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Optimal Control Using Nonholonomic Integrators

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Abstract— **This paper addresses the optimal control of nonholonomic systems through provably correct discretization of the system dynamics. The essence of the approach lies in the discretization of the Lagrange-d'Alembert principle which results in a set of forced discrete Euler-Lagrange equations and discrete nonholonomic constraints that serve as equality constraints for the optimization of a given cost functional. The method is used to investigate optimal trajectories of wheeled robots.**

I. Introduction

We study motion planning for mechanical systems with *nonholonomic* (e.g. rolling) constraints. That is, we consider the problem of computing the controls necessary to move the system from an initial state to a goal state while optimizing some criterion such as the total energy consumed or the distance travelled while satisfying task constraints such as obstacles in the environment.

For this purpose, we develop optimization algorithms based on structure-preserving numerical integrators derived from *discrete mechanics*. In particular, we employ a discrete *Lagrange-d'Alembert* principle which provides a variational framework to study mechanical systems with constraints and external forces. The resulting discrete systems inherit the structure of the continuous systems and respect the work-energy balance which is important for numerical accuracy and stability.

The method is illustrated using two simple car-like models. Experimental comparison with a standard approach (i.e. direct collocation) demonstrates the computational advantages of the proposed framework.

A. Related work

Trajectory design and motion control of robotic systems have been studied from many different perspectives. A particularly interesting point of view is the geometric approach, that uses symmetry and reduction techniques (see [1], [2]). The authors of [3] use reduction by symmetry to simplify the optimal control problem and provide a framework for computing motions for general nonholonomic systems. A related approach, applied to an underwater eel-like robot, involves finding approximate solutions using truncated basis of cyclic input functions [4]. Other applications of symmetry in robotics include motion planning of dynamic variable inertia mechanical systems [5], improving the precision and efficiency of randomized kinodynamic planning [6], efficient motion planning using maneuver primitives [7], [8].

The systems we consider fall in the class of underactuated nonlinear systems with drift. The motion planning problem for these systems cannot be solved in closed form and one has to resort to numerical optimization techniques. One such approach relies on the differential flatness of systems [9], [10] or on performing system inversion [11]. Several motion planners have been implemented in mobile robotics to optimize trajectories among obstacles. We mention the path deformation method in [12] that could be extended to systems with drift; the method based on standard parametrized optimal control for autonomous navigation [13]; and various methods applied to wheeled robots [14], [15], [16], [17]; as well as a nonlinear optimization method for local planning in a randomized search framework [18] (see [19] for additional references).

The recent work on Discrete Mechanics and Optimal Control (DMOC) ([20], [21], [22]) employs a discretization strategy different from the standard optimization methods (i.e. collocation, shooting, or multiple shooting, see [23] for a taxonomy of various methods). It is based on *variational integrators* [24] which, unlike standard integration methods, can preserve energy, momenta, and symplectic structure, or in the presence of forces, compute the exact change in these quantities. The approach can be extended to the nonholonomic case to yield *nonholonomic integrators* (see [25], [26]). We use such integrators in a DMOC framework as a basis for this work.

In the context of mobile robotics, our proposed method may seem related to existing variational approaches such as [3] and [27], where the continuous equations of motion are first obtained, enforced as constraints, and *subsequently* discretized. However, the main difference is that in the DMOC framework variational principles determining the system dynamics are discretized *first* so as to ensure provably-good numerical behavior; the resulting discrete equations are then used as constraints along with a discrete cost function to form the control problem.

B. The Optimization problem

We consider a finite-dimensional mechanical system with configuration space Q and a distribution $\mathcal{D}$ that describes the nonholonomic constraints of interest. The distribution $\mathcal{D}$ is a collection of linear subspaces denoted $\mathcal{D}_q \subset T_qQ$ of the tangent space T_qQ for each $q \in Q$. A curve $q(t) \in Q$ is said to satisfy the nonholonomic constraint if $\dot q(t) \in \mathcal{D}_{q(t)}$ for all t.

The system is required to move from an initial state $(q(0) = q_0, \dot q(0) = \dot q_0)$ to a final state $(q(T) = q_T, \dot q(T) = \dot q_T)$ during a time interval $[0, T]$ under the influence of a control force $f(t)$ while minimizing a cost function:

$$J(q, \dot q, f) = \int_0^T C(q(t), \dot q(t), f(t))dt \tag{1}$$

The motion must satisfy the nonholonomic Lagrange-d'Alembert principle (L is the Lagrangian, $L : TQ \to \mathbb{R}$):

$$\delta \int_0^T L(q(t), \dot q(t))\mathrm{d}t \; + \; \int_0^T f(t) \cdot \delta q(t)\,\mathrm{d}t \; = \; 0 \tag{2}$$

for variations $\delta q(t)$ such that $\delta q(0) = \delta q(T) = 0$ and $\delta q(t) \in \mathcal{D}_{q(t)}$ for each $t \in [0, T]$. Additional nonlinear equality or inequality constraints might be imposed in the form $H(q(t)) \geq 0$.

II. Discretization of Nonholonomic Systems

In this section we extend the discrete optimal control formulation of [20] to systems with nonholonomic constraints. The system is discretized by replacing the state space TQ with $Q \times Q$ [24]. Thus, a velocity vector $(q, \dot q) \in TQ$ is represented by a pair of points $(q_0, q_1) \in Q \times Q$. A path $q : [0, T] \to Q$ is replaced by a discrete path $q_d : \{kh\}_{k=0}^N \to Q$, $Nh = T$. Discrete configurations are denoted $q_k = q_d(kh)$. Similarly, a continuous force $f : [0, T] \to T^*Q$ is replaced by a discrete force $f_d : \{kh\}_{k=0}^N \to T^*Q$ with corresponding notation $f_k = f_d(kh)$. Based on this discretization, the nonholonomic constraint distribution $\mathcal{D} \subset TQ$ is replaced with $\mathcal{D}_d \subset Q \times Q$ such that $(q_k, q_{k+1}) \in D_d$ for all $k = 0, ..., N-1$.

A. Discrete Nonholonomic Lagrange-d'Alembert Principle

The Lagrangian action integral (2) can then be approximated on a time interval $[kh, (k+1)h]$ by the *discrete Lagrangian* $L_d : Q \times Q \to \mathbb{R}$ according to

$$L_d(q_k, q_{k+1}) \approx \int_{kh}^{(k+1)h} L(q(t), \dot q(t))\mathrm{d}t \tag{3}$$

The virtual work in (2) can be approximated using

$$f_k^- \cdot \delta q_k + f_k^+ \cdot \delta q_{k+1} \approx \int_{kh}^{(k+1)h} f(t) \cdot \delta q(t)\mathrm{d}t \tag{4}$$

where $f_k^-, f_k^+ \in T^*Q$ are called *left* and *right discrete forces*.

The discrete version of the Lagrange-d'Alemebert principle becomes:

$$\delta \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}) + \sum_{k=0}^{N-1} f_k^- \cdot \delta q_k + f_k^+ \cdot \delta q_{k+1} = 0 \tag{5}$$

such that $\delta q_k \in \mathcal{D}_{q_k}$, $(q_k, q_{k+1}) \in \mathcal{D}_d$ for all $k = 0, ..., N-1$ and $\delta q_0 = \delta q_N = 0$.

Assume that the space $\mathcal{D}$ is defined by m functions $\omega^a : TQ \to R$, $a = 1, ..., m$ that are linear in the velocities and satisfy $\omega^a(q, \dot q) = 0$. One can always select coordinates $q = (r, s)$ such that the functions can be expressed as $\omega^a(q, \dot q) = \dot s^a + A^a_\alpha(r, s)\dot r^\alpha$, $\alpha = n - m$, which is equivalent to constraining the variations according to $\delta s^a + A^a_\alpha \delta r^\alpha = 0$[1]. Forces can be expressed in the corresponding dual basis as $f = (f_\alpha, f_a)$ and we assume that $f_a = 0$, i.e. forces enter only in the r-coordinates. The resulting equations after substituting the constraint are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot r} - \frac{\partial L}{\partial r} - f = A(r, s)\left(\frac{d}{dt}\frac{\partial L}{\partial \dot s} - \frac{\partial L}{\partial s}\right) \tag{6}$$

Assume that the functions ω^a are approximated using corresponding discrete constraint functions $\omega^a_d : Q \times Q \to \mathbb{R}$. Then (5) becomes equivalent to the *discrete nonholonomic Euler-Lagrange equations*:

$$\begin{aligned} &\frac{\partial L_k}{\partial r_k} + \frac{\partial L_{k-1}}{\partial r_k} + f_k^- + f_{k-1}^+ \\ &\qquad = A(r_k, s_k)\left(\frac{\partial L_k}{\partial s_k} + \frac{\partial L_{k-1}}{\partial s_k}\right) \\ &\omega^a_d(r_k, s_k, r_{k+1}, s_{k+1}) = 0, \end{aligned} \tag{7}$$

for $k = 0, ..., N-1$, $a = 1, ..., m$, where $L_k := L_d(r_k, s_k, r_{k+1}, s_{k+1})$. The above formulation avoids the use of Lagrange multipliers.

B. Discrete Optimization Problem

The cost function is approximated on each trajectory segment (q_k, q_{k+1}) using

$$C_d(q_k, q_{k+1}, f_k, f_{k+1}) \approx \int_{kh}^{(k+1)h} C(q, \dot q, f)dt \tag{8}$$

yielding the total cost

$$J_d(q_d, f_d) = \sum_{k=0}^{N-1} C_d(q_k, q_{k+1}, f_k, f_{k+1})$$

Velocity boundary conditions $\dot q(0) = \dot q_0$ and $\dot q(T) = \dot q_T$ are enforced using

$$\begin{aligned} &\frac{\partial L}{\partial \dot r}(q_0, \dot q_0) + \frac{\partial L_0}{\partial r_0} + f_0^- \\ &\qquad = A(r_0, s_0)\left(\frac{\partial L}{\partial \dot s}(q_0, \dot q_0) + \frac{\partial L_0}{\partial s_0}\right) \\ &\frac{\partial L}{\partial \dot r}(q_T, \dot q_T) - \frac{\partial L_{N-1}}{\partial r_N} - f_{N-1}^+ \\ &\qquad = A(r_N, s_N)\left(\frac{\partial L}{\partial \dot s}(q_N, \dot q_N) - \frac{\partial L_{N-1}}{\partial s_N}\right) \end{aligned} \tag{9}$$

In summary, we have the following constrained nonlinear optimization problem:

Compute: q_d, f_d

minimizing $\sum_{k=0}^{N-1} C_d(q_k, q_{k+1}, f_k, f_{k+1})$

subject to:

$q(0) = q_0, \;\; q(T) = q_T$

Equations (7)

Equations (9)

$H(q_k) \geq 0,$

[1] Using the summation convention $a_i b^i := \sum_i a_i b^i$

for $k = 0, ..., N-1$.

C. Algorithm Construction

In order to construct an optimization algorithm we have to define the discrete cost function C_d, the discrete Lagrangian L_d, the constraints ω_d and the discrete force f_d consistently. We choose the *midpoint rule* according to which

$$\begin{aligned}
&C_d(q_k, q_{k+1}, f_k, f_{k+1}) \\
&\qquad = hC(q_{k+\frac{1}{2}}, \Delta q_k, f_{k+\frac{1}{2}}), \\
&L_d(q_k, q_{k+1}) = hL(q_{k+\frac{1}{2}}, \Delta q_k), \\
&\omega_d^a(q_k, q_{k+1}) = \omega^a(q_{k+\frac{1}{2}}, \Delta q_k), \\
&\int_{kh}^{(k+1)h} f(t)\cdot \delta q(t)\mathrm{d}t \approx h f_{k+\frac{1}{2}} \cdot \delta q_{k+\frac{1}{2}} \\
&\qquad = \frac{h}{4}(f_k + f_{k+1})\cdot \delta q_k + \frac{h}{4}(f_k + f_{k+1})\cdot \delta q_{k+1},
\end{aligned}$$

using the notation $q_{k+\frac{1}{2}} := \frac{q_k + q_{k+1}}{2}$ and $\Delta q_k := \frac{q_{k+1} - q_k}{h}$. The left and right forces then become $f_k^- = f_k^+ = \frac{h}{4}(f_k + f_{k+1})$. The midpoint rule is second order accurate. Higher order integrators using composition methods or symplectic partitioned Runga-Kutta methods can also be constructed [24].

III. Reduced Discretization

The discrete equations derived in the previous section might become singular or the choice of coordinates might not be globally valid. When symmetries are present such problems can be avoided by applying reduction. In this section we use the reduced integrators derived in [25] to formulate a reduced optimization framework for Chaplygin systems that are relevant to car-like vehicles.

A. Reduced Lagrange-d'Alembert Equations

Assume that we are given a Lie group G acting to Q. We can pick local coordinates $q = (r, g)$, $q \in Q$, $r \in M$, $g \in G$, where $M = Q/G$ is the *shape space*. Assume that the Lagrangian L and constraint distribution $\mathcal{D}$ are invariant under the induced action of G on TQ. Then we can define the *reduced Lagrangian* $\ell : TQ/G \to \mathbb{R}$ satisfying $L(r, \dot r, \dot g, g) = \ell(r, \dot r, g^{-1}\dot g)$, and the *constrained reduced Lagrangian* $\ell_c : \mathcal{D}/G \to \mathbb{R}$, such that $\ell(r, \dot r, g^{-1}\dot g) = \ell_c(r, \dot r)$. The main point is that the Lagrange-D'Alembert equations on TQ induce well-defined *reduced Lagrange-D'Alembert equations* on $\mathcal{D}/G$, vector fields in $\mathcal{D}$ are also G-invariant and define reduced vector fields on $\mathcal{D}/G$ [28].

Whenever the group directions (the set of vector fields obtained by differentiating the group flow) complement the constraints we have the *principle kinematic case* or the *Chaplygin* case. In this case there is a principal connection one-form

$$\mathcal{A}(r, g)\cdot(\dot r, \dot g) = Ad_g(g^{-1}\dot g + \mathcal{A}_{loc}(r)\dot r), \tag{10}$$

where $\mathcal{A}_{loc}$ is the local form of the connection. This connection defines the evolution of the group variables in terms of the shape variables. It can be derived directly from the constraints. Since $\mathcal{A}(q)\cdot \dot q = 0$ the constrained Lagrangian is given by

$$\ell_c(r, \dot r) = \ell(r, \dot r, -\mathcal{A}_{loc}(r)\dot r)$$

Assuming that the control forces are restricted to the shape, i.e. $f : TM \to T^*M$ the continuous equations of motion are

$$\frac{\partial \ell_c}{\partial r} - \frac{d}{dt}\frac{\partial \ell_c}{\partial \dot r} + f = \hat f \tag{11}$$

$$\dot g = -g\mathcal{A}_{loc}(r)\dot r, \tag{12}$$

where the forces $\hat f : TM \to T^*M$ arise from the curvature of $\mathcal{A}_{loc}$ and are defined by

$$\hat f_\beta = \frac{\partial l}{\partial \xi_b}\left(\frac{\partial \mathcal{A}^b_\beta}{\partial r^\alpha} - \frac{\partial \mathcal{A}^b_\alpha}{\partial r^\beta} - C^b_{ac}\mathcal{A}^a_\alpha \mathcal{A}^c_\beta\right)\dot r^\alpha, \tag{13}$$

with $\xi = -\mathcal{A}_{loc}(r)\cdot \dot r$, $\mathcal{A}^b_\alpha$ the components of $\mathcal{A}_{loc}$, and C^b_{ac} are the structure constants of the Lie algebra defined by $[e_a, e_c] = C^b_{ac}e_b$ (see [1] for details and an example). Equations (11) are independent of $g \in G$ and determine the unconstrained evolution of the system in the shape space M. Curves in M can be *lifted* to Q using (12) to produce a unique curve in Q [29]. This fact allows us to reduce the optimal control problem from Q to M and after finding optimal trajectories in M to lift them back to Q.

Next we apply the discrete Lagrange-d'Alemebert principle in the reduced space M. The integral of l_c is approximated by the *discrete constrained reduced Lagrangian* $L_d^* : M \times M \to \mathbb{R}$ [25]. Then we obtain the discrete reduced equations of motion and discretized constraints:

$$\begin{aligned}
&D_2L_d^*(r_{k-1}, r_k) + D_1L_d^*(r_k, r_{k+1}) + f^+_{k-1} + f^-_k \\
&\qquad = \hat f^+_{k-1} + \hat f^-_k \\
&w_d(r_k, g_k, r_{k+1}, g_{k+1}) = 0,
\end{aligned} \tag{14}$$

and velocity boundary conditions (corresponding to (9))

$$\begin{aligned}
&D_2\ell_c(r_0, \dot r_0) + D_1L_d^*(r_0, r_1) + f_0^- = \hat f_0^- \\
&D_2\ell_c(r_N, \dot r_N) - D_2L_d^*(r_{N-1}, r_N) - f^+_{N-1} = \hat f^+_{N-1}
\end{aligned} \tag{15}$$

which determine the complete evolution of the system.

When constructing a reduced algorithm with the midpoint rule we set

$$L_d^*(r_k, r_{k+1}) = h\, l_c\left(r_{k+\frac{1}{2}}, \Delta r_k\right) \tag{16}$$

$$\hat f_k^\pm = \frac{h}{2}\hat f\left(r_{k+\frac{1}{2}}, \Delta r_k\right) \tag{17}$$

The equation for $\hat f_k^\pm$ was derived assuming linear dependence of the connection on the base point [25]. While a more general formulation exists, for the purpose of this paper we assume that it is a valid approximation. For the car-like examples that we consider the connection is linear in the base point and the linearity assumption is satisfied.

Using the exponential map to define the midpoint (along the flow) between two configurations in G, the constraint equation in (14) becomes

$$g_k^{-1}g_{k+1} - \exp(h\xi_k) = 0,$$

and the cost function

$$C_d(r_k, r_{k+1}, g_k, f_k, f_{k+1}) \\ = hC(r_{k+\frac{1}{2}}, \Delta r_k, g_{k+\frac{1}{2}}, g_{k+\frac{1}{2}}\xi_k),$$

where $\xi_k = -\mathcal{A}_{loc}(r_{k+\frac{1}{2}}) \cdot \Delta r_k$ and $g_{k+\frac{1}{2}} = g_k \exp(\frac{h}{2}\xi_k)$.

B. Reduced Discrete Optimization Problem

The reduced optimization problem can be formulated as follows

Compute: r_d, f_d

minimizing $\sum_{k=0}^{N-1} C_d(r_k, r_{k+1}, g_k, f_k, f_{k+1})$

subject to:

$r(0) = r_0,\ \ g(0) = g_0,\ \ r(T) = r_T,\ \ g(T) = g_T,$

Equations (14)

Equations (15)

$H(r_k, g_k) \geq 0,$

for $k = 0, ..., N-1$, $r_d = \{r_i\}_{i=0}^N$. Group variables g_k need not be included as part of the optimization state vector since they can be reconstructed from shape trajectories internally during optimization.

IV. WHEELED VEHICLES APPLICATIONS

A. Models

1) Two wheeled robot: Consider the two wheeled mobile robot [1] controlled by applying torque to each wheel independently assuming there is roll without slip. The configuration space is $Q = S^1 \times S^1 \times SE(2)$ with coordinates $q = (\phi_R, \phi_L, x, y, \theta)$, where (ϕ_R, ϕ_L) are the rotation angles of the right and left wheels and $(x, y, \theta) \in SE(2)$ are the position and orientation. The robot is controlled with right and left wheel torques τ_R and τ_L respectively. The Lagrangian is

$$L(q, \dot{q}) = \frac{1}{2}J_w(\dot{\phi}_R^2 + \dot{\phi}_L^2) + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \quad (18)$$

where m is the mass, and J and J_w are the moments of inertia, ρ is the wheel radius and d is the distance from the wheel to the center of mass which is assumed to coincide with the center of rotation.

The system is symmetric with respect to actions of the group $G = SE(2)$. The shape space is described by coordinates $r = (\phi_R, \phi_L) \in Q/G$. The matrix representation of the local connection in (10):

$$[\mathcal{A}_{loc}(r)] = \begin{bmatrix} -\frac{\rho}{2} & -\frac{\rho}{2} \\ 0 & 0 \\ \frac{\rho}{2\omega} & -\frac{\rho}{2\omega} \end{bmatrix} \quad (19)$$

The constrained Lagrangian is

$$\ell_c = \frac{1}{2}C\left(\dot{\phi}_R^2 + \dot{\phi}_L^2\right) + D\dot{\phi}_R\dot{\phi}_L, \quad (20)$$

where

$$C = \left(J_\omega + \frac{m\rho^2}{4} + \frac{J\rho^2}{4d^2}\right), \ D = \left(\frac{m\rho^2}{4} - \frac{J\rho^2}{4d^2}\right) \quad (21)$$

Substituting the Lagrangian and the connection in equations (14), we see from (13) that the forces $\hat{f}$ vanish and the reduced discrete equations of motion become:

$$\begin{aligned} &C\left(\Delta\phi_{Rk} - \Delta\phi_{Rk-1}\right) + D\left(\Delta\phi_{Lk} - \Delta\phi_{Lk-1}\right) \\ &\quad = \frac{h}{4}\left(\tau_{Rk-1} + 2\tau_{Rk} + \tau_{Rk+1}\right) \\ &D\left(\Delta\phi_{Rk} - \Delta\phi_{Rk-1}\right) + C\left(\Delta\phi_{Lk} - \Delta\phi_{Lk-1}\right) \\ &\quad = \frac{h}{4}\left(\tau_{Lk-1} + 2\tau_{Lk} + \tau_{Lk+1}\right) \\ &x_{k+1} = x_k + \frac{v_k}{\omega_k}(\sin(\theta_k + h\omega_k) - \sin(\theta_k)) \\ &y_{k+1} = y_k + \frac{v_k}{\omega_k}(-\cos(\theta_k + h\omega_k) + \cos(\theta_k)) \\ &\theta_{k+1} = \theta_k + h\omega_k, \end{aligned}$$

where $v_k = \frac{\rho}{2}\left(\Delta\phi_{Rk} + \Delta\phi_{Lk}\right)$, $\omega_k = \frac{\rho}{2d}\left(\Delta\phi_{Lk} - \Delta\phi_{Rk}\right)$

2) Simple Car: The simple car is controlled using rear wheel torque u^ψ and torque u^σ steering the front wheels. The configuration space is $Q = S^1 \times S^1 \times SE(2)$ with coordinates $q = (\psi, \sigma, x, y, \theta)$, where $(x, y, \theta) \in SE(2)$ are the position and orientation of the car, ψ is the rolling angle of the rear wheels, and σ is defined as $\sigma = \tan(\phi)$ where ϕ is the orientation of the front wheels relative to the car orientation θ, i.e. the steering angle. The model assumes that the distance between the left and the right wheels is negligible, such as in a bicycle model (e.g. [30]). The Lagrangian is

$$L(q, \dot{q}) = \frac{1}{2}I\dot{\psi}^2 + \frac{1}{2}J\dot{\sigma}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}K\dot{\theta}^2 \quad (22)$$

where m is the mass, I and J are the moments of inertia, l is the distance between front and rear wheel axles, and ρ is the radius of the wheels. We choose to parametrize the steering angle in order to avoid a nonlinear term in the constraint connection as derived below and to avoid the computation of tan during optimization.

The Lagrangian and constraints (see [1]) are again invariant under action of the group $G = SE(2)$. The shape coordinates are now $r = (\psi, \sigma) \in M$. The matrix representation of the local connections in (10) is

$$[A_{loc}(r)] = \begin{bmatrix} -\rho & 0 \\ 0 & 0 \\ -\frac{\rho}{l}\sigma & 0 \end{bmatrix} \quad (23)$$

The constrained Lagrangian is

$$\ell_c(r, \dot{r}) = \frac{1}{2}\left(I + m\rho^2 + \frac{K\rho^2\sigma^2}{l^2}\right)\dot{\psi}^2 + \frac{1}{2}\left(\frac{J}{(1+\sigma^2)^2}\right)\dot{\sigma}^2$$

Using (13) one can compute

$$\hat{f} = \left[-\frac{K\rho^2\sigma\dot{\psi}\dot{\sigma}}{l^2}, \frac{K\rho^2\sigma\dot{\psi}^2}{l^2}\right] \quad (24)$$

The resulting discrete equations of motion are found by expressing the discrete Lagrangian, forces, and constraints in terms of the corresponding quantities ℓ_c, A_{loc}, and $\hat{f}$.

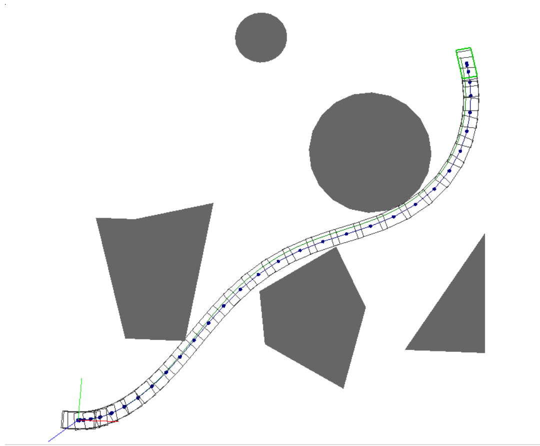

Fig. 1. Differential drive optimized trajectory with $N = 32$

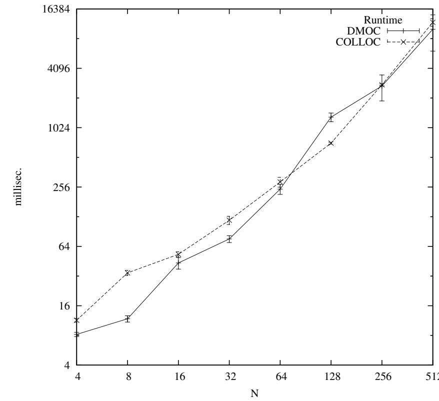

Fig. 2. Runtime comparison

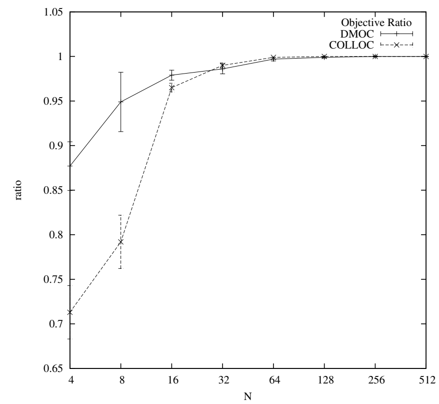

Fig. 3. Objective function ratio

B. Experiments

The goal is to find a trajectory between an initial and final state (fig. 1) with minimum control effort (i.e. the objective function is the sum of the squares of controls). The task constraints include bounds on the accelerations, bounds on the velocities, and obstacle avoidance. These conditions are expressed as inequality constraints. We only consider obstacles represented by arbitrary polygons and ellipses, although the method can be extended to any obstacles. The experimental results described below are averaged over runs in 10 different environments created by randomly perturbing the obstacles parameters (e.g. vertices, centers, size).

The optimization is performed using an SQP solver which requires an initial guess. An A^* algorithm is used to plan a shortest distance path in (x, y) space and generate approximate values for the remaining coordinates. The solver is then able to transform this path into an executable trajectory and optimize it further. The resulting solution is not guaranteed to be globally optimal but there is strong evidence that it is a good optimum within its homotopy class.

The method is compared against standard direct collocation with Hermite-Simpson discretization [23] of the continuous reduced equations. We are intersted in finding how well the method performs as a function of the discretization resolution, i.e. the number of time steps N used. The four criteria tested are: runtime (fig. 2), ratio of convergence to the optimal objective value (fig. 3), goal error after executing the trajectory (fig. 4), and average error between computed and executed trajectories (fig. 5). We use a fourth order Runga-Kutta solver with 10000 time steps as a ground truth. The resulting errors provide insight into how well the system dynamics is preserved by each method.

The runtime of the two algorithms is comparable; DMOC slightly more efficient at smaller time steps [2]. DMOC converges to its optimal objective value faster. DMOC exhibits less error at bigger time steps and hence the executed paths (by a fine integration) reach the goal closer. Therefore, computational advantage can be gained by using DMOC at reduced resolution.

[2]At $N = 128$ the runtime for DMOC consistently jumps. This condition was related to sudden large memory allocation by the SQP solver only at this specific value.

V. Conclusions and Future Work

The paper proposes a new method for solving the motion planning problem for nonholonomic mechanical systems. It is based on discretizing the Lagrange-d'Alembert principle to derive discrete equations of motion serving as equality constraints in a numerical optimization scheme. Reduction by symmetry in the principal kinematic case is employed to simplify the optimal control formulation. Our experiments for car-like robots suggest that the method is a good alternative to standard collocation. It would be uesful to further optimize the approach based on ideas from, e.g. [9], [27], [4].

A major limitation of the approach is that it only provides a locally optimal solution. Nevertheless, it could be employed as an efficient local optimization method since it has to ability to converge to a good approximation with relatively few time steps. Another obvious application is the refinement of suboptimal trajectories computed from discrete or sampling-based motion planners.

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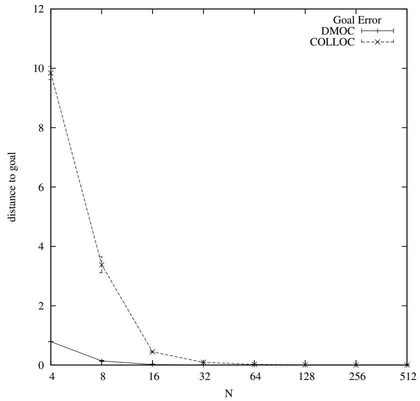

Fig. 4. Final state achievement error

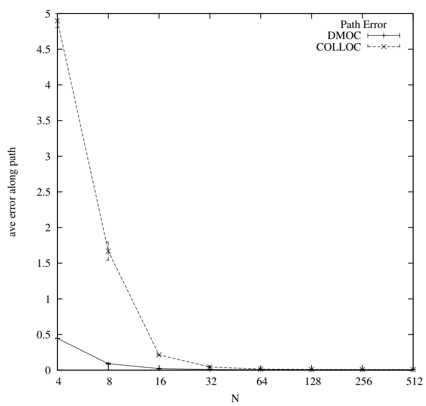

Fig. 5. Trajectory execution error

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