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Non-Collision Conditions in Multi-agent Robots Formation using Local Potential Functions

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***Abstract*— An analysis of the convergence and non-collision conditions of a formation control strategy for multi-agent robots based on local potential functions is presented. The goal is to coordinate a group of agents, considered as points in plane, to achieve a particular formation. The control law is designed using local attractive forces only where every agent knows the position of another two agents reducing the requirements of the control law implementation. The control law guarantees the convergence to the desired formation but does not avoid inter-agent collisions. A set of necessary and sufficient non-collision conditions based on the explicit solution of the closed-loop system is derived. The conditions allow to conclude from the initial conditions whether or not the agents will collide. The formal proof is presented for the case of three agents. The result is extended to the case of formations of three unicycles.**

I. INTRODUCTION

During the last 20 years, Multi-agent Robots Systems (MARS) have found a wide range of applications in terrestrial, spatial and oceanic explorations. MARS appear as a new research area [1]. Some advantages can be obtained from the collective behavior of MARS. For instance, the kind of tasks that can be accomplished are inherently more complex than those a single robot can accomplish. Also, the system becomes more flexible and fault-tolerant. The range of applications includes toxic residues cleaning, transportation and manipulation of large objects, alertness and exploration, searching and rescue tasks and simulation of biological entities behaviors.

The field of MAS encompasses different research areas [2]. Motion coordination is one of most important areas, specifically formation control [3], [4]. The goal is to coordinate a group of mobile agents to achieve a particular formation avoiding inter-agent collisions. It is assumed that every agent detects the positions of certain agents to converge to its desired position. The main intention is to achieve desired global behaviors through local interactions [5].

There exist different formation control strategies based on the group architecture, environment or the origin of the cooperation [1], [6]. Local potential functions is one of the most important because the control laws can be designed in decentralized manner using artificial potential functions [7], [8]. In a formation control strategy, the convergence to the desired formation and the avoidance of inter-agent collisions are two fundamental requirements. One approach to tackle the first requirement consists on applying the negative gradient of an attractive potential function (APF) as control signal to each robot, steering every agent to the minimum of this potential function. The APF is designed according to the desired inter-agent distances of a particular formation. A formation control law based in APF only, guarantees the convergence to the desired formation, but inter-agent collisions can occur. One approach to solve the collision problem consists on adding repulsive potential functions (RPF) [8], [9] designed in decentralized manner. The main disadvantage of this strategy is that the convergence to the desired formation is not guaranteed for all initial conditions because the robots can be trapped at undesired equilibrium points. Also, the analysis to calculate this equilibria and the trajectories which do not converge to the desired formation is very complex.

To simplify the analysis of convergence using RPF and to ensure collision-free trajectories at same time, we use a simple formation control strategy based on APF only, which is common in the literature [9]. Then, our principal result is to obtain necessary and sufficient conditions for non-collision based on the exact solution of the closed-loop system. The main idea is to predict, since the initial positions, whether the agents will collide or not. Doing this, the convergence and the non-collision requirements are satisfied in a subset of initial conditions within the workspace. The result has an important application in experimental work where the analysis of initial conditions can be done off-line and the robots can be protected of undesired shocks. In this paper, this preliminary result and its geometric interpretation are studied for the case of three agents. After that, the result is extended to the case of formation control of three unicycles.

The paper is organized as follows. Section II introduces a formal problem statement. Section III describes the formation control strategy for agents, considered as points in plane, using APF. Convergence to the desired formation is demonstrated. In Section IV, non-collision conditions are obtained for the case $n = 3$. In Section V, this result is extended to the case of formation control of unicycles. Section VI presents numerical simulations. Finally, concluding remarks are presented in Section VII.

II. PROBLEM STATEMENT

Denote by $\{R_1, ..., R_n\}$, a set of n agents moving in plane with positions $z_i(t) = [x_i(t), y_i(t)]^T$, $i = 1, ..., n$. The kinematic model of each agent or robot R_i is described by

$$\dot{z}_i = u_i, \quad i = 1, ..., n, \tag{1}$$

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where $u_i = [u_{i1}, u_{i2}]^T \in \Re^2$ is the velocity along the X and Y axis of i-th robot. Let $N_i \subset \{R_1, ..., R_n\}$ denote the subset of robots which can be detected by R_i. Every N_i is a static set defined off-line. Let z_i^* be the desired relative position of R_i in a particular formation. Every z_i^* is established according to inter-agent distance, then $z_i^* = \gamma_i(N_i)$.

For instance, it is possible to establish the desired inter-agent distance for every agent according to position of one neighbor agent with $z_i^* = \gamma_i(z_{i+1})$. Then

$$\begin{aligned} z_i^* &= z_{i+1} + c_{i+1}, \quad i = 1, ..., (n-1) \\ z_n^* &= z_1 + c_1 \end{aligned} \tag{2}$$

where $c_i = [h_i, v_i]^T \in \Re^2$ is a vector which represents the desired distance with respect to R_i within formation. Thus, the desired relative position of R_i is established according to the position of R_{i+1} modulo a certain displacement.

Problem Statement. The control objective is to design a control law $u_i(t) = g_i(N_i(t))$ for every robot R_i, such that $\lim_{t\to\infty}(z_i - z_i^*) = 0$, $i = 1, ..., n$.

Definition 1: The desired relative position of n mobile agents given by (2) is said to be a closed-formation if

$$\sum_{i=1}^{n} c_i = 0. \tag{3}$$

This condition means that the desired formation is a closed polygon.

Definition 2: The centroid of positions is defined by

$$\bar{z}(t) = \sum_{i=1}^{n} z_i(t). \tag{4}$$

For completeness, the following definition is introduced.

Definition 3: Let $V : \Re^n \to \Re$ be a smooth function, the gradient of V is defined by

$$\nabla V(x) = \left[\frac{\partial V}{\partial x_1}, ..., \frac{\partial V}{\partial x_n}\right] \tag{5}$$

III. CONTROL STRATEGY

For system (1), a global potential function is defined by

$$V(z_1, .., z_n) = \sum_{i=1}^{n} V_i \tag{6}$$

where $V_i = \|z_i - z_i^*\|^2$. The function V is positive definite and reaches its global minimum ($V = 0$) when $z_i = z_i^*$, $i = 1, ..., n$. Using this function, we define a control law given by

$$u = [u_1, ..., u_n]^T = -\frac{1}{2} k (\nabla V)^T \tag{7}$$

Theorem 1: Consider the system (1) and the control law (7). Suppose that $k > 0$ and the desired formation is a closed-formation. Then, in the the closed-loop system (1)-(7) the agents converge exponentially to the desired formation, i.e. $\lim_{t\to\infty}(z_i - z_i^*) = 0$, $i = 1, ..., n$. Moreover, the centroid of positions of the n agents remains constant, i.e. $\bar{z}(t) = \bar{z}(0), \forall t \geq 0$.

The proof of Theorem 1 requires a preliminary lemma.

Lemma 1: Let $A \in \Re^{m\times m}$ and $\Delta_r(A)$, $r = 1, ..., m$ the determinant of A with the last $m - r$ rows and columns removed, then A is negative definite if and only if $\Delta_r(A) < 0$ for r odd and $\Delta_r(A) > 0$ for r even.

Proof of Theorem 1. The closed-loop system (1)-(7) has the form

$$\dot{z} = k((A \otimes I_2)z + c), \tag{8}$$

where $z = [z_1, ..., z_n]^T$, $\otimes$ denotes the Kronecker product, $c = [(c_2 - c_1), (c_3 - c_2), \cdots, (c_1 - c_n)]^T$, I_2 is the 2×2 identity and

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 & -2 \end{bmatrix}.$$

A change of coordinates for the closed-system (8) is defined by

$$\begin{bmatrix} e \\ \bar{z} \end{bmatrix} = (Q \otimes I_2)z - c_q, \tag{9}$$

where $e = [e_1, ..., e_{n-1}]^T$,

$$Q = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & & \ddots & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} & \frac{1}{n} \end{bmatrix}, c_q = \begin{bmatrix} c_2 \\ c_3 \\ \vdots \\ c_n \\ 0 \end{bmatrix}.$$

We observe that $e_i = z_i - z_i^*$, $i = 1, ..., n-1$ are the error coordinates of the first $n-1$ agents whereas $\bar{z}$ is the centroid of position defined by (4). The dynamics of the coordinates (9) is given by

$$\begin{bmatrix} \dot{e} \\ \dot{\bar{z}} \end{bmatrix} = k\left\{\left(\begin{bmatrix} \tilde{A} & 0 \\ 0 & 0 \end{bmatrix} \otimes I_2\right) e + \tilde{c}_q\right\}, \tag{10}$$

where $\tilde{A} \in \Re^{(n-1)\times(n-1)}$ and $\tilde{c}_q \in \Re^{n\times 1}$ have the form

$$\tilde{A} = \begin{bmatrix} -3 & 0 & -1 & -1 & -1 & \cdots & -1 & -1 & -1 \\ 1 & -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & \ddots & & & & \vdots & \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ -1 & -1 & -1 & -1 & -1 & \cdots & -1 & 0 & -3 \end{bmatrix},$$

$$\tilde{c}_q = -\left[(c_1 + ... + c_n), 0, \cdots, (c_1 + ... + c_n), 0\right]^T.$$

Due to condition (3), $\tilde{c}_q = 0$. Then, the dynamics of the new coordinates is given by

$$\begin{aligned} \dot{e} &= k(\tilde{A} \otimes I_2)e \\ \dot{\bar{z}} &= 0. \end{aligned} \tag{11}$$

We observe that $\dot{\bar{z}} = 0$ and the dynamics of the coordinates (10) is reduced to dimension $n - 1$. Therefore, the centroid of positions is given by the initial locations of agents, i.e. $\bar{z}(t) = \bar{z}(0)$ and remains constant for all $t \geq 0$.

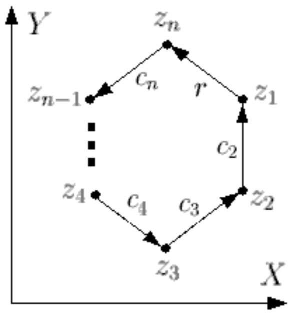

Fig. 1. Positions of agents at equilibrium point

The equilibrium point of the system (11) is $e = 0$. Using Lemma 1, we conclude that $\tilde{A}$ is negative definite since the principal minors of $\tilde{A}$ have the form $\Delta_1(\tilde{A}) = -3$, $\Delta_i(\tilde{A}) = (-1)^i(|\Delta_{i-1}(\tilde{A})| + i + 1)$, $i = 2, ..., n-2$ and $\Delta_{n-1}(\tilde{A}) = (-1)^{n-1}n^2$. Therefore, the trajectories of e_i, $i = 1, ..., n-1$ converge exponentially to zero. This means that the first $n-1$ agents converge to the desired relative position in the formation. Now, let us analyze the distance between robots R_n and R_1. Fig. 1 shows the positions of all agents when e_i, $i = 1, ..., n-1$ converge to zero. By inspection, it follows that $c_2 + c_3 + ... + c_n + r = 0$. Due to condition (3), we deduce that $r = c_1$. Thus, $e_i = 0$, $i = 1, ..., n-1$ implies that $z_n = z_n^*$. Then, we conclude that all agents converge to the desired formation. ■

Remark 1: The closed-loop system (8) is the same that the obtained for a kind of undirected formations graphs given in [9]. The difference is that the design of desired formation (definitions of z_i^*) is simpler than [9]. Also, equation (8) was obtained from the negative gradient of a global function (6). Also, in [9], matrix A is the Laplacian matrix of a kind of undirected formation graph. However, we present a different proof based on the error coordinates.

IV. Non-collision conditions

The control law (7) guarantees that the agents converge exponentially to the desired formation but the inter-agent collisions can occur from some initial agent positions. Collision-free trajectories are defined by

$$f(t) = \|z_i(t) - z_j(t)\|^2 \quad > \quad d^2, \forall t > 0, i \neq j, \quad (12)$$

where d is the diameter of the circle that every agent occupies in the plane. In this section, we analyze the case $n = 3$.

Proposition 1: Suppose that $n = 3$. Then, the trajectories of the closed-loop system (1)-(7) are given by

$$z(t) = \frac{1}{3}\left[(B \otimes I_2)z_0 + s(t)c\right], \quad (13)$$

where $s(t) = 1 - e^{-3kt}$, $z_0 = [z_{10}, z_{20}, z_{30}]^T$, $c = [(c_2 - c_1), (c_3 - c_2), \cdots, (c_1 - c_n)]^T$ and

$$B = \begin{bmatrix} 3 - 2s(t) & s(t) & s(t) \\ s(t) & 3 - 2s(t) & s(t) \\ s(t) & s(t) & 3 - 2s(t) \end{bmatrix}.$$

Proof: A change of coordinates for the closed-loop system (8) is defined by

$$p \quad = \quad (T \otimes I_2)z - c_t, \quad (14)$$

where $T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, $c_t = \begin{bmatrix} c_2 \\ c_3 \\ 0 \end{bmatrix}$.

The dynamics in the new coordinates is

$$\dot{p} \quad = \quad k\left((\tilde{A}_t \otimes I_2)p + \tilde{c}_t\right), \quad (15)$$

where $\tilde{A}_t = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 1 & 2 & 0 \end{bmatrix}$, $\tilde{c}_t = \begin{bmatrix} -c_2 - c_3 - c_1 \\ -c_2 - c_3 - c_1 \\ c_2 + c_3 + c_1 \end{bmatrix}$.

Due to condition (3), $\tilde{c}_t = 0$. Then, the dynamics of the new coordinates is given by $\dot{p} = k(\tilde{A}_t \otimes I_2)p$. We can diagonalize the matrix $\tilde{A}_t \otimes I_2$ through the following similarity transformation

$$(D \otimes I_2) = (P^{-1} \otimes I_2)(\tilde{A}_t \otimes I_2)(P \otimes I_2), \quad (16)$$

where $P = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{3}{2} & 0 & -\frac{1}{2} \\ 1 & 1 & 0 \end{bmatrix}$, $D = k\begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

Then, the solution of (15) is given by

$$p(t) = (PE(t)P^{-1} \otimes I_2)p_0, \quad (17)$$

where $p_0 = p(0)$ is the vector of initial conditions of coordinates p and

$$E(t) = \begin{bmatrix} e^{-3kt} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-3kt} \end{bmatrix}.$$

After some algebra and defining $s(t) = 1 - e^{-3kt}$, the solution in original coordinates can be written as in (13). ■

The desired relative positions of every agent at $t = 0$ are given by $z_{10}^* = z_{20} + c_2$, $z_{20}^* = z_{30} + c_3$ and $z_{30}^* = z_{10} + c_1$. Using this notation, we establish our main result.

Theorem 2: Consider the dynamics of two agents R_i and R_j of the closed-loop system (1)-(7) and suppose that

1) $k > 0$, $n = 3$
2) $\|c_j\|^2 > d^2$, $j = 1, 2, 3$
3) $\|z_{i0} - z_{j0}\|^2 > d^2$, $i \neq j$

Then, anyone of the three following conditions is sufficient to guarantee non-collision, i.e. $f(t) > d^2$, $\forall t > 0$:

i) $(z_{i0}^* - z_{i0})^T(z_{j0} - z_{i0}) \leq 0$.
ii) $(z_{i0}^* - z_{i0})^T(z_{j0} - z_{i0}) \geq \|z_{i0}^* - z_{i0}\|^2$.
iii) $\|z_{i0}^* - z_{i0}\|^2 > (z_{i0}^* - z_{i0})^T(z_{j0} - z_{i0}) > 0$ and

$$\|z_{j0} - z_{i0}\|^2 - \frac{\left((z_{i0}^* - z_{i0})^T(z_{j0} - z_{i0})\right)^2}{\|z_{i0}^* - z_{i0}\|^2} > d^2 \quad (18)$$

Moreover, if $f(t) > d^2$ then necessarily one of the conditions i)-iv) is satisfied.

Hypothesis 2) means that the desired distance between agent R_i and R_j must be greater than d. Hypothesis 3) means that agents do not collide at $t = 0$.

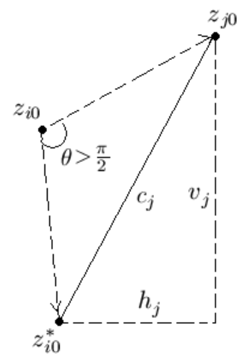

Fig. 2. Agents positions in space in case $\left(z_{i0}^* - z_{i0}\right)^T \left(z_{j0} - z_{i0}\right) < 0$

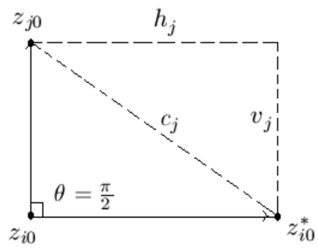

Fig. 3. Agents positions in space in case $\left(z_{i0}^* - z_{i0}\right)^T \left(z_{j0} - z_{i0}\right) = 0$

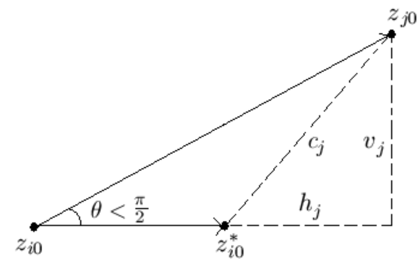

Fig. 4. Agents positions in space in case $\left(z_{i0}^* - z_{i0}\right)^T \left(z_{j0} - z_{i0}\right) \geq \left\|z_{i0}^* - z_{i0}\right\|^2$

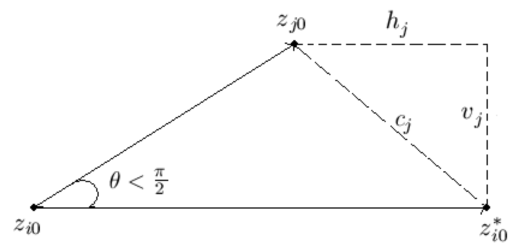

Fig. 5. Agents positions in space in case $\left\|z_{i0}^* - z_{i0}\right\|^2 > \left(z_{i0}^* - z_{i0}\right)^T \left(z_{j0} - z_{i0}\right) > 0$

Proof: (sufficiency) Replacing the explicit solution of two agents z_i, z_j in the condition of collision-free trajectories (12), we obtain $f(t) = s^2(t) \left\|z_{i0}^* - z_{i0}\right\|^2 - 2s(t) \left(z_{i0}^* - z_{i0}\right)^T \left(z_{j0} - z_{i0}\right) + \left\|z_{j0} - z_{i0}\right\|^2$.

Analyzing the function $s(t) = 1 - e^{-3kt}$, we observe that $s(t) \in [0, 1)$. Therefore, $f_i = f(0) = \|z_{j0} - z_{i0}\|^2$ and $f_f = \lim_{t\to\infty} f(t) = \|c_j\|^2$. By hypothesis 2) and 3) $f_i, f_f > d^2$. The derivative of $f(t)$ is given by

$$\frac{d}{dt} f(t) = 6ke^{-3kt}\eta(t), \qquad (19)$$

where $\eta(t) = \left\|z_{i0}^* - z_{i0}\right\|^2 s(t) - \left(z_{i0}^* - z_{i0}\right)^T \left(z_{j0} - z_{i0}\right)$. The derivative of $f(t)$ vanishes only when $\eta(t) = 0$.

proof of i). If $\left(z_{i0}^* - z_{i0}\right)^T \left(z_{j0} - z_{i0}\right) \leq 0$, then $\eta(t)$ is positive and $f(t)$ is monotonously increasing, therefore, $f(t) > d^2$, $\forall t \geq 0$. Fig. 2 and 3 show the position of the agents in these cases.

proof of ii). If $\left(z_{i0}^* - z_{i0}\right)^T \left(z_{j0} - z_{i0}\right) \geq \left\|z_{i0}^* - z_{i0}\right\|^2$, then $\eta(t)$ never crosses by zero, and we conclude that $f(t)$ is monotonously decreasing. Therefore, $f_i > f_f$ and $f_f > d^2$ implies that $f(t) > d^2$, $\forall t \geq 0$. Fig. 4 shows this case.

proof of iii). If $\left\|z_{i0}^* - z_{i0}\right\|^2 > \left(z_{i0}^* - z_{i0}\right)^T \left(z_{j0} - z_{i0}\right) > 0$, then $\eta(t)$ is negative, crosses by zero at time instant t_α and, after that, it is positive. Calculating t_α we obtain

$$t_\alpha = -\frac{1}{k} \ln \left(1 - \frac{\left(z_{i0}^* - z_{i0}\right)^T \left(z_{j0} - z_{i0}\right)}{\left\|z_{i0}^* - z_{i0}\right\|^2}\right) \qquad (20)$$

Evaluating $f(t_\alpha) = \left\|z_{j0} - z_{i0}\right\|^2 - \frac{\left(\left(z_{i0}^* - z_{i0}\right)^T \left(z_{j0} - z_{i0}\right)\right)^2}{\left\|z_{i0}^* - z_{i0}\right\|^2}$.

By condition (18), it is satisfied that $f(t) > d^2$ when $\eta(t_\alpha) = 0$. The distance between agents R_i and R_j decreases to t_α. After that, the distance increases up to c_j. Fig. 5 shows this case.

■

Proof: (necessity). It is necessary to prove that if $f(t) > d^2$ then necessarily i), ii) or iii) hold. Previously, we have shown that $f_i, f_f > d^2$ and that $\frac{d}{dt} f(t)$ vanishes only when $\eta(t) = 0$. Because $s(t)$ is monotonously increasing, the behavior of $f(t)$ can be anyone of the next following cases.

1) $f(t)$ is monotonously increasing because $\eta(t)$ never crosses by zero and remains on the positive interval. This occurs only when condition *i)* is satisfied.
2) $f(t)$ is monotonously decreasing because $\eta(t)$ never crosses by zero and remains on the negative interval. This occurs only when condition *ii)* is satisfied.
3) $f(t)$ is decreasing, crosses by zero at time instant t_α and after that, becomes increasing. This occurs only when condition *iii)* is satisfied.

■

V. Extension to Formation Control of Unicycles

In this section, we extend the analysis of Section IV to the case of formations of unicycles. The kinematic model of each agent R_i, as shown in Fig. 6, is given by

$$\begin{bmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \end{bmatrix} = R_i(\theta_i) \begin{bmatrix} v_i \\ w_i \end{bmatrix}, \quad i = 1, ..., n \qquad (21)$$

where v_i is the linear velocity of the midpoint of the wheels

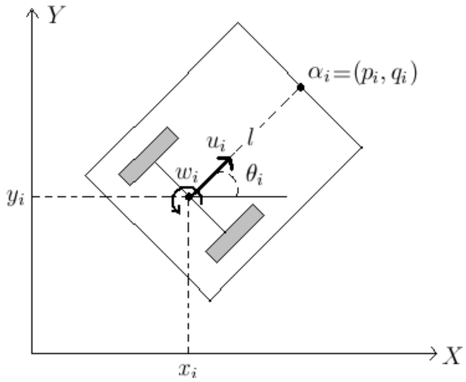

Fig. 6. Kinematic model of unicycles

axis, w_i the angular velocity of the robot and

$$R_i(\theta_i) = \begin{bmatrix} \cos\theta_i & 0 \\ \sin\theta_i & 0 \\ 0 & 1 \end{bmatrix}.$$

It is know [10] that the dynamical system (21) can not be stabilized by continuous and time-invariant control law. Because of this restriction, in the rest of the paper, we will analyze the dynamics of the coordinates $\alpha_i = (p_i, q_i)$ shown in Fig. 6. The coordinates α_i are given by

$$\alpha_i = \begin{bmatrix} p_i \\ q_i \end{bmatrix} = \begin{bmatrix} x_i + l\cos(\theta_i) \\ y_i + l\sin(\theta_i) \end{bmatrix}. \tag{22}$$

The dynamics of (22) are given by $\dot{\alpha}_i = A_i(\theta_i)[v_i, w_i]^T$ where $A_i(\theta_i) = \begin{bmatrix} \cos\theta_i & -l\sin\theta_i \\ \sin\theta_i & l\cos\theta_i \end{bmatrix}$ is the so-called decoupling matrix of every R_i. The decoupling matrix is non-singular because $\det(A(\theta_i)) = l \neq 0$. Following the control strategy of the Section III, the desired inter-agent distance for the n unicycles are established by $\alpha_i^* = \alpha_{i+1} + c_{i+1}$, $i = 1, ..., n-1$ and $\alpha_n^* = \alpha_1 + c_1$. Then, a global potential function is defined by

$$\tilde{V}(\alpha_1, .., \alpha_n) = \sum_{i=1}^{n} \tilde{V}_i \tag{23}$$

where $\tilde{V}_i = \|\alpha_i - \alpha_i^*\|^2$. Thus, the functions $\tilde{V}_i$ are similar to functions V_i but depending on coordinates α_i instead of coordinates z_i. Then, the formation control law is given by

$$\begin{aligned} u &= [v_1, w_1, \cdots, v_n, w_n]^T \qquad (24) \\ &= -\frac{1}{2}k \begin{bmatrix} A^{-1}(\theta_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A^{-1}(\theta_n) \end{bmatrix} \left(\nabla\tilde{V}\right)^T \end{aligned}$$

Corollary 1: Consider the system (21) and the control law (24). Suppose that $k > 0$. Then, in the closed-loop system (21)-(24), the agents converge to the desired formation, i.e. $\lim_{t\to\infty}(\alpha_i - \alpha_i^*) = 0$.

Proof: The dynamics of the coordinates α_i for the closed-loop system (21)-(24) is given by

$$\dot{\alpha}_i = -\frac{1}{2}kA_i(\theta_i)A^{-1}(\theta_i)\left(\frac{\partial\gamma_i}{\partial\alpha_i}\right) = -\frac{1}{2}k\frac{\partial\tilde{V}_i}{\partial\alpha_i}. \tag{25}$$

The closed-loop system of the coordinates α_i has the form

$$\dot{\alpha} = k((A \otimes I_2)\alpha + c), \tag{26}$$

where $\alpha = [\alpha_1, ..., \alpha_n]^T$ and A, I_2, C were previously defined for system (8). It is clear, that the closed-loop system (26) is the same as (8) for the case of point agents. The result follows. ∎

Remark 2: The idea of controlling coordinates α_i instead of the center of the wheels axis is frequently found in the mobile robot literature in order to avoid singularities in the control law.

Remark 3: The control law (24) steers the coordinates α_i to a desired position. However, the angles θ_i remain uncontrolled. These angles do not converge to any specific value. Thus, the control law (24) is to be considered as a formation control without orientation.

VI. Numerical Simulations

Figures 7 and 8 show a simulation for the closed-loop system (1)-(7) for $n = 3$, $d = 2$ and $k = 1$. The desired formation is a triangle with sides length equal to 5. In Fig. 8, d_{ij} is the distance between agent i and j. The initial conditions are given by $z_{10} = [2, 6]^T$, $z_{20} = [2, 2]^T$ and $z_{30} = [-6, 8]^T$. Initially, the agents satisfy $(z_{10}^* - z_{10})^T(z_{20} - z_{10}) = -1.3205 < 0$, $(z_{20}^* - z_{20})^T(z_{30} - z_{20}) = 140$, $\|z_{20}^* - z_{20}\|^2 = 205$, $(z_{30}^* - z_{30})^T(z_{10} - z_{30}) = 96.6603$ and $\|z_{30}^* - z_{30}\|^2 = 150.3205$. We observe that the initial positions of R_1 and R_2 satisfy case *i)* of the Theorem 2. The distance d_{12} increases and the agents do not collide. The conditions for agents R_2 and R_3 or R_3 and R_1 fall in case *iii)*. The conditions of non-collision are satisfied in both cases because $\|z_{30} - z_{20}\|^2 - \frac{\left((z_{20}^* - z_{20})^T(z_{30} - z_{20})\right)^2}{\|z_{20}^* - z_{20}\|^2} = 4.3902 > d^2$ and $\|z_{10} - z_{30}\|^2 - \frac{\left((z_{30}^* - z_{30})^T(z_{10} - z_{30})\right)^2}{\|z_{30}^* - z_{30}\|^2} = 5.8448 > d^2$. We observe in Fig. 8 that d_{23} and d_{31} decrease to some value greater than 2 and then they increase up to the desired value but their values never are less than d^2.

Figures 9 and 10 show another simulation for the same triangle formation but different initial conditions which are $z_{10} = [6, 6]^T$, $z_{20} = [0, -4]^T$ and $z_{30} = [-6, 0]^T$. Now, the agents initially satisfy $(z_{10}^* - z_{10})^T(z_{20} - z_{10}) = 77.6987 < 0$, $\|z_{10}^* - z_{10}\|^2 = 44.3975$, $(z_{20}^* - z_{20})^T(z_{30} - z_{20}) = 82$, $\|z_{20}^* - z_{20}\|^2 = 137$, $(z_{30}^* - z_{30})^T(z_{10} - z_{30}) = 184.0192$ and $\|z_{30}^* - z_{30}\|^2 = 213.0385$. We observe that the initial positions of R_1 and R_2 satisfy the case *ii)* of the Theorem 2. Even though the distance d_{12} in Fig. 10 decreases, the agents do not collide. The other conditions satisfy the case *iii)* where $\|z_{30} - z_{20}\|^2 - \frac{\left((z_{20}^* - z_{20})^T(z_{30} - z_{20})\right)^2}{\|z_{20}^* - z_{20}\|^2} = 2.9197 < d^2$ and $\|z_{10} - z_{30}\|^2 - \frac{\left((z_{30}^* - z_{30})^T(z_{10} - z_{30})\right)^2}{\|z_{30}^* - z_{30}\|^2} = 21.0471 > d^2$. Thus, R_2 and R_3 collide whereas R_3 and R_1 do not collide. We observe in Fig. 10 that d_{23} decreases reaching a value less than d^2 whereas d_{31} never is less than d^2.

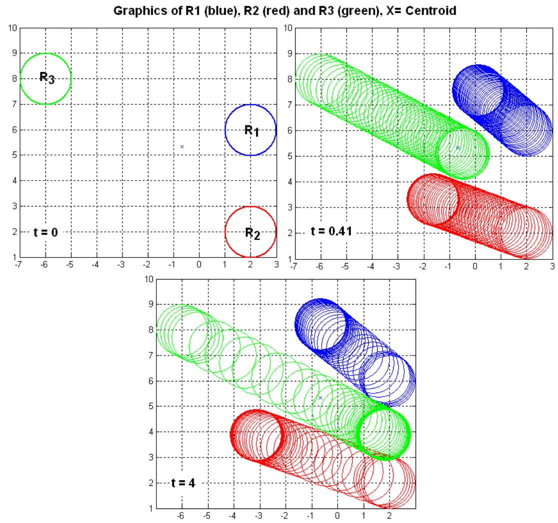

Fig. 7. Agents trajectories in plane

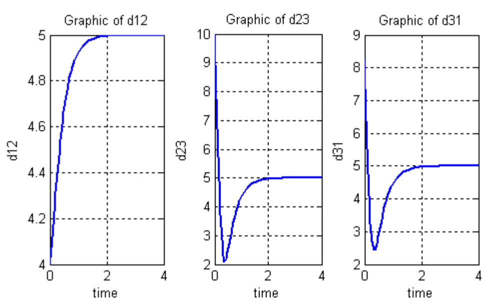

Fig. 8. Inter-agent distances

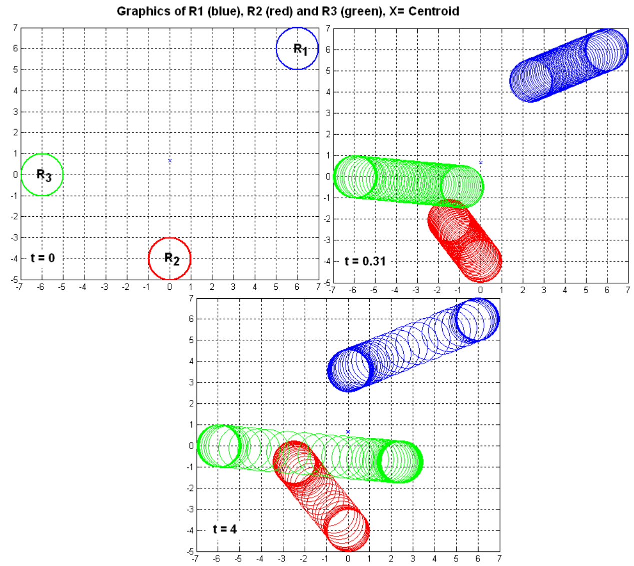

Fig. 9. Agents trajectories in plane

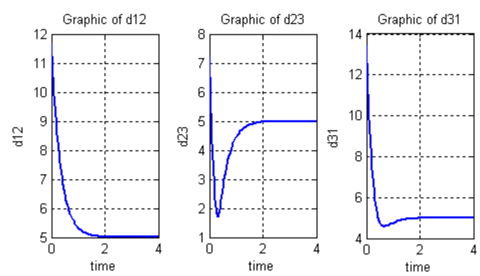

Fig. 10. Inter-agent distances

VII. Conclusions

An analysis of non-collision conditions of a simple formation control strategy for multi-agent robots is presented. The formation control is based on attractive potential functions which ensures the convergence to the desired formation but agents might collide. To ensure collision-free trajectories, a set of necessary and sufficient conditions, based on the exact solution of the closed-loop system is presented. The main idea is to predict, from the initial conditions whether or not the agents will be collide. The main result has an application in experimental work where the analysis of these conditions can be achieved off-line and the robots can be protected of undesired shocks. The control strategy is of easy implementation and avoids the complexity of repulsive forces as non-collision strategy. The result is extended to the case of three unicycles.

VIII. ACKNOWLEDGMENTS

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References

[1] Y. U. Cao, A.S. Fukunaga, A.B. Kahng, Cooperative mobile robotics: Antecedents and directions, *Autonomous Robotics*, vol. 4(1), 1997, pp. 34-46.

[2] T. Arai, E. Pagello, L. E. Parker, Guest Editorial Advances in Multi-robot systems, *IEEE Transactions on Robotics and Automation*, vol. 18(5), 2002, pp. 655-661.

[3] Y. Q. Chen, Z. Wang, Formation control: A Review and a New Consideration, *International Conference on Intelligent Robots and Systems*, 2005, pp. 3181 - 3186.

[4] J.P. Desai, J.P Ostrowski, V. Kumar, Modeling and control of formations of nonholonomic mobile robots, *IEEE Transactions on Robotics and Automation*, vol. 6(17), 2001, pp. 905-908.

[5] Z. Lin, M. Broucke, B. Francis, Local Control Strategies for Groups of Mobile Autonomous Agents, *IEEE Transactions on Automatic Control*, vol. 49(4), 2004, pp. 622-629.

[6] T. Balch, R.C. Arkin, Behavior-based formation control for multirobot teams, *IEEE Transactions on Robotics and Automation*, vol. 14(3), 1998, pp. 926-939.

[7] N. E. Leonard, E. Fiorelli, Virtual Leaders, Artificial Potentials and Coordinated Control of Groups, *IEEE Conference on Decision and Control*, 2001, pp. 2968-2973.

[8] H. G. Tanner, A. Kumar, Towards Decentralization of Multi-robot Navigation Functions, *IEEE Conference on Robotics and Automation*, 2005, pp. 4132-4137.

[9] D. V. Dimarogonas, K. J. Kyriakopoulos, Distributed cooperative control and collision avoidance for multiple kinematic agents *IEEE Conference on Decision and Control*, 2006, pp. 721-726.

[10] R. Brockett, Asymptotic Stability and Feedback Stabilization, in R. Brockett, R.S. Millman, H. J. Sussmann, Eds. *Differential Geometric Control Theory*, Birkhauser, 1983.