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Latency Analysis of Coalescence for Robot Groups

Sameera Poduri and Gaurav S. Sukhatme

***Abstract*— Coalescence is the problem of isolated mobile robots independently searching for peers with the goal of forming a single connected network. This is important because communication is a necessary requirement for several collaborative robot tasks. In this paper, we consider a scenario where the robots do not have any information about the environment or positions of other robots and perform a random walk search. We show through probabilistic analysis that as the number of isolated robots N increases, the expected Coalescence Time decreases as $1/\sqrt{N}$. Simulations results are presented to validate this analysis.**

I. Introduction

Mobile robots rely on wireless communication for collaborative tasks such as formations, collective exploration and coverage, behaviors like construction and box pushing, *etc.* In some applications, the robots start in a disconnected state and attaining connectivity with each other is the first step to begin a task. Robots could lose connectivity with the network due to irregularities in wireless communication. We can also imagine scenarios where the cost of maintaining connectivity is very high and so the robots split into smaller groups for some part of the task and rejoin later. Algorithms to establish network connectivity are therefore very important.

This paper focuses on a scenario where disconnected robots either individually or in small groups search for their peers and form large connected components. This is similar to the phenomenon of coalescence in fluids - small fluid drops merge together to from larger drops. We refer to this problem of network formation as *coalescence*. While coalescence can be considered a variation of the rendezvous problem, we use the term coalescence to emphasize that the connected component spreads as more robots join it. This spread plays an important role in determining latency. The basic idea is that robots need not be collocated to remain connected. If they remain spread out, the disconnected robots will have a better chance of discovering them. We are interested in understanding the nature of the Coalescence Time and in particular its variation with the number of isolated robots. Our analysis uses an approximate model for the communication spread of the connected component and shows that the Coalescence Time decays as $\frac{1}{\sqrt{N}}$ with the number of robots N. On the other hand, if we ignore the communication spread and assume that the robots collocate when they form a connected component then the Coalescence Time grows as $\ln(n)$.

The robots' search is modeled as a simple (memoryless) random walk. The robots do not have any knowledge about the environment or locations of other robots. We assume that there is a single, stationary base station whose identity (but not location) is known to all robots. The robots terminate their search when they are connected to the base station either directly or via other robots. This is illustrated in figure 1. It is possible to develop coalescence strategies without using a base station. For example, the robots could perform a random walk and coalesce every time they meet any other robot until all robots form a single component. However, such a strategy is extremely hard to analyze. The simplified strategy with the base station allows us to understand the nature of Coalescence Time and the insights gained can be used to design more sophisticated coalescence algorithms.

The remainder of this paper is organized as follows. Section II discusses the related research, section III formulates the problem, analysis of hitting times is presented in section IV and section V verifies these models in simulation. We conclude with a discussion of our contributions in section VI.

II. Related Work

Rendezvous is the problem of multiple mobile robots meeting at a point in an unknown environment. Several distributed algorithms have been proposed. Some of these assume connectivity [1], [2]. If the communication graph is an RNG then rendezvous algorithms have been proposed that can deal with partial communication loss over some of the links [3]. In the absence of connectivity, the robots can explore the environment building a map of the landmarks [4] where other robots are likely to visit. This is similar in spirit to the Coalescence problem where the base station is a landmark whose location is unknown. However, our focus is on forming a connected component while allowing the robots to remain "spread out" so that the time for coalescence is reduced.

Several algorithms have been proposed to maintain connectivity of a wireless network [5], [6] given that the initial network is connected. Coalescence algorithms complement these by providing a way to regain connectivity in the event of node-failure, *etc.* . In [7], the variation in connectivity of a network of robots performing random walks on a lattice has been analyzed.

Recently, the role of mobility in increasing sensing coverage of a network has been analyzed [8]. In particular, it

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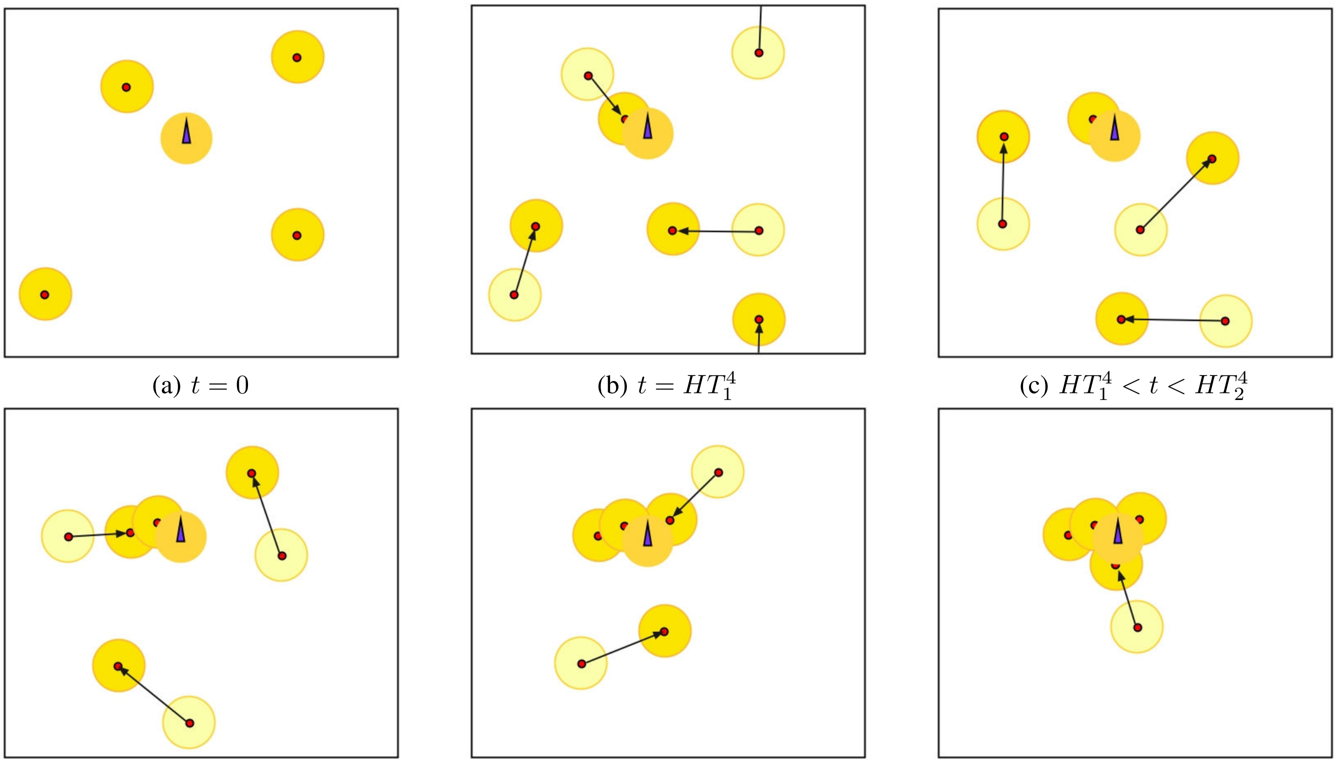
S. Poduri sameera@usc.edu and G. S. Sukhatme gaurav@usc.edu are with the Robotic Embedded Systems Laboratory, Computer Science Department, University of Southern California, 941 W. 37th Place, Los Angeles, CA 90089, USA



(a) $t = 0$ (b) $t = HT_1^4$ (c) $HT_1^4 < t < HT_2^4$

(d) $t = HT_2^4$ (e) $t = HT_3^4$ (f) $t = HT_4^4 = CT(N = 4)$

Fig. 1. Illustration of Coalescence for $N = 4$. The blue cone represents the base station, dark red circles represent robots and the yellow discs around them are their communication regions. The arrows show the random walk steps with the faded discs being the robot positions at the previous time instant. As robots join the base station's network, the area of the connected component grows and the robots that are still searching have a better chance of hitting it.

has been shown that the time taken to detect an intruder decreases. In our problem, the robots have to detect the base station. However, there are two key differences. First, in [8], each robot chooses a random direction initially always moves along that direction (with a varying speed) whereas in our case, the robots change the direction of motion after every fixed step length (while keeping the speed constant). Second, in [8], the robots continue to move after a detection event whereas in our case, when the robots hit the connected component, they stop moving and join the connected component.

In random walks literature, a coalescing random walk refers to a system of particles that coalesce when they hit other particles while performing a random walk [9]. Several asymptotic properties such as the time for convergence have been analyzed. Here, a group of coalesced particles is equivalent to one particle and this analysis does not capture the increase in communication spread.

III. Definitions and Problem Formulation

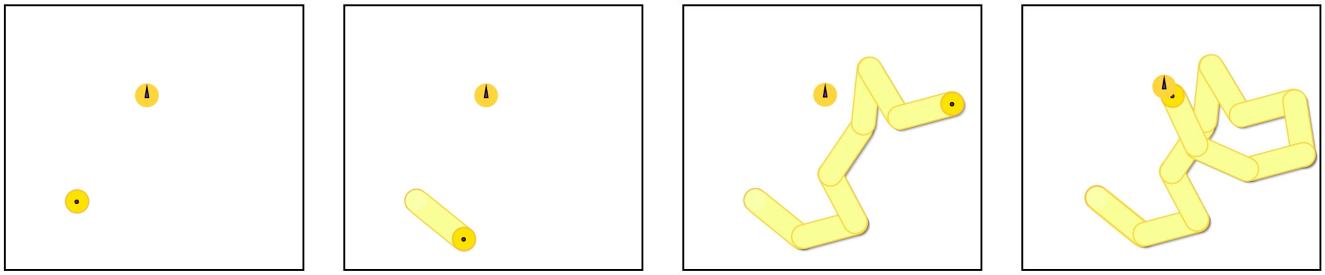
A robot can communicate directly with any robot that is within its communication range R. Two robots are *connected* if they can communicate directly or via other robots. **Coalescence** is defined as initially isolated robots coming together to form a single connected component. In this work, we address the following problem.

Given a bounded domain $\mathcal{D} \subset \Re^2$ and N isolated robots that start at unknown locations and have no knowledge about the structure of $\mathcal{D}$, how long will it take for them to coalesce into a single connected component?

Definition: The Coalescence Time ($CT(n)$) is the time taken for n isolated robots to coalesce. Clearly, the Coalescence Time will be a function of size of domain ($\mathcal{D}$), number of robots (n), communication range (R), and the robots' motion strategy.

A *base station* is assumed to be randomly placed in the domain. Initially, the connected component consists only of the base station. Robots are distributed independently and uniformly, *i.e.* according to the Poisson point process. They start independent random walks in search of the connected component. When a robot comes into the communication range of any robot in the connected component, it stops. Then it is also part of the connected component. This is illustrated in figure 1. The idea is that robots need not be collocated to stay connected. By forming a connected component while allowing robots to remain "spread out", the Coalescence Time is reduced.

Motion Model: Each robot moves independently of the others. It chooses a direction θ_i uniformly distributed in $[0, 2\pi)$ and moves for a distance ℓ in that direction after which it chooses another direction. When it gains connectivity with the base station, or any robots that are connected to it, it stops moving.



(a) $t = 0$ (b) $t = 1$ (c) $t = 7$ (d) $HT_1^1 = t = 11$

Fig. 2. Illustration of random walk model for $N = 1$. The blue cone and dark red circle are the base station and the isolated robot respectively along with their communication discs. The light yellow region is the area that the robot has "covered" in its search for the base station.

Each robot is assumed to be a point so that it does not obstruct any other robot. We do not assume any localization or information about the environment.

IV. Analysis of Coalescence Time

Let $HT_1^N \leq HT_2^N \leq \cdots \leq HT_N^N$ be the $1^{st}, 2^{nd}, \cdots, N^{th}$ hitting times for a group of size N. These are the times at which the $1^{st}, 2^{nd}, \cdots, N^{th}$ robots "hit" or join the connected component of the base station. The Coalescence Time, $CT(n = N) = HT_N^N$.

The hitting times HT_m^N will depend on the communication spread of the connected component - greater the communication spread, smaller the hitting times. Let $\mathcal{S}(m)$ represent the shape of the connected component after m robots have joined the base station. This is the union of the communication discs of the base station and the m robots in the connected component at $t = HT_m^N$. At this time instant, the remaining $N - m$ isolated robots will be Poisson distributed over the domain because of the initial Poisson distribution and independent random motion model. Let $\mathcal{W}_m^i(t)$ represent the walk traced by an isolated robot i, t time units after HT_m^N. Note that random walk traced by this robot before HT_m^N can be ignored because the random walk model is memoryless.

Our first step is to consider the case when there is only one robot in the domain *i.e.* $(N = 1)$ and estimate its hitting time, HT_1^1.

A. Hitting time for $N = 1$, HT_1^1

At $t = 0$, the connected component consists of the base station alone. Therefore, $S(m = 0)$ is the communication disc of the base station with area πR^2. At $t = HT_1^1$, the robot's walk $\mathcal{W}_0^1(t)$ intersects $S(m = 0)$. This is illustrated in figure 2. It is easier to solve for HT_1^1 by considering the robot's perspective. The hitting time will be unchanged if the robot remains stationary and the base station performs a random walk, *i.e.* if $\mathcal{S}(m = 0)$ traces $\mathcal{W}_0^1(t)$.

Let $\mathcal{S}(m = 0) \oplus \mathcal{W}_0^1(t)$ represent the shape obtained by moving $\mathcal{S}(m = 0)$ along $\mathcal{W}_0^1(t)$. The hitting time, HT_1^1 can be estimated by finding the probability of the robot lying in this shape. Since the robots are uniformly distributed, this is given by

$$\mathcal{P}(HT_1^1 \leq t) = \frac{\mathcal{A}[\mathcal{S}(m = 0) \oplus \mathcal{W}_0^1(t)]}{\mathcal{W}(\mathcal{D})} \quad (1)$$

where $\mathcal{A}[\mathcal{X}]$ is the area of a shape $\mathcal{X}$.

For $m = 0$, $\mathcal{S}(m = 0)$ is the communication disc of the base station and $\mathcal{A}[\mathcal{S}(m = 0)] = \pi R^2$. Estimating $\mathcal{A}[\mathcal{S}(m) \oplus \mathcal{W}_0^1(t)]$ for general m is quite complex and hence we look for an approximation.

Let $\alpha(t) = \frac{\mathcal{A}[\mathcal{S}(m=0) \oplus \mathcal{W}_0^1(t)]}{\mathcal{A}[\mathcal{D}]}$

We compute $\alpha(t)$ approximately by assuming that the shape $\mathcal{S}(m = 0) \oplus \mathcal{W}_0^1(t)$ is uniformly distributed over $\mathcal{D}$. Strictly speaking, this assumption is not correct but simulations presented in the next section indicate that it is a reasonably good approximation. The uniform distribution of $\mathcal{S}(m = 0) \oplus \mathcal{W}_0^1(t))$ implies that with each new step, the addition to the area $\alpha(t)$ is proportional to "open" area in the domain *i.e.* the area that is not covered by $\mathcal{S}(m = 0) \oplus \mathcal{W}_0^1(t)$. We can express this as the following recursive equation.

$$\begin{aligned} \alpha(t+1) - \alpha(t) &\approx (1 - \alpha(t)) \frac{2R\ell}{\mathcal{A}[\mathcal{D}]} \\ \Rightarrow \alpha(t) &\approx 1 - \left(1 - \frac{2R\ell}{\mathcal{A}[\mathcal{D}]}\right)^t \end{aligned} \quad (2)$$

Since $\mathcal{P}(HT_1^1 \leq t) = \alpha(t)$, the hitting time has an exponential distribution with mean

$$E[HT_1^1] = \frac{\mathcal{A}[\mathcal{D}]}{2R\ell} \quad (3)$$

For $N = 1$, this is also the Coalescence Time.

$$E[CT(n = 1)] = E[HT_1^1] = \frac{\mathcal{A}[\mathcal{D}]}{2R\ell} \quad (4)$$

B. First hitting time for N robots, HT_1^N

Each of the N isolated robots performs a random walk with a hitting time T_i, $i = 1, 2, \cdots, N$. Any of these N robots could be the first to hit the base station and therefore the first hitting time, HT_1^N will be the minimum of all the hitting times.

$$HT_1^N = \min\{T_1, T_2, ..., T_N\} \quad (5)$$

$T_1, T_2, \cdots, T_N$ have identical exponential distributions with expected hitting time of $E[T_i] = \frac{\mathcal{A}[\mathcal{D}]}{2R\ell}$ (from Eqn. 3). Moreover, the N walks are independent of each other.

The first hitting time being their minimum will also have an exponential distribution with

$$E[HT_1^N] = \frac{1}{N} \cdot \frac{\mathcal{A}[\mathcal{D}]}{2R\ell} \tag{6}$$

Therefore, larger the number of robots, smaller the first hitting time which matches intuition.

C. $m^{th}(m > 1)$ hitting time for N robots, HT_m^N

Consider the scenario when $m-1$ robots have joined the base station network. The remaining $N-m+1$ robots are uniformly distributed over $\mathcal{D}$ and perform independent random walks $\mathcal{W}_m^i(t)$ to hit the connected component that has a shape $\mathcal{S}(m-1)$. Since the random walks are memoryless, we can ignore the random walk steps prior to $t = HT_{m-1}^N$. This scenario is equivalent to that in the previous subsection with a more complex shape of the connected component, $\mathcal{S}(m-1)$. We expect that the time for the next hit, $\Delta HT_m^N = HT_m^N - HT_{m-1}^N$ will have an exponential distribution with an expected value of the form (similar to $E[HT_1^N]$)

$$E[\Delta HT_m^N] = \frac{1}{(N-m+1)} \frac{\mathcal{A}[\mathcal{D}]}{2 \cdot R_{m-1} \cdot l} \tag{7}$$

where R_{m-1} is the effective radius of $\mathcal{S}(m-1)$ such that if $\mathcal{S}(m-1)$ were replaced by a disk of radius R_{m-1} and moved along the random walk path of an isolated robot, i, then the area $\mathcal{A}[\mathcal{S}(m-1) \oplus \mathcal{W}_{m-1}^i(t)]$ remains unchanged. If we can estimate R_{m-1}, we can easily find HT_m^N and our analysis will be complete. Unfortunately, this is very complex because R_{m-1} depends not only on the area of the connected component but also its shape. We will consider two models for R_{m-1}.

1) *Case 1:* Assume that the connected robots collocate so that $R_{m-1} = R$. Then we have

$$E[\Delta HT_m^N] \approx \frac{1}{(N-m+1)} \frac{\mathcal{A}[\mathcal{D}]}{2R\ell} \tag{8}$$

The hitting time HT_m^N will have an exponential distribution with

$$\begin{aligned} E[HT_m^N] &= E[HT_1^N] + \sum_{i=2}^{m} E[\Delta HT_i^N] \\ &\approx E[HT_1^N] + \sum_{i=2}^{m} \frac{1}{(N-i+1)} \frac{\mathcal{A}[\mathcal{D}]}{2R\ell} \\ &= \frac{\mathcal{A}[\mathcal{D}]}{2R\ell} \sum_{i=1}^{m} \frac{1}{(N-i+1)} \end{aligned} \tag{9}$$

The Coalescence Time, $CT(n=N) = HT_N^N$ can be estimated as

$$\begin{aligned} E[CT(n=N)] = E[HT_N^N] &\approx \frac{\mathcal{A}[\mathcal{D}]}{2R\ell} \sum_{i=1}^{N} \frac{1}{(N-i+1)} \\ &= E[CT(n=1)] \sum_{i=1}^{N} \frac{1}{(N-i+1)} \\ &= \Theta(\ln N) \end{aligned} \tag{10}$$

2) *Case 2:* Assume that $R_{m-1} \approx (m) \cdot R$. By repeating the above steps, we get

$$E[\Delta HT_m^N] \approx \frac{1}{(N-m+1)} \frac{1}{m} \frac{\mathcal{A}[\mathcal{D}]}{2R\ell} \tag{11}$$

$$\begin{aligned} E[HT_m^N] &= E[HT_1^N] + \sum_{i=2}^{m} E[\Delta HT_i^N] \\ &\approx E[HT_1^N] + \sum_{i=2}^{m} \frac{1}{(N-i+1)} \frac{1}{i} \frac{\mathcal{A}[\mathcal{D}]}{2R\ell} \\ &= \frac{\mathcal{A}[\mathcal{D}]}{2R\ell} \sum_{i=1}^{m} \frac{1}{(N-i+1)} \frac{1}{i} \end{aligned} \tag{12}$$

Finally, the Coalescence Time, $CT(n=N) = HT_N^N$ can be estimated as

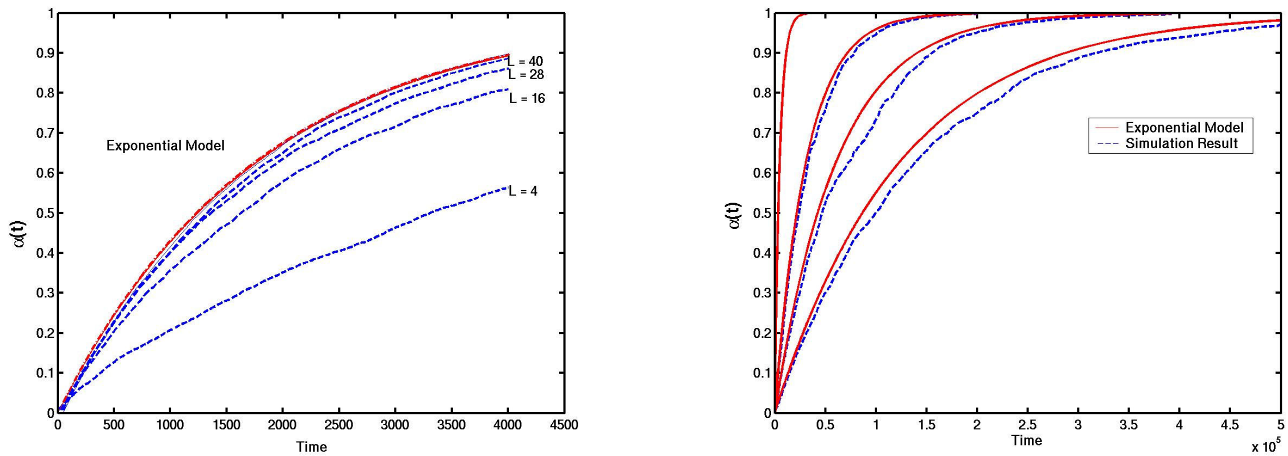
$$\begin{aligned} E[CT(n=N)] = E[HT_N^N] &\approx \frac{\mathcal{A}[\mathcal{D}]}{2R\ell} \sum_{i=1}^{N} \frac{1}{(N-i+1)} \frac{1}{i} \\ &= E[CT(n=1)] \sum_{i=1}^{N} \frac{1}{(N-i+1)} \frac{1}{i} \\ &= \Theta\Big(\frac{1}{\sqrt{N}}\Big) \end{aligned} \tag{13}$$

In case 1, the Coalescence Time increases as $\ln N$ whereas in case 2, it decreases as $\frac{1}{\sqrt{N}}$. The difference illustrates the effect of the increase in the communication spread of the connected component.

V. Simulations

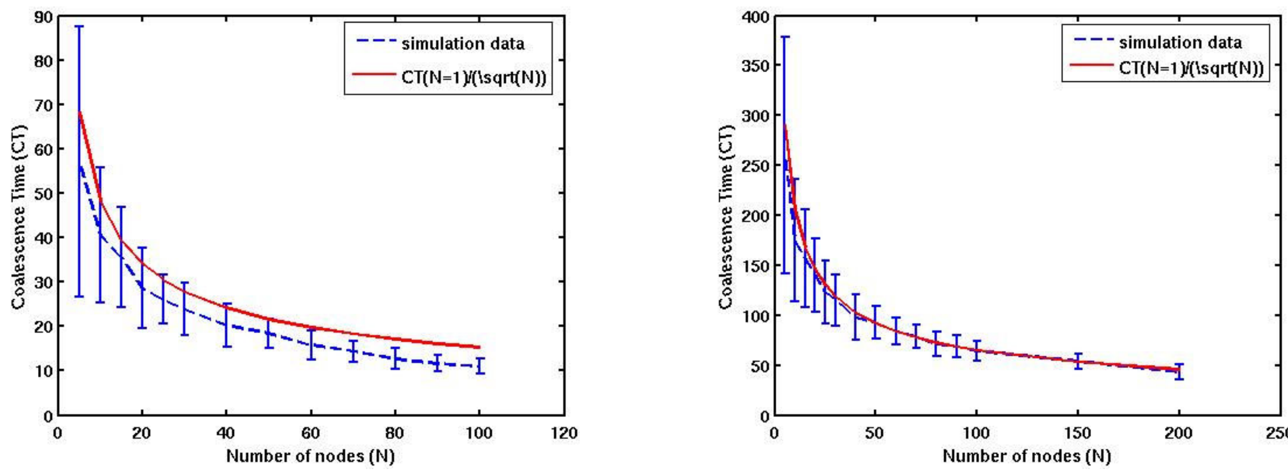
Simulations were conducted in MATLAB to verify the analysis. The domain used is a 2D torus. The parameters to the program are number of robots N, step size ℓ, communication rage R and torus side T. The robots and the base station start at random positions uniformly distributed over the torus. Every robot picks a random direction and moves with a uniform speed of 1 unit per time step. At each time step, the robot checks if it is within a distance R of connected component, in which case, it terminates its random walk. After ℓ time steps, all the isolated robots pick a new direction. The simulation stops when all the robots are connected.

We first conducted experiments with a single robot to verify our model for $\alpha(t) \approx 1 - \left(1 - \frac{2R\ell}{A(D)}\right)^t$ (Eqn 2). To measure the area covered by the robot's random walk, we



(a) (b)

Fig. 3. Verification of the exponential model derived for $\alpha(t)$. Plots show $\alpha(t)$ vs. time averaged over 100 runs for varying (a) Step sizes and (b) Domain sizes



(a) (b)

Fig. 4. The mean Coalescence Time decreases as $1/\sqrt{N}$. The graph shows the result of 100 runs with $l = 100$ and $R = 20$ on a torus of side (a) 500 and (b) 1000

discretized the domain into a fine grid and flagged all the grid cells that were within R of the robot's random walk. Note that the grid was only used to enable area computation and the random walk model was not modified. Figure 3a plots $\alpha(t)$ vs t for a range of values for the step length, ℓ. The experiments show that the expression in Eqn. 2 is an upper bound for $\alpha(t)$. $\alpha(t)$ varies with step length. It is low for small step lengths ($\ell = R$) but quickly approaches the bound ($\ell = 10 \cdot R$). Figure 3b shows that the expression for $\alpha(t)$ holds for varying sizes of the domain.

Our next set of experiments simulated Coalescence and measured Coalescence Time as a function of N. Figure 4 compares the mean Coalescence Time to the analytical expression derived above $\frac{E[CT(n=1)]}{\sqrt{N}}$. Surprisingly, even though the analysis was approximate, there is a very good match with the simulation results.

Lastly, we study the effect of varying the step length ℓ on the Coalescence Time. Figure 5 plots the Coalescence Time for a fixed domain size ($T = 500$) and number of robots ($N = 100$) with the step length varying from a very small value ($\ell = R = 10$) to the torus side ($\ell = T = 500$). The Coalescence Time does not vary significantly with the step length. This was expected because according to Figure 3a for $\ell \approx 10 \cdot R$, $\alpha(t)$ converges to the exponential distribution. However a step length that is much larger than the torus size will not be efficient because it will lead to the robot's path overlapping itself and the Coalescence Time will increase.

VI. Discussion and Conclusions

We consider the problem of coalescence, *i.e.* isolated robots independently searching for peers and forming a single connected component. Coalescence strategies can

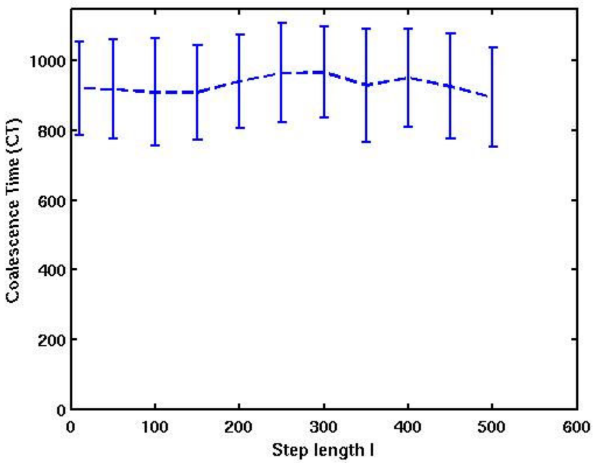


Fig. 5. Variation Coalescence Time (CT) with step length for $T = 500$, $R = 10$ and $N = 100$ averaged over 100 runs

complement algorithms for robot collaboration that require a connected network. In the absence of localization or any information about the environment, the robots can perform a simple random walk search till they are within the communication range of other robots. We show using probabilistic analysis that for such a strategy, Coalescence Time decays as $1/\sqrt{N}$ with the number of robots, N. Even though the analysis is approximate, there is a surprisingly good match with the simulation results.

There are a number open problems and extensions to this work. In the absence of a base station, what strategy will guarantee the formation of a single connected component? Is it possible to achieve this without the knowledge of the total number of robots? If the robots form intermediate "blobs" that perform the random walk and later merge to form the connected component, would the time for coalescence decrease?

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