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A Systematic Approach to Stiffness Analysis of Parallel Mechanisms

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***Abstract*— A systematic approach to compute total stiffness of parallel manipulators is proposed. Link stiffness and joint stiffness can be calculated simultaneously. Stiffness of passive joint (joint without actuator) is temporally assumed as nonzero variable in the procedure and infinite stiffness can be treated. A simple parallel manipulator is used as an example to illustrate the process and its correctness. Then validity of the proposed approach is confirmed by comparing to FEA and conventional joint stiffness method. The total link compliance (inverses of total stiffness) calculated by the developed approach and its counterpart obtained by FEA model are very close. The total joint stiffness computed by the developed approach and its correspondence derived from conventional joint stiffness method are totally the same. Stiffness performance of parallel force redundancy mechanisms is evaluated. These results indicate that this scheme is feasible and effective.**

I. INTRODUCTION

Fast mechanisms have been developed using merits of parallel mechanism, for example, DELTA [1], HEXA [2], RP-1AH [3], and NINJA [4]. In designing such a highly efficient mechanism, optimizing techniques are required to pursue the higher acceleration. For example, a two-step optimum design method is descried in [4].

In designing these mechanisms, in order to solve the contradiction between acceleration and accuracy, the calculation of stiffness is very important. The used methods for stiffness calculation can be classed into 3 categories: conventional joint stiffness analysis, the Finite Element Analysis (FEA), and the matrix structural analysis.

The first of them, conventional joint stiffness analysis is based on the calculation of Jacobian matrix that relating the joint displacement in joint space to Top-plate deflection in Cartesian space. In this method, only active joint stiffness is considered and links of mechanism are assumed strictly rigid. Some of related previous works can be found in [5-8].

FEA is reliable for calculating the stiffness. For example, the FEA model is adopted to characterize robot static rigidity and natural frequencies in [9] and it is used to validate analytical model in [10]. However, FEA does not establish the analytical relationship between stiffness, dimensions and free shape of the mechanism. It does not fit the optimization techniques in which the performance index should be represented as a function of design parameters like widths of links. What is more, these models might be re-meshed repeatedly and this results in high computational expenses.

The third of them, the matrix structural analysis incorporates the main ideas of FEA, but operates with rather large elements - flexible beams describing the manipulator structure [11]. It is used for parallel mechanism stiffness matrix calculation in [12]. However, the clear symbolic relations required for the parametric stiffness is not provided.

In this paper, a proper systematic way, which is based on the matrix structural analysis, to obtain the overall stiffness, taking into account the passive joints, active joints and links simultaneously, is proposed. The characteristics of this systematic approach are: (a) Stiffness of passive joints, active joints and links are all calculated simultaneously; (b) Infinite stiffness can be treated in the procedure and is shown in the final result. The advantages of this approach are: (1) the final stiffness result is a function of design parameters and is very useful for optimization design. Many different methods based on matrix structural analysis have been proposed in the literature. However, it is difficult with these methods to obtain symbolic results due to the use of very large matrices which are difficult to inverse. This method makes it possible to only use 6x6 matrices (or 3x3 in planar cases) and to sum them, which will lead to usable symbolic expressions. (2) It is effective and efficient to get the total stiffness by writing program like using MATLAB symbol toolbox since this approach is a systematical mathematical one.

The paper is organized as follows: In section 2, the statement of the problem is described. Section 3 develops the steps of the systematic stiffness analysis approach and an illustrative example is given. Section 4 shows the correspondence in results between the proposed approach and the FEA models. In section 5, the method is validated by parallel manipulators with force redundancy and compared with conventional joint stiffness analysis. The conclusion is given at the end of the paper.

II. PROBLEM STATEMENT

A parallel structure is composed by several serial arms, as shown in Figure 1. To explain the main idea of this study and strategy of treating infinite stiffness and zero stiffness (e.g., stiffness of passive joint shown in Fig. 1), we will first calculate the total stiffness of a simple example.

Fig. 2 shows a simple parallel mechanism and its spring model. In order to deal with infinite stiffness and zero stiffness, we will use stiffness to describe the stiffness of each joint (k_{11} and k_{21} are joint stiffness of each joint respectively) and compliance to describe the compliance of each link (c_{11} and c_{21} are link compliance of each link respectively). Denote C_i and $K_i (= C_i^{-1})$ $(i = 1, 2)$ as the compliance and stiffness of each sub-arm, respectively. K is defined as total stiffness of the parallel mechanism.

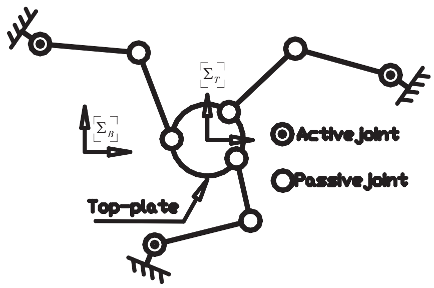

Fig. 1. A parallel manipulator including passive joint

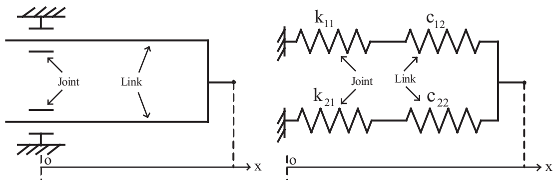

Fig. 2. Parallel mechanism and its spring model

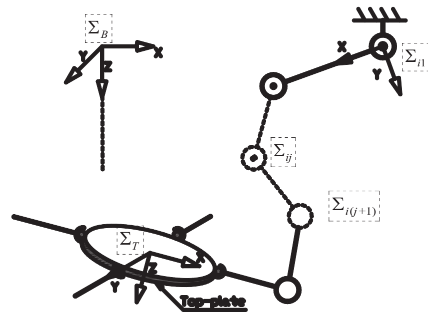

Fig. 3. One sub-arm of a parallel manipulator

Compliance of a serial mechanism is the summation of compliance of components, i.e.:

$$C_i = \frac{1}{k_{i1}} + c_{i2} \tag{1}$$

Here, no problem will happen even passive joint exists because all these variables are described by symbols in (1).

Whereas, the stiffness of a parallel mechanism is the summation of stiffness of components in the mechanism, i.e.:

$$K = K_1 + K_2 \tag{2}$$

From the above definition, the overall stiffness of the simple parallel mechanism, in which passive joint stiffness is included, is derived as follow:

$$K = \frac{k_{11}}{1+k_{11}c_{12}} + \frac{k_{21}}{1+k_{21}c_{22}} \tag{3}$$

After getting the overall stiffness of the mechanism shown in (3), we can treat both infinite and zero stiffness as follow:

(a) Infinite stiffness (e.g., any link of the mechanism is assumed as perfectly rigid) can be treated by setting its compliance as zero. For example, $c_{12} = 0$.

(b) Zero stiffness (e.g., one of the joint is passive joint) can be treated by substituting its symbol as zero.For example, substituting 0 for k_{21} in (3) does cause no trouble.

Based on the above idea, in the followed approach, the element of joint compliance matrix will be described by stiffness of each element, and the element of link compliance matrix will be described by compliance of each element. Equation (1) is adopted to calculate the compliance of each sub-arm, and then (2) is used to get the overall stiffness of parallel mechanism.

III. Systematic Approach to Stiffness Analysis

In this section, the general kind of parallel mechanisms and nomenclatures for the 3 dimensional mechanisms are introduced (Note that, the same nomenclatures are used for the planar mechanism but the size of nomenclatures is not shown all). Mechanical element compliance, in which both joint compliance and link compliance are included, is defined relating to each local coordinate. The systematic approach to get the total stiffness of a parallel manipulator is presented and the procedure is then checked by a simple example.

A. Parallel Mechanisms to be Discussed

Fig. 3 shows one sub-arm of a parallel manipulator and the nomenclatures are shown as follows:

n: Number of sub-arms of a general parallel manipulator
m_i:Number of joints of the i-th sub-arm.i=1, 2,…,n
Σ_B: Base coordinate frame,the origin O_B
Σ_{ij}: The ij-th coordinate frame, the origin O_{ij}. j=1, 2,…,m_i
Σ_T: Top-plate coordinate frame, the origin O_T
$\boldsymbol{F} \in R^6$: Force applied at the top-plate, expressed in Σ_B
${}^T\boldsymbol{F} \in R^6$: Force applied at the top-plate, expressed in Σ_T
$\boldsymbol{F}_{ij} \in R^6$: Force acting at the origin O_{ij}, which is equivalent to force $\boldsymbol{F}$ and expressed in frame Σ_{ij}
$\Delta\boldsymbol{r} \in R^6$: Top-plate deflection due to $\boldsymbol{F}$, expressed in Σ_B
${}^T\Delta\boldsymbol{r} \in R^6$: Top-plate deflection due to $\boldsymbol{F}$, expressed in Σ_T
$\Delta\boldsymbol{r}_{ij} \in R^6$: Small deflection due to force $\boldsymbol{F}_{ij}$, expressed in Σ_{ij}
${}^B\boldsymbol{R}_T \in R^{3\times3}$: Rotational matrix from Σ_B to Σ_T
$\boldsymbol{C}_{Li(j-1)} \in R^{6\times6}$: The $i(j-1)$-th link compliance expressed in its local frame $\Sigma_{i(j-1)}$(Note that, Σ_{i0} and Σ_B are the same coordinate and $\boldsymbol{C}_{Li0} = 0$ as the joints included in the i-th sub arm is m_i while links number is $m_i - 1$)
$\boldsymbol{C}^*_{Jij} \in R^{6\times6}$: The ij-th joint compliance expressed in Σ_{ij}
$\boldsymbol{C}^*_{ij} \in R^{6\times6}$: The ij-th compliance, expressed in frame Σ_{ij}
${}^B\boldsymbol{C}^*_{ij} \in R^{6\times6}$:The ij-th mechanical element compliance, expressed in frame Σ_B
${}^B\boldsymbol{C}^*_{i} \in R^{6\times6}$: The i-th sub-arm compliance, expressed in Σ_B
${}^B\boldsymbol{K}^*_{r} \in R^{6\times6}$: Overall stiffness in which passive joint is included
${}^B\boldsymbol{K}_{r} \in R^{6\times6}$: Total stiffness of the mechanism

B. Compliance Definition of Mechanical Element

According to the material strength theory, the force and moment at the endpoint are used while the coordinate is located at the base point. However, the joint force is acting at local coordinate whose origin is the base point. Therefore, we should make the definition of the ij-th compliance $\boldsymbol{C}^*_{ij}$.

The ij-th compliance $\boldsymbol{C}^*_{ij}$ is defined as follow:

$$\Delta\boldsymbol{r}_{ij} = \boldsymbol{C}^*_{ij}\boldsymbol{F}_{ij} \tag{4}$$

As shown in Fig. 4, the deflection on the ij-th coordinates $\Delta\boldsymbol{r}_{ij}$ is the summation of the ij-th joint deflection and the $i(j-1)$-th link deflection, that is

$$\Delta\boldsymbol{r}_{ij} = \Delta\boldsymbol{r}_{Jij} + \Delta\boldsymbol{r}_{Lij} \tag{5}$$

where
$\Delta\boldsymbol{r}_{Jij}$:the ij-th joint deflection, expressed in Σ_{ij}
$\Delta\boldsymbol{r}_{Lij}$:the $i(j-1)$-th link deflection, expressed in Σ_{ij}

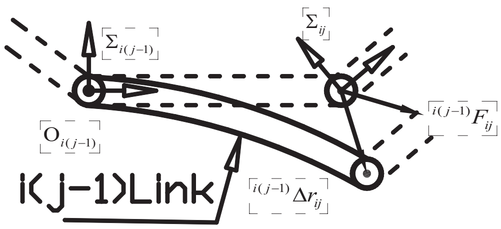

Fig. 4. One sub-arm of a parallel manipulator

The relation between ij-th joint deflection and ij-th joint compliance is expressed as below

$$\Delta \boldsymbol{r}_{Jij} = \boldsymbol{C}^*_{Jij} \boldsymbol{F}_{ij} \tag{6}$$

Based on material science, the followed equation can be obtained.

$$^{i(j-1)}\Delta \boldsymbol{r}_{Lij} = \boldsymbol{C}_{Li(j-1)} {}^{i(j-1)}\boldsymbol{F}_{ij} \tag{7}$$

where

$^{i(j-1)}\Delta \boldsymbol{r}_{Lij}$:$i(j-1)$-th link deflection, expressed in $\Sigma_{i(j-1)}$

$^{i(j-1)}\boldsymbol{F}_{ij}$: Force $\boldsymbol{F}_{ij}$, expressed in $\Sigma_{i(j-1)}$

By using the rotation matrix, the following equations can be gotten

$$^{i(j-1)}\Delta \boldsymbol{r}_{Lij} = \begin{bmatrix} ^{i(j-1)}\boldsymbol{R}_{ij} & 0 \\ 0 & ^{i(j-1)}\boldsymbol{R}_{ij} \end{bmatrix} \Delta \boldsymbol{r}_{Lij} \tag{8}$$

$$^{i(j-1)}\boldsymbol{F}_{ij} = \begin{bmatrix} ^{i(j-1)}\boldsymbol{R}_{ij} & 0 \\ 0 & ^{i(j-1)}\boldsymbol{R}_{ij} \end{bmatrix} \boldsymbol{F}_{ij} \tag{9}$$

Combining (4)-(9), finally we get the ij-th compliance $\boldsymbol{C}^*_{ij}$, as below

$$\boldsymbol{C}^*_{ij} = \boldsymbol{C}^*_{Jij} + \begin{bmatrix} ^{i(j-1)}\boldsymbol{R}_{ij} & 0 \\ 0 & ^{i(j-1)}\boldsymbol{R}_{ij} \end{bmatrix}^{-1} \boldsymbol{C}_{Li(j-1)} \begin{bmatrix} ^{i(j-1)}\boldsymbol{R}_{ij} & 0 \\ 0 & ^{i(j-1)}\boldsymbol{R}_{ij} \end{bmatrix} \tag{10}$$

For a planar mechanism, the ij-th compliance is given by

$$\boldsymbol{C}^*_{ij} = \boldsymbol{C}^*_{Jij} + {}^{i(j-1)}\boldsymbol{R}^{-1}_{ij} \boldsymbol{C}_{Li(j-1)} {}^{i(j-1)}\boldsymbol{R}_{ij} \tag{11}$$

The link deformation and joint deformation of mechanical element are taken into account by one equation. Moreover, one can ignore deformation of some elements by setting their compliance as zero.

C. Procedure of Calculating the Total Stiffness of Parallel Mechanisms

This subsection describes steps of the proposed approach. Setting the coordinates is the first step. Then finding compliance of mechanical element introduced in the above subsection is the second step. Deriving compliance of ij-th mechanical module and deriving compliance of i-th sub arm are the followed two steps. In the fifth step, overall stiffness formulation including passive joint stiffness is derived. Finally, total stiffness of the mechanism can be obtained by substituting passive joint stiffness as zero.

1) 3 Dimensional Mechanism:

[**step1**] Setting the ij-th coordinates Σ_{ij}

D-H coordinates frames Σ_{ij} are assigned.

[**step2**] Finding compliance of mechanical element $\boldsymbol{C}^*_{ij}$

By using (10) (Equation (11) is for planar mechanism), all of the mechanical elements compliance can be obtained.

[**step3**] Deriving compliance of ij-th mechanical module $^B\boldsymbol{C}^*_{ij}$

$^B\boldsymbol{C}^*_{ij}$ is defined as below

$$\Delta \boldsymbol{r} = {}^B\boldsymbol{C}^*_{ij} \boldsymbol{F} \tag{12}$$

The relation between displacement of Σ_{T} and Σ_{ij}, as analyzed in [13], is given by

$$\Delta \boldsymbol{r}_{ij} = \boldsymbol{J}_{ijT} {}^T\Delta \boldsymbol{r} \tag{13}$$

where

$$\boldsymbol{J}_{ijT} = \begin{bmatrix} ^{ij}\boldsymbol{R}_T & [^{ij}\boldsymbol{P}_T \times]^{ij}\boldsymbol{R}_T \\ 0 & ^{ij}\boldsymbol{R}_T \end{bmatrix} \tag{14}$$

The notation $[.\times]$ denotes, for an arbitrary three-dimensional vector,$\boldsymbol{a} = \begin{bmatrix} a_x & a_y & a_z \end{bmatrix}^{\mathrm{T}}$,$[\boldsymbol{a}\times] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$.

The Jacobian matrix (14) relates to displacement and orientation of frame Σ_{T} and Σ_{ij} with respect to Σ_{B}. It can be determined from the homogeneous transform $^{ij}\boldsymbol{T}_T$ ($^{ij}\boldsymbol{R}_T$ and $^{ij}\boldsymbol{P}_T$ are included) relating to Σ_{T} and Σ_{ij}, and it is independent of Σ_{B}.

Then the equivalent force relation is given as the following

$$\boldsymbol{F}_{ij} = (\boldsymbol{J}^{-1}_{ijT})^{\mathrm{T}}\, {}^T\boldsymbol{F} \tag{15}$$

By using the rotational matrix $^B\boldsymbol{R}_T$, the following equation can be easily gotten

$$\Delta \boldsymbol{r} = \begin{bmatrix} ^B\boldsymbol{R}_T & 0 \\ 0 & ^B\boldsymbol{R}_T \end{bmatrix} {}^T\Delta \boldsymbol{r} \tag{16}$$

$$\boldsymbol{F} = \begin{bmatrix} ^B\boldsymbol{R}_T & 0 \\ 0 & ^B\boldsymbol{R}_T \end{bmatrix} {}^T\boldsymbol{F} \tag{17}$$

From combination of (4), (12)-(13), (15)-(17), the compliance of ij-th mechanical module is derived

$$^B\boldsymbol{C}^*_{ij} = \begin{bmatrix} ^B\boldsymbol{R}_T & 0 \\ 0 & ^B\boldsymbol{R}_T \end{bmatrix} \boldsymbol{J}^{-1}_{ijT} \boldsymbol{C}^*_{ij} (\boldsymbol{J}^{-1}_{ijT})^{\mathrm{T}} \begin{bmatrix} ^B\boldsymbol{R}_T & 0 \\ 0 & ^B\boldsymbol{R}_T \end{bmatrix}^{-1} \tag{18}$$

Substituting (14) into (18) and simplifying it, reduces to

$$^B\boldsymbol{C}^*_{ij} = \begin{bmatrix} ^B\boldsymbol{R}_{ij} & -^B\boldsymbol{R}_{ij}[^{ij}\boldsymbol{P}_T \times] \\ 0 & ^B\boldsymbol{R}_{ij} \end{bmatrix} \boldsymbol{C}^*_{ij} \begin{bmatrix} ^B\boldsymbol{R}_{ij} & -^B\boldsymbol{R}_{ij}[^{ij}\boldsymbol{P}_T \times] \\ 0 & ^B\boldsymbol{R}_{ij} \end{bmatrix}^{\mathrm{T}} \tag{19}$$

[**step4**] Deriving compliance of i-th sub arm

The compliance of a serial sub arm is the summation of the compliance of component elements, i.e.

$$^B\boldsymbol{C}^*_i = \sum_{j=1}^{m_i} {}^B\boldsymbol{C}^*_{ij} \tag{20}$$

[**step5**] Deriving overall stiffness of the mechanism

Stiffness of a parallel mechanism is the summation of stiffness of sub arms, i.e.

$$^B\boldsymbol{K}^*_r = \sum_{i=1}^{n} ({}^B\boldsymbol{C}^*_i)^{-1} \tag{21}$$

[**step6**] Deriving total stiffness

The total stiffness of the mechanism is obtained by substituting the passive joint stiffness K_{JPij} as zero.

$$^B\boldsymbol{K}_r = {}^B\boldsymbol{K}^*_r \,|_{K_{JPij}=0} \tag{22}$$

2) Planar Mechanism Formulation Deduction:

The deduction procedure for planar mechanisms is exactly the same as the one for 3D mechanisms. Therefore only some equations which are different from those for 3D mechanisms will be introduced in this section.

In step3, the Jacobian matrix (14) has the form

$$\boldsymbol{J}_{ijT} = \begin{bmatrix} {}^{ij}\boldsymbol{R}_T & -[{}^{ij}\boldsymbol{P}_T \times]^{\mathrm{T}} \\ 0 & 1 \end{bmatrix} \tag{23}$$

The notation $[.\times]$ denotes, for an arbitrary two-dimensional vector, $\boldsymbol{a} = \begin{bmatrix} a_x & a_y \end{bmatrix}^{\mathrm{T}}$, $[\boldsymbol{a}\times] = \begin{bmatrix} -a_y & a_x \end{bmatrix}$.

For a planar manipulator, this $\boldsymbol{J}_{ijT}$ can also be determined from the homogeneous transform ${}^{ij}\boldsymbol{T}_T$. Note that, here ${}^{ij}\boldsymbol{R}_T \in \mathrm{R}^{2\times 2}$, and ${}^{ij}\boldsymbol{P}_T \in \mathrm{R}^{2\times 1}$. Their relation to ${}^{ij}\boldsymbol{T}_T$ is described as below

$${}^{ij}\boldsymbol{T}_T = \begin{bmatrix} {}^{ij}\boldsymbol{R}_T & {}^{ij}\boldsymbol{P}_T \\ 0 & 1 \end{bmatrix} \tag{24}$$

Equation (16) and (17) should be written as

$$\Delta\boldsymbol{r} = {}^{B}\boldsymbol{R}_T{}^{T}\Delta\boldsymbol{r} \tag{25}$$

$$\boldsymbol{F} = {}^{B}\boldsymbol{R}_T{}^{T}\boldsymbol{F} \tag{26}$$

Finally, the compliance of ij-th mechanical module ${}^{B}\boldsymbol{C}^{*}_{ij}$ of a planar mechanism can be obtained as

$${}^{B}\boldsymbol{C}^{*}_{ij} = \begin{bmatrix} {}^{B}\boldsymbol{R}_{ij} & {}^{B}\boldsymbol{R}_{ij}[{}^{ij}\boldsymbol{P}_T\times]^{\mathrm{T}} \\ 0 & 1 \end{bmatrix} \boldsymbol{C}^{*}_{ij} \begin{bmatrix} {}^{B}\boldsymbol{R}_{ij} & {}^{B}\boldsymbol{R}_{ij}[{}^{ij}\boldsymbol{P}_T\times]^{\mathrm{T}} \\ 0 & 1 \end{bmatrix}^{\mathrm{T}} \tag{27}$$

D. An Example for Illuminating the Procedure

The planar mechanism shown in Fig.5 is a 2 DOF planar mechanism and it is used to illuminate the procedure. Its total stiffness is calculated according to the steps of the proposed approach. Note that, the compliance and stiffness matrices are pose independent. For simplicity, the following results are all gotten under the assumption that the mechanism just moves into the position as shown in Fig.5.

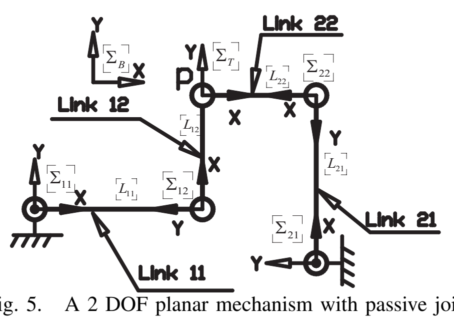

Fig. 5. A 2 DOF planar mechanism with passive joint

(a) The D-H coordinates Σ_{ij} (i=1, 2; j=1, 2, 3) are assigned. (Here, coordinate Σ_{13} and Σ_{23} are same as Σ_{T}.)

(b) The $i(j-1)$-th link stiffness matrix $\boldsymbol{C}_{Li(j-1)}$, expressed in their local coordinates $\Sigma_{i(j-1)}$, is 0 (when j=1),and $\begin{bmatrix} C_{Li(j-1)a} & 0 & 0 \\ 0 & C_{Li(j-1)b} & C_{Li(j-1)c} \\ 0 & C_{Li(j-1)c} & C_{Li(j-1)d} \end{bmatrix}$ (when $j \neq 1$). As shown in the nomenclatures definition section, $\boldsymbol{C}^{*}_{Jij}$ denotes the ij-th joint compliance (Size of $\boldsymbol{C}^{*}_{Jij}$ is 3 by 3 as this is a planar case. Note that, the size of other nomenclatures for planar mechanism will not be expressed in the following sections).

where $\boldsymbol{C}^{*}_{Jij} = \mathrm{diag}(0, 0, K^{-1}_{Jij})$ K_{Jij}:Stiffness of each joint

$C_{Li(j-1)a} = \frac{L_{i(j-1)}}{E_{i(j-1)}A_{i(j-1)}}$, $C_{Li(j-1)b} = \frac{{L_{i(j-1)}}^3}{3E_{i(j-1)}I_{i(j-1)}}$

$C_{Li(j-1)c} = \frac{{L_{i(j-1)}}^2}{2E_{i(j-1)}I_{i(j-1)}}$, $C_{Li(j-1)d} = \frac{L_{i(j-1)}}{E_{i(j-1)}I_{i(j-1)}}$

$L_{i(j-1)}$, $E_{i(j-1)}$, $A_{i(j-1)}$, $I_{i(j-1)}$ are the length, Young's modulus of elasticity, the original cross-sectional area, second moment of area of $i(j$-1)-th link, respectively.

Using (11), we compute all mechanical element compliances, expressed in Σ_{ij} , of the 1st sub-arm as follow.

$$\boldsymbol{C}^{*}_{11} = \boldsymbol{C}^{*}_{J11} \qquad \boldsymbol{C}^{*}_{12} = \begin{bmatrix} C_{L11b} & 0 & C_{L11c} \\ 0 & C_{L11a} & 0 \\ C_{L11c} & 0 & C_{L11d}+C_{J12} \end{bmatrix}$$

$$\boldsymbol{C}^{*}_{13} = \begin{bmatrix} C_{L12b} & 0 & -C_{L12c} \\ 0 & C_{L12a} & 0 \\ -C_{L12c} & 0 & C_{L12d}+C_{J13} \end{bmatrix} \tag{28}$$

The three results above are the compliance, with respect to their local coordinate, and they make it clear that both link and joint deformation of each mechanical element can be expressed in one equation after transformation.

(c) Substituting these mechanical element compliance, shown by (28), into (27) respectively, we get

$${}^{B}\boldsymbol{C}^{*}_{11} = \begin{bmatrix} L^2_{12}C_{J11} & -L_{11}L_{12}C_{J11} & -L_{12}C_{J11} \\ -L_{11}L_{12}C_{J11} & L^2_{11}C_{J11} & L_{11}C_{J11} \\ -L_{12}C_{J11} & L_{11}C_{J11} & C_{J11} \end{bmatrix}$$

$${}^{B}\boldsymbol{C}^{*}_{12} = \begin{bmatrix} C_{L11a}+L^2_{12}(C_{L11d}+C_{J12}) & -L_{12}C_{L11c} & -L_{12}(C_{L11d}+C_{J12}) \\ -L_{12}C_{L11c} & C_{L11b} & C_{L11c} \\ -L_{12}(C_{L11d}+C_{J12}) & C_{L11c} & C_{L11d}+C_{J12} \end{bmatrix}$$

$${}^{B}\boldsymbol{C}^{*}_{13} = \boldsymbol{C}^{*}_{13} \tag{29}$$

(d) Substituting equations of (29) into (20), we get

$${}^{B}\boldsymbol{C}^{*}_{1} = {}^{B}\boldsymbol{C}^{*}_{11} + {}^{B}\boldsymbol{C}^{*}_{12} + {}^{B}\boldsymbol{C}^{*}_{13} \tag{30}$$

(e) Compliance of 2nd sub-arm ${}^{B}\boldsymbol{C}^{*}_{2}$ can be obtained similarly by the way of getting ${}^{B}\boldsymbol{C}^{*}_{1}$. Substituting in (21), we get the overall stiffness of the mechanism ${}^{B}\boldsymbol{K}^{*}_{r}$.

(f) Finally, the total stiffness is obtained by using (22).

For easily seeing the validity and effectiveness of this procedure, several results under some special conditions are shown in the following section.

[Condition1] All the links are rigid

$$({}^{B}\boldsymbol{C}^{*}_{1})^{-1} = \begin{bmatrix} \frac{K_{J12}+K_{J13}}{L^2_{12}} & \frac{K_{J12}}{L_{11}L_{12}} & \frac{K_{J13}}{L_{12}} \\ \frac{K_{J12}}{L_{11}L_{12}} & \frac{K_{J11}+K_{J12}}{L^2_{11}} & 0 \\ \frac{K_{J13}}{L_{12}} & 0 & K_{J13} \end{bmatrix} \tag{31}$$

$$({}^{B}\boldsymbol{C}^{*}_{2})^{-1} = \begin{bmatrix} \frac{K_{J21}+K_{J22}}{L^2_{21}} & \frac{-K_{J22}}{L_{21}L_{22}} & 0 \\ \frac{-K_{J22}}{L_{21}L_{22}} & \frac{K_{J22}+K_{J13}}{L^2_{22}} & \frac{K_{J13}}{L_{22}} \\ 0 & \frac{K_{J13}}{L_{22}} & K_{J13} \end{bmatrix} \tag{32}$$

where K_{Jij} is the ij-th joint stiffness and is inverse of C_{Jij}

From this result, we easily know passive joint stiffness can be treated. If all the passive joint stiffness are substituted by zeros, the active joint stiffness is as follows:

$${}^{B}\boldsymbol{K}_r = \begin{bmatrix} \frac{K_{J21}}{L^2_{21}} & 0 & 0 \\ 0 & \frac{K_{J11}}{L^2_{11}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{33}$$

This result is totally the same with the one obtained by conventional joint stiffness calculation method.

[Condition2] All of joint stiffness including the active joint are assumed zero

$$^{B}\boldsymbol{K}_r = 0 \tag{34}$$

At this time, the total stiffness of the mechanism is zero no matter when the stiffness of links is zero or infinite. It means total stiffness of the mechanism is always zero if all the joints are passive joints. This is accordant with the practice fact.

[Condition3] all the links are assumed rigid except link11.

$$^{B}\boldsymbol{K}_r = \begin{bmatrix} \frac{K_{J21}}{L_{21}^2} & 0 & 0 \\ 0 & \frac{K_{J11}}{L_{11}^2 + K_{J11} C_{L11b}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{35}$$

The above three conditions are used to check the proposed procedure since active joint stiffness, link stiffness, passive joint stiffness, infinite stiffness are all included. Joint stiffness and link stiffness can be treated simultaneously. The same active joint stiffness as the one obtained from conventional joint stiffness analysis method and the obvious physics meaning indicates the proposed procedure is correct.

IV. Comparison with FEA

The total link compliance (inverses of total stiffness), in which all joints are assumed as rigid, obtained by this developed approach and its counterpart derived from FEA model, are compared.

A. Description of the FEA Model

MSC.visualNastran 4D is used to implement these models. The shape of all links is assumed as rectangle. Unit forces are applied at point P and defined in the global coordinates. Then position displacements in the x-direction, y-direction, with respect to the global coordinate, are obtained. At the same time, the orientation displacement of the mechanism can be achieved by computing the deformation of boundary points.

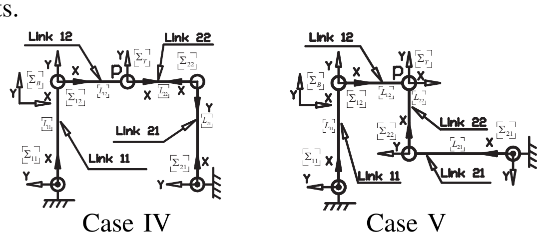

Fig. 6. Mechanism structure of Case IV, and Case V

TABLE I

Design Parameters of Each Case

Symbol	Case I	Case II	Case III	Case IV	Case V
Height	16 mm	4 mm	4 mm	16 mm	16 mm
Width	12 mm	12 mm	6 mm	12 mm	12 mm
Length	$L_{11} = L_{21} = 100$ mm, $L_{12} = L_{22} = 120$ mm				
E	Modulus of Elasticity: $E = 2.0 \times 10^{11}$ pa				

With the purpose of covering all the diverse parameters of the mechanism, 5 different cases are studied. Three FEA models (Case I, case II, and case III) of the analyzed example, shown by Fig.5, are built with different links parameters. Fig.6 shows structures of Case IV and Case V. All links parameters of each case are shown in Table I.

B. Case Studies Results and Comparison

Table II-VI (Position displacement units: m, orientation displacement units: rad) show the results of different cases.These results computed by the proposed approach and those corresponding results obtained from FEA model are very close to each other.Errors between these elements of the results are less than 3.1% with respect to the results derived from FEA model. This confirms the validity of the proposed approach.

TABLE II

Comparison Case:I

Method	Compliance matrix
Proposed approach	$\begin{bmatrix} 1.1517 & 0.2056 & -5.237 \\ 0.2056 & 1.5632 & -12.649 \\ -5.237 & -12.649 & 322.14 \end{bmatrix} \times 10^{-7}$
FEA	$\begin{bmatrix} 1.1171 & 0.2035 & -5.193 \\ 0.2035 & 1.5498 & -12.555 \\ -5.212 & -12.475 & 316.81 \end{bmatrix} \times 10^{-7}$

TABLE III

Comparison Case:II

Method	Compliance matrix
Proposed approach	$\begin{bmatrix} 7.2239 & 1.2824 & -32.647 \\ 1.2824 & 9.8661 & -80.505 \\ -32.647 & -80.505 & 2049.5 \end{bmatrix} \times 10^{-6}$
FEA	$\begin{bmatrix} 7.0124 & 1.2543 & -31.635 \\ 1.2544 & 9.6805 & -79.862 \\ -32.025 & -80.025 & 2137.5 \end{bmatrix} \times 10^{-6}$

TABLE IV

Comparison Case:III

Method	Compliance matrix
Proposed approach	$\begin{bmatrix} 14.448 & 2.5648 & -65.294 \\ 2.5648 & 19.732 & -161.01 \\ -65.294 & -161.01 & 4099.0 \end{bmatrix} \times 10^{-6}$
FEA	$\begin{bmatrix} 14.165 & 2.5375 & -64.008 \\ 2.5372 & 19.503 & -160.33 \\ -64.725 & -160.25 & 4132.5 \end{bmatrix} \times 10^{-6}$

TABLE V

Comparison Case:IV

Method	Compliance matrix
Proposed approach	$\begin{bmatrix} 0.9614 & 0 & 4.3228 \\ 0 & 1.3551 & 0 \\ 4.3228 & 0 & 227.88 \end{bmatrix} \times 10^{-7}$
FEA	$\begin{bmatrix} 0.9622 & 0 & 4.3976 \\ 0 & 1.3423 & 0 \\ 4.4181 & 0 & 226.25 \end{bmatrix} \times 10^{-7}$

Some reasons for the mismatch shown in the results might be due to the following. (1) Very small error exists between the position of point P in the FEA model and very exactly the position of point P in the proposed approach. (2) FEA models have to be re-meshed again and again. The default mesh size should be set before doing the FEA simulation. This produces very small errors as the default mesh size can not be set as very small value as it costs much long time

TABLE VI

COMPARISON CASE:V

Method	Compliance matrix
Proposed approach	$\begin{bmatrix} 1.1517 & -0.2056 & -5.237 \\ -0.2056 & 1.5632 & 12.649 \\ -5.237 & 12.649 & 322.14 \end{bmatrix} \times 10^{-7}$
FEA	$\begin{bmatrix} 1.1171 & -0.2043 & -5.179 \\ -0.2043 & 1.5546 & 12.56 \\ -5.2025 & 12.579 & 317.2 \end{bmatrix} \times 10^{-7}$

to finish the process. (3) The angle displacements (elements of third row of each matrix) are computed by finding the deformation of the boundary points in the FEA model.

V. STIFFNESS ANALYSIS OF PARALLEL MECHANISM WITH FORCE REDUNDANCY

Total stiffness of a mechanisms with force redundancy, shown in Fig.7, is given. To compare with the results obtained by conventional joint stiffness method, all the links compliance and passive joint stiffness are then substituted by zero. The two results calculated by these two methods are totally the same, which validate the proposed approach.

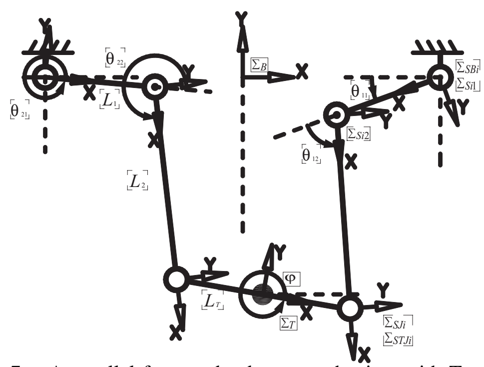

Fig. 7. A parallel force redundancy mechanism with Top-plate

The D-H coordinate frames are assigned. Joint variables are θ_{ij} $(i=1,2; j=1,2)$ and orientation of Top-plate is φ. Length of i1-th, i2-th link, Top-plate are L_1 , L_2 and L_T respectively. The overall stiffness is obtained. Then total joint stiffness is gotten by substituting all the link compliances as zero. The result stiffness matrix is very complex. For simplicity, the following position is chosen for comparison. When the mechanism moves to the pose as below

$$\theta_{11}=0, \theta_{12}=\frac{\pi}{2}, \theta_{21}=2\pi, \theta_{22}=\frac{3\pi}{2}, \varphi=0$$

and if all passive joint stiffness are substituted by zero, the active joint stiffness is gotten as follows:

$$ {}^{B}\boldsymbol{K}_r=\begin{bmatrix} \frac{K_{J12}+K_{J22}}{L_2^2} & \frac{K_{J12}-K_{J22}}{L_1L_2} & \frac{L_T(K_{J12}+K_{J22})}{2L_1L_2} \\ \frac{K_{J12}-K_{J22}}{L_1L_2} & \frac{K_{J11}+K_{J12}+K_{J21}+K_{J22}}{L_1^2} & \frac{L_T(K_{J11}+K_{J22})}{2L_1^2} \\ \frac{L_T(K_{J12}+K_{J22})}{2L_1L_2} & \frac{L_T(K_{J11}+K_{J22})}{2L_1^2} & \frac{L_T^2(K_{J11}+K_{J12}+K_{J21}+K_{J22})}{4L_1^2} \end{bmatrix} \quad (36)$$

Equation (36) is the same as the stiffness results calculated by conventional joint stiffness method.

The proposed approach is effective for both non-redundant mechanism and force redundant mechanism. The stiffness performance can be improved by introducing force redundancy. Take (36) for example, we can distribute $(1,1)$ element of a desired ${}^{B}\boldsymbol{K}_r$ such that $K_{J12}+K_{J22}$ is constant. However, we should pay attention to the control scheme when designing a real mechanism benefits from force redundancy since K_{J12} or K_{J22} will be produced by certain control scheme. That is, the active joints should not all adopt position control scheme with the integral motion in order to avoid big internal force caused by a small positional incoincidence.

VI. CONCLUSION

The main results obtained in this paper summarized as follows.

1) A systematic parametric approach, which is fit for optimal design, to compute the static stiffness of robots with/without force redundancy is proposed. It can compute link stiffness and joint stiffness simultaneously. Stiffness of passive joints and rigid links can be dealt with.

2) The validity of the proposed approach is confirmed by comparing to FEA and conventional joint stiffness method. A comparative analysis between the proposed method and FEA model shows that the mechanism total link compliance results obtained by them are much closed. Analysis of the stiffness matrix of a planar parallel force redundant manipulators is presented. The obtained results show that there are no differences between the active joint stiffness results obtained by conventional joint stiffness method and the proposed approach.

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