) \\ | & & | \end{pmatrix} \quad (18)$$

where $\mathbf{f}_{Li}(\underline{\mathbf{x}})$ is the i^{th} component of $\mathbf{f}_L(\underline{\mathbf{x}}) = -D^{-1}\mathcal{G}$ and $\mathbf{g}_{Lj}(\underline{\mathbf{x}})$ is the j^{th} column vector of $\mathbf{G}_L(\underline{\mathbf{x}}) = D^{-1}A(\mathbf{x})$. $[\mathbf{g}_{L1}(\mathbf{x}), \cdots, \mathbf{g}_{Ln}(\mathbf{x})] = -D^{-1}A(\mathbf{x})$, where the columns of $A(\mathbf{x})$ represent the different direction of the cable. Hence, $\| V_{\underline{\mathbf{x}}}\mathbf{g}(\underline{\mathbf{x}}) \| = 0$ if

$$\begin{pmatrix} \dot{x}_e + \lambda_x(x_e - x_d) \\ \dot{y}_e + \lambda_y(y_e - y_d) \\ \dot{\theta}_e + \lambda_\theta(\theta_e - \theta_d) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (19)$$

On substitution of Eq. (19) into Eq. (17),

$$V_{\underline{\mathbf{x}}}\mathbf{f}(\underline{\mathbf{x}}) = -\lambda_x\eta_x(x_e - x_d)^2 - \lambda_y\eta_y(y_e - y_d)^2 - \lambda_\theta\eta_\theta(\theta_e - \theta_d)^2. \quad (20)$$

$\dot{V} < 0$ when $\| V_{\underline{\mathbf{x}}}\mathbf{g}(\underline{\mathbf{x}}) \| = 0$ for $\forall \mathbf{x} \neq \mathbf{x}_d$. Hence, the proposed CLF is well defined.

IV. CLF POSITIVE CONTROLLER

For a 3 DOF system with n-cable inputs given by Eq. (13), the time derivative of V is given by

$$\dot{V} = V_{\underline{\mathbf{x}}}\mathbf{f} + \sum_{i=1}^{n} V_{\underline{\mathbf{x}}}\mathbf{g}_i u_i. \quad (21)$$

In order to design the n controls u_i, we introduce n weighting parameters w_i as follows:

$$\dot{V} = \sum_{i=1}^{n}\big(V_{\underline{\mathbf{x}}}\mathbf{f}w_i + V_{\underline{\mathbf{x}}}\mathbf{g}_i u_i\big), \quad (22)$$

where $\sum_{i=1}^{n} w_i = 1$.

We can rewrite Eq. (22) in polar coordinates (r, ξ) as follows:

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{n}\big(V_{\underline{\mathbf{x}}}\mathbf{f}w_i + V_{\underline{\mathbf{x}}}\mathbf{g}_i u_i\big) \\ &= \sum_{i=1}^{n} r_i(sin\xi_i + cos\xi_i u_i) \end{aligned} \quad (23)$$

where $r_i = \sqrt{(V_{\underline{\mathbf{x}}}\mathbf{f}w_i)^2 + (V_{\underline{\mathbf{x}}}\mathbf{g}_i)^2}$, $sin\xi_i = \frac{V_{\underline{\mathbf{x}}}\mathbf{f}w_i}{\sqrt{(V_{\underline{\mathbf{x}}}\mathbf{f}w_i)^2 + (V_{\underline{\mathbf{x}}}\mathbf{g}_i)^2}}$, $cos\xi_i = \frac{V_{\underline{\mathbf{x}}}\mathbf{g}_i}{\sqrt{(V_{\underline{\mathbf{x}}}\mathbf{f}w_i)^2 + (V_{\underline{\mathbf{x}}}\mathbf{g}_i)^2}}$.

Remark 1: To obtain $\dot{V} < 0$, we choose the following for each u_i,

$$\begin{aligned} u_i &> -tan\xi_i, \quad \tfrac{\pi}{2} < \xi_i \leq \tfrac{3\pi}{2} \\ u_i &< -tan\xi_i, \quad \tfrac{3\pi}{2} < \xi_i \leq 2\pi \end{aligned} \quad (24)$$

Note that for $0 < \xi_i \leq \frac{\pi}{2}$, no choice of u_i ensures $\dot{V} < 0$. For the multi-cable system of Eq. (13), the goal is to derive control inputs that respect the positive constraints and guarantee asymptotic closed-loop stability.

Theorem 1: Consider the nonlinear system of Eq. (13), for which a CLF V exists. Then, a family of *m* nonlinear state feedback controllers of the form,

$$u_i = \begin{cases} -tan(\xi_i) + \epsilon_i, & \frac{\pi}{2} \leq \xi_i \leq \pi \\ -\xi_i + 2\pi, & \pi \leq \xi_i \leq 2\pi \end{cases} \quad (25)$$

ensure the following: (1) satisfy positivity of the inputs, (2) enforce asymptotic stability of the closed loop system.

Proof:

In order to prove the stability of the closed loop system and the positivity of the input, it is enough to show that $\dot{V} < 0$ and $u_i > 0$.

- For $\xi_i \in (\frac{\pi}{2}, \pi)$, since $u_i = -tan\xi_i + \epsilon_i > -tan\xi_i > 0$, we have $\dot{V} < 0$ and $u_i > 0$. Here, ϵ_i is a positive value.
- For $\forall\xi_i \in (\pi, \frac{3\pi}{2})$, any positive u_i is feasible. Hence, $u_i = -\xi_i + 2\pi > 0$ is a suitable choice.
- Since $\xi_i < tan\xi_i$ for $\forall\xi_i \in (0, \frac{\pi}{2})$, it implies that $0 < u_i = -\xi_i + 2\pi < tan(-\xi_i + 2\pi) = -tan\xi_i$ for $\forall\xi_i \in (\frac{3\pi}{2}, 2\pi)$.

This complete the proof of Theorem 1.

A. Determination of Parameters w_i

w_i in Eq. (23) are free parameters. We know that when the states reach the goal, $(V_{\underline{\mathbf{x}}}\mathbf{g}_i, V_{\underline{\mathbf{x}}}\mathbf{f}w_i)$ is located at the origin. Hence, it is desirable that using the free parameters w_i, the following cost is minimized.

$$\begin{aligned} J &= \sum_{i=1}^{n}(V_{\underline{\mathbf{x}}}\mathbf{f}w_i)^2 + (V_{\underline{\mathbf{x}}}\mathbf{g}_i)^2 \\ &= (V_{\underline{\mathbf{x}}}\mathbf{f})^2(w_1^2 + \cdots + w_n^2) + (V_{\underline{\mathbf{x}}}\mathbf{g}_i)^2 \\ s.t. & \quad w_1 + w_2 + \cdots + w_n = 1 \end{aligned} \tag{26}$$

The minimizing solution of J is obtained by growing the radius of a hypersphere $w_1^2 + \cdots + w_n^2 = k$ until it touches the hyperplane $w_1 + w_2 + \cdots + w_n = 1$. Geometrically, it can be shown that the minimal solution is attained when $w_1 = w_2 = \cdots = w_n = \frac{1}{n}$.

B. Secondary Controller Over Infeasible region

If at all times, $(V_{\underline{\mathbf{x}}}\mathbf{g}_i, V_{\underline{\mathbf{x}}}\mathbf{f}w_i)$ lies in the $2nd$-$4th$ quadrants (see Path A in Fig. 4), it suffices to implement only the CLF control law of Eq. (25). However, from Eq. (23), we observe a feasible solution $u_i > 0$ is not possible if $\xi_i \in (0, \frac{\pi}{2})$. Hence, the infeasible region Φ_i^c, where $V_{\mathbf{x}}\mathbf{f}w_i > 0$ and $V_{\mathbf{x}}\mathbf{g}_i > 0$ needs to be characterized. When $(V_{\mathbf{x}}\mathbf{g}_i, V_{\mathbf{x}}\mathbf{f}w_i)$ traverses to an infeasible region, see Path B in Fig. 4, a switching controller is applied to ensure closed loop stability and positivity of the input. In this phase, the control law is chosen to make the system's behavior as follows:

$$\ddot{\mathbf{x}} + \alpha(\mathbf{s}) = 0 \quad for\ \forall\xi_i \in (0, \frac{\pi}{2}). \tag{27}$$

This behavior is achieved by a secondary control law given by

$$\mathbf{u} = \underbrace{A(\mathbf{x})^{\dagger}\mathcal{G}}_{\mathbf{u}_s} - A(\mathbf{x})^{\dagger}D\underline{\alpha}(\mathbf{s}), \tag{28}$$

where $A(\mathbf{x})^{\dagger} = A^T(AA^T)^{-1}$ is the pseudo inverse of matrix A and $\mathbf{s}$ is selected as follows:.

$$\begin{aligned} \mathbf{s} &= [\, s_1 \quad s_2 \quad s_3 \,]^T \\ &= \begin{pmatrix} \dot{x}_e + \lambda_x(x_e - x_{ed}) \\ \dot{y}_e + \lambda_y(y_e - y_{ed}) \\ \dot{\theta}_e + \lambda_\theta(\theta_e - \theta_{ed}) \end{pmatrix} \\ \underline{\alpha}(\mathbf{s}) &= [\, \alpha(s_1) \quad \alpha(s_2) \quad \alpha(s_3) \,]^T \end{aligned} \tag{29}$$

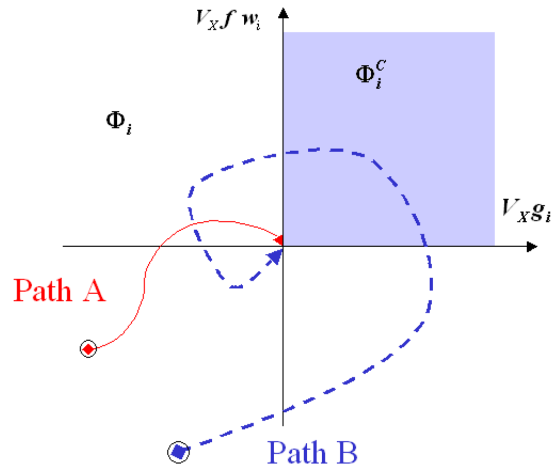

Fig. 4. A sketch of the infeasible region Φ_i^c and a feasible region Φ_i. Path A travels over the feasible region, wheras, Path B enters the infeasible region. For Path B, a secondary controller is required to obtain the performance.

Note that

$$\alpha(s_i) = \begin{cases} c & s_i \geq 0 \\ -c & s_i < 0, \end{cases} \tag{30}$$

where c is a positive constant. The first term $\mathbf{u}_s$ in Eq. (28) is determined only by the geometry of the cable robot. Hence, under the assumption that the cable robot moves within a geometrically feasible workspace, there always exists c which is not zero and makes all components of $\mathbf{u}$ positive.

Furthermore, without loss of generality, the convergence of Eq. (27) can be verified by integrating the component equation. Since all components of $\mathbf{x}$ have the same behavior, we illustrate using the first component, namely, x. For $\ddot{x} = -\alpha(s)$, the solution of the equation is

$$\tfrac{1}{2}\dot{x}^2 = \mp c\ x + c_1. \tag{31}$$

This behavior of Eq. (27) is described pictorially in Fig. 5. From the initial point (A), the state travels along the curve 1 until it hits a control surface $s = 0$. When the state intersects B on the sliding surface, it switches to the curve 2. However, since the curve 2 starting from point B propagates within a region $s > 0$, the trajectory follows along the control surface.

Remark 2: Switching between CLF controller and secondary controller happens when the locus of $(V_{\mathbf{x}}\mathbf{f}w_i, V_{\mathbf{x}}\mathbf{g}_i)$ approaches the first quadrant.

Remark 3: Narrower the width of parabola determined by c in Fig. 5, it has a higher excursion from the goal. Hence, one would expect that once the state leaves the infeasible region, Lyapunov function may increase. To achieve a monotonic decrease of Lyapunov function, the following additional condition is required,

$$V(\mathbf{x}(t_k^{\dagger})) \leq V(\mathbf{x}(t_k)) \tag{32}$$

where t_k denotes the time when a state enters an infeasible region and $t_k^{\dagger}$ is the time when it switches back to CLF control mode. Eq. (32) is required for the system to be Lyapunov stable.

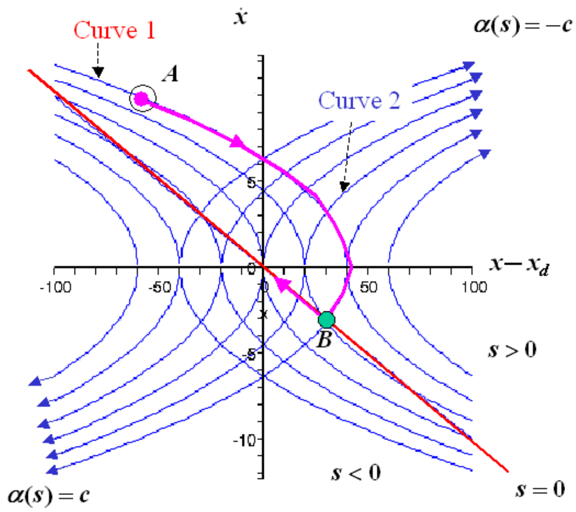

Fig. 5. A sketch of a typical trajectory for Eq. (31). Starting from A, the state propagates to an intersection point B, and then slides along a switching line $s = 0$ toward the goal.

C. Illustrative Example

In order to illustrate the results of this procedure, we present a example.

1) 3-Cable Case:

Control Lyapunov Function: A parameterized CLF candidate can be given by

$$\begin{aligned} V &= \tfrac{1}{2}\Big[\big(\dot{x}_e + \lambda_x(x_e - x_d)\big)^2 + \eta_x(x_e - x_d)^2\Big] \\ &+ \tfrac{1}{2}\Big[\big(\dot{y}_e + \lambda_y(y_e - y_d)\big)^2 + \eta_y(y_e - y_d)^2\Big] \\ &+ \tfrac{1}{2}\Big[\big(\dot{\theta}_e + \lambda_\theta(\theta_e - \theta_d)\big)^2 + \eta_\theta(\theta_e - \theta_d)^2\Big], \end{aligned} \quad (33)$$

with

$$V_{\underline{\mathbf{x}}} = \left[\tfrac{\partial V}{\partial x_e} \quad \tfrac{\partial V}{\partial y_e} \quad \tfrac{\partial V}{\partial \theta_e} \quad \tfrac{\partial V}{\partial \dot{x}_e} \quad \tfrac{\partial V}{\partial \dot{y}_e} \quad \tfrac{\partial V}{\partial \dot{\theta}_e} \right]. \quad (34)$$

$$\begin{aligned} V_{\underline{\mathbf{x}}}\mathbf{f}(\underline{\mathbf{x}}) = &\big(\lambda_x \dot{x}_e + \lambda_x^2(x_e - x_d) + \eta_x(x_e - x_d)\big)\dot{x}_e \\ &+ \big(\lambda_y \dot{y}_e + \lambda_y^2(y_e - y_d) + \eta_y(y_e - y_d)\big)\dot{y}_e \\ &+ \big(\lambda_z \dot{\theta}_e + \lambda_\theta^2(\theta_e - \theta_d) + \eta_\theta(\theta_e - \theta_d)\big)\dot{z}_e \\ &+ \big(\dot{x}_e + \lambda_x(x_e - x_d)\big)\mathbf{f}_{L1}(\underline{\mathbf{x}}) \\ &+ \big(\dot{y}_e + \lambda_y(y_e - y_d)\big)\mathbf{f}_{L2}(\underline{\mathbf{x}}) \\ &+ \big(\dot{\theta}_e + \lambda_\theta(\theta_e - z_d)\big)\mathbf{f}_{L3}(\underline{\mathbf{x}}) \end{aligned} \quad (35)$$

$$V_{\underline{\mathbf{x}}}\mathbf{g}(\underline{\mathbf{x}}) = \begin{pmatrix} \dot{x}_e + \lambda_x(x_e - x_d) \\ \dot{y}_e + \lambda_y(y_e - y_d) \\ \dot{\theta}_e + \lambda_\theta(\theta_e - \theta_d) \end{pmatrix}^T \begin{pmatrix} | & | & | \\ \mathbf{g}_{L1}(\underline{\mathbf{x}}) & \mathbf{g}_{L2}(\underline{\mathbf{x}}) & \mathbf{g}_{L3}(\underline{\mathbf{x}}) \\ | & | & | \end{pmatrix} \quad (36)$$

where $\mathbf{f}_{Li}(\underline{\mathbf{x}})$ is the i^{th} component of $\mathbf{f}_L(\underline{\mathbf{x}})$ and $\mathbf{g}_{Lj}(\underline{\mathbf{x}})$ is the j^{th} column vector of $\mathbf{g}_L(\underline{\mathbf{x}})$.

CLF Control Design: Based on $V_{\underline{\mathbf{x}}}\mathbf{f}(\underline{\mathbf{x}})$ and $V_{\underline{\mathbf{x}}}\mathbf{g}(\underline{\mathbf{x}})$, the parameters $w_1 = w_2 = w_3 = \frac{1}{3}$. The steps of CLF control design were described in detail in Section III.

TABLE I
SYSTEM PARAMETERS IN MKS UNIT

Sys. Parameter	Value	Sys. Parameter	Value
r_{OA_1}	(1.5,0)	r_{OA_2}	(0,0)
r_{OA_3}	(0,0.5)	ϵ	π
λ_i	1	T_s	1 msec
η_i	1	a	0.12
g	9.8	m	20
α_i	$\frac{2\pi}{3} * (i-1)$	b_i	a

Secondary Control Design: The proposed CLF controller is infeasible when $(V_x g, V_x f)_i$ comes in the first quadrant. During the time, the assistive controller is turned on until the states enter the feasible region. The desired closed loop dynamics of Eq. (27) requires $\mathbf{u}$ to be:

$$\mathbf{u} = \underbrace{A(\mathbf{x})^{-1}\mathcal{G}}_{\mathbf{u}_s} - A(\mathbf{x})^{-1}D\underline{\alpha}(\mathbf{s}), \quad (37)$$

where

$$\begin{aligned} \mathbf{s} &= \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix}^T \\ &= \begin{pmatrix} \dot{x}_e + \lambda_x(x_e - x_d) \\ \dot{y}_e + \lambda_y(y_e - y_d) \\ \dot{\theta}_e + \lambda_\theta(\theta_e - \theta_d) \end{pmatrix}, \\ \underline{\alpha}(\mathbf{s}) &= \begin{bmatrix} \alpha(s_1) & \alpha(s_2) & \alpha(s_3) \end{bmatrix}^T. \end{aligned} \quad (38)$$

Note that

$$\alpha(s_i) = \begin{cases} c & , s_i \geq 0 \\ -c & , s_i < 0 \end{cases} \quad (39)$$

where c is a positive constant. We know that as long as a cable robot moves over a geometrically feasible workspace, there always exists c which is not zero and makes all components of $\mathbf{u}$ positive. The condition to make $\mathbf{u} \geq \mathbf{0}$ is

$$\begin{aligned} min(\mathbf{u}) &\geq min(\mathbf{u}_s) - \sqrt{\lambda_{max}((AA^T)^{-1})}\lambda_{max}(D)c \\ &\geq min(\mathbf{u}_s) - \frac{max(m, Iz)}{\sqrt{\lambda_{min}(AA^T)}}c. \end{aligned} \quad (40)$$

In the worst case, the bound on c is quantitatively obtained as follows:

$$0 < c \leq \frac{\sqrt{\lambda_{min}(AA^T)}}{max(m, Iz)} min(\mathbf{u}_s), \quad (41)$$

where $min(\mathbf{u}) = min(u_1, u_2, u_3)$.

Simulation: The control objectives are to: (1) stabilize the end-effector of the cable robot at the set point and (2) maintain the cable tension to be positive. The system parameters are listed in Table 1. We consider a specific move of the end-effector from $\underline{\mathbf{x}}_0 = [0.2, 0.2, 0, 0, 0, 0]^T$ to $\underline{\mathbf{x}}_d = [0.6, 0.6, -10^o, 0, 0, 0]^T$. All signals are in MKS unit. Fig. 6 shows the trajectories of the state $\mathbf{x}$, the desired signal $\mathbf{x}_d$, the control commands u_i, $i = 1, \cdots, 3$, with $c = 1$. Note that the condition to select c is given by Eq. (40). We observe that the controller successfully stabilizes the state of the system initialized at $\mathbf{x}_0$ at the desired steady state $\mathbf{x}_d$ as shown in Fig. 6. In the plots of Fig. 6, the control inputs are not smooth. This is expected since during the transition between CLF controller and assistive controller, the continuity of inputs is not guaranteed. Apart from the transition, the secondary controller itself results in an input-chattering behavior.

Fig. 7 shows the locus of the coordinate $(V_{\mathbf{x}}\mathbf{g}_i, V_{\mathbf{x}}\mathbf{f}w_i),\ i = 1, 2, 3$. These profiles move over the safe region R_A and infeasible region R_B during the simulation. When $(V_{\mathbf{x}}\mathbf{g}_i, V_{\mathbf{x}}\mathbf{f}w_i),\ i = 1, 2, 3$ lie in R_A, CLF control mode is used. Once the states are in R_B, the assistive controller is made active. The transition between these controllers ensures continuous decay of Lyapunov function as shown in Fig. 7. The boundary lines in Fig. 7 have angles $\xi_{min} = \frac{\pi}{2}(1+0.001)$ and $\xi_{max} = \frac{3\pi}{2} + \frac{\pi}{5}$, respectively.

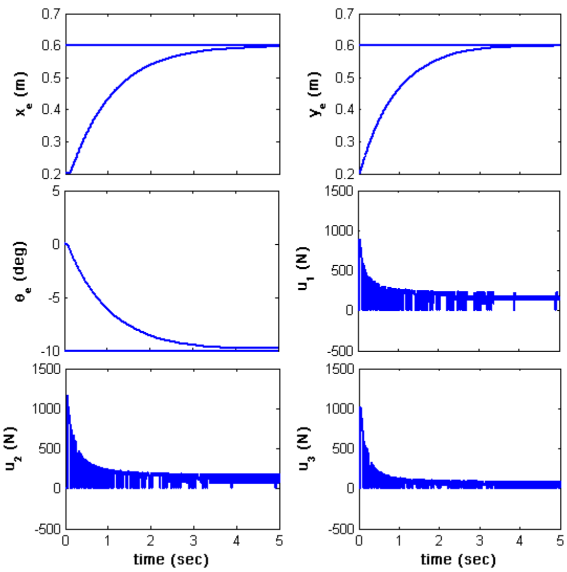

Fig. 6. Closed-loop state and input profile. The controller successfully stabilizes the state of the system from $\mathbf{x}_0$ to the desired steady state $\mathbf{x}_d$, while maintaining positive inputs.

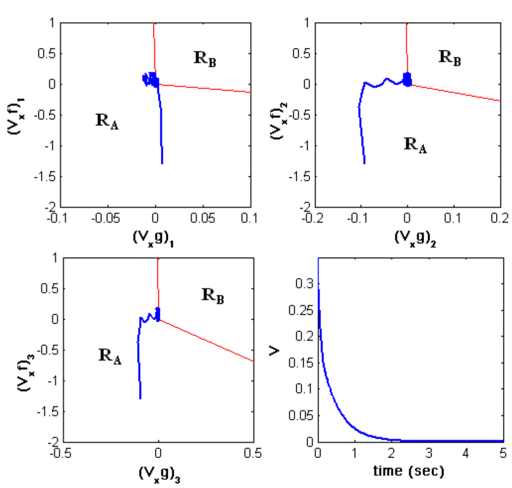

Fig. 7. The locus of $(V_x g, V_x f)_i,\ i = 1, 2, 3$. Lyapunov function decreases smoothly over time, indicating the stability in the presence of switching CLF and secondary controller.

V. Conclusions

This work proposed a novel nonlinear feedback control method for a broad class of cable suspended robots with input constraints. The salient features of our approach are as follows: First, we have proposed a valid CLF for feedback control of a cable robot. Second, two constructive control design for cable robots with multi-dimensional input have been shown via CLF, along with a proper selection of control gains that ensure positive control. In the study of first control design, a secondary controller was implemented to address the limitation of CLF controller. The transition between these two controllers assures input positivity and global stability. The effectiveness of this controller has been verified through computer simulations for one-dimensional and 3DOF planar cable robots. This study offers new insights into CLF-based nonlinear control of systems with multiple inputs and has the potential to be an useful tool in the design and analysis of constrained nonlinear system.

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