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Caging Rigid Polytopes via Finger Dispersion Control

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Abstract— **The object caging problem focuses on designing a formation of fingers that keeps an object within a bounded space without immobilizing it. This paper addresses the problem of designing such formation for object represented by a polytope in any finite dimensional workspace and for any specified number of pointed finger. Our goal is to characterize all the caging sets, each of which corresponds to a largest connected set of initial formations of fingers guaranteed to cage the object, *up to* maintaining a certain class of real-valued measurement induced by the whole fingers' formation below a critical value. In our previous works, such measurement is simply the distance between two fingers (the formation). We found that it is possible to apply the framework based on graph search from the previous works to broader classes of measurements. In this paper, we introduce two of measurements, called *dispersion* and *concentration* and propose a generalized approach to query and to report all caging sets with respect to a given dispersion or concentration.**

I. INTRODUCTION

Originally, the caging problem was posed by Kuperberg in [1] as a problem of designing an algorithm for finding a formation of points that prevents a polygon from moving arbitrarily far from a position. In the past few decades, the concept of caging has been applied to a number of manipulation and related problems such as transportation using mobile robots, part feeding, and grasping [2].

A number of works have proposed solutions for caging problems with different workspace, finger types, caging methods used, number of fingers in the system, and constrains imposed on the object,. Rimon and Blake [3] applied the stratified Morse theory to solve the problem of caging an object in the plane with 1-DOF two-fingers gripper and introduced the notion of caging set (also known as capture region [4]), a set of system configurations (e.g., finger formation) that can prevent the object from escaping. They proposed a solution applies to general planar objects, either polygonal or curved by employing numerical computations. Pipattanasomporn and Sudsang [5], Vahedi and Stappen [6] have independently developed $O(n^2 \log n)$ algorithms for characterizing all two-finger squeezing and stretching cage sets of a polygon with n vertices. They also provided data structures for querying whether a given finger placement forms a cage in $O(\log n)$. Recently, the former group together with Vongmasa [7] extended the previous algorithm to three dimensions while the latter group [8] further developed an approach to cage a convex object with three fingers in two dimensions.

In two finger squeezing or stretching caging, one could establish a cage via setting up a certain finger formation on two opposite concave sections and maintain the distance between the fingers, preventing them from moving too far apart (for squeezing caging) or too close to each other (for stretching caging). In this paper, we propose a solution to object caging problem given any number of controllable point fingers by extending the idea of caging by maintaining the distance between two fingers to caging by maintaining a real-valued measurement induced by the whole formation of fingers. It can be shown that keeping such measurement below an appropriate value after setting up the formations appropriately guarantees to block the object from escaping, caging the object. We call the stated measurement *dispersion* which is a characteristic of the whole formation of fingers that only depends on the relative finger positions within the formation. The more loose the formation of fingers is, the larger its dispersion. Naturally, the following functions provide behaviors meeting our criteria: (a) the sum of squared distance of adjacent fingers in a circular formation: d_2, and (b) the maximum squared distance of adjacent fingers in a circular formation: d_∞, i.e.

$$\begin{aligned} d_2(\boldsymbol{x}) &\mapsto \|\boldsymbol{x}_\phi - \boldsymbol{x}_1\|^2 + \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^2 + ... + \|\boldsymbol{x}_{\phi-1} - \boldsymbol{x}_\phi\|^2, \\ d_\infty(\boldsymbol{x}) &\mapsto \sup\{\|\boldsymbol{x}_\phi - \boldsymbol{x}_1\|^2, \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^2, ..., \|\boldsymbol{x}_{\phi-1} - \boldsymbol{x}_\phi\|^2\}; \end{aligned}$$

where $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_\phi)$ is the formation of ϕ fingers and x_i denotes the position of the i-th finger in such formation. Controlling multiple fingers with some dispersion such as d_∞ allows each finger (or robots) to easily maintain the cage by keeping the distances to its adjacent fellows below a critical value.

This paper presents a solution to such caging problem. The solution generalizes that of our previous works [5] and [7], furthermore applicable to a system any given number of fingers in any dimensional workspace. We propose a) a framework for characterizing all caging sets with respect to a given measurement meeting the criteria along with b) a method to query which caging set corresponds to a given formation if one exists. Like our previous works, instead of performing a search directly on the configuration space, we construct a certain class of roadmap, called caging roadmap, designed to incorporate every formation that potentially leads to a distinct caging formation and perform a graph search algorithm on such roadmap.

The rest of the paper is organized as follows. In the next section, we state our assumptions and review necessary backgrounds concerning the caging sets, the critical measurement,

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and the paths as these are fundamental tools for analyzing and developing caging roadmaps in Section III. A general framework for constructing caging roadmaps, along with an algorithm to report all caging sets with respect to a dispersion or a concentration and to query such caging sets are given in Section IV. Finally, in Section V, we conclude the paper with some discussion and future works.

II. Assumptions and Definitions

We assume that the object $\mathcal{P}$ is a rigid body without internal inaccessible region such that $\mathcal{P}$ can be represented by a bounded simplicial η-complex, $\mathcal{P} \subseteq \mathbb{R}^\eta$ (implying that $\mathbb{R}^\eta \backslash \mathcal{P}$ is a connected component), and all ϕ fingers in the system are dimensionless points. We are interested in finding a condition that guarantees to prevent the object from moving arbitrarily far from the fingers by controlling the dispersion of fingers. For example, suppose that we initially setup a finger formation that immobilizes an object as shown in Fig. 1(i). The object will remain caged as long as the formation dispersion is maintained below some critical value δ. If the formation dispersion is allowed to grow indefinitely, at some point where the dispersion gets larger than the critical value, the object is no longer caged – possibly be brought arbitrarily far from the fingers without penetrating any of them. Alternatively, an object may be caged by maintaining the dispersion greater than some critical value, or maintaining *concentration* of fingers below the value if the fingers start at an appropriate formation (see Fig. 1(ii)). Given an initial formation, a measurement f is used in limiting possible formations so as to cage an object. All possible formations (relative to object's frame of reference) that appear as the object is caged by limiting the possible formation's measurement below a critical value comprises a *caging set with respect to the measurement* f. We aim to identify all of such distinct caging sets and evaluate their critical values given a measurement f, a dispersion or a concentration. It is noted that the union of all caging sets with respect to a measurement is generally a subset of the union of all caging sets, following the definition of caging set in [3]. It has been shown in [6] that for a system with two-fingers, caging sets w.r.t. d_2 and $-d_2$ are all the caging sets while this is not true when more fingers involved.

We define *dispersion* as a convex function that maps from $\mathbb{R}^{\eta\phi}$ (a finger formation) to $\mathbb{R}$ such that the measurement is preserved under rigid transformations of formation i.e. given a dispersion d, for any $(x_1, x_2, ..., x_\phi) \in \mathbb{R}^{\eta\phi}$ such that $x_i \in \mathbb{R}^\eta$, and for any rigid transformation $T : \mathbb{R}^\eta \to \mathbb{R}^\eta$, $d(\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_\phi) = d(T(\boldsymbol{x}_1), T(\boldsymbol{x}_2, ..., T(\boldsymbol{x}_\phi))$. Both d_2 and d_∞, exemplified in the introduction, are also convex function since addition and supremum operators preserve convexity and every squared norm of formation difference is a convex function due to its hessian positive semi-definiteness [9]. In addition, they are invariant to rigid transformations of formation since they only depends on the distances between fingers preserved over rigid transformations. We also define *concentration* in the same manner except that it is a concave function; for example, $-d_2$ and $-d_\infty$ are both concentration

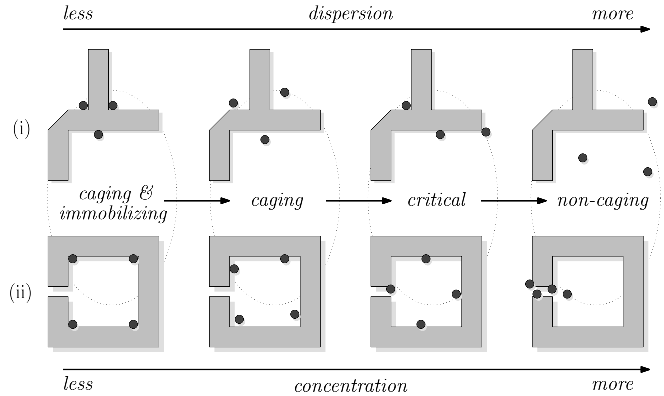

Fig. 1. Classification of formations by (i) dispersion and (ii) concentration.

functions. In this paper, we deal with caging by maintaining dispersion and concentration which are resemble to squeezing and stretching caging, respectively. For convenience, let us refer to dispersion and concentration as $f_\vee$ and $f_\wedge$. If the context applies to both dispersion and concentration, we usually drop the subscripts from $f_\vee$ and $f_\wedge$, i.e. f, called *measurement*.

Let us consider the entire system in the object's frame of reference. Under this frame, the object is fixed near the origin and whether the object is caged depends solely on the ability to moves the fingers arbitrarily far from the object. For the object to escape from a formation of fingers, an *unbounded path* representing such escape motion of fingers must exist. Here, we can define the workspace $\mathcal{W} \subseteq \mathbb{R}^\eta$ as a set of points not occupied by $\text{open}(\mathcal{P})$, and the configuration space as $\mathcal{C} = \mathcal{W}^\phi \subseteq \mathbb{R}^{\eta\phi}$, the set of all valid finger formations. The configuration space $\mathcal{C}$ is connected since $\mathcal{W}$ is connected. We say that ζ is a *path* on $\mathbf{X} \subseteq \mathcal{C}$ if ζ is a continuous function that maps an interval of $\mathbb{R}$ to $\mathbf{X}$. For convenience, we assume that the domain of a path is the open interval $(0, 1)$ and refer to its initial and final point as $\zeta(0)$ and $\zeta(1)$, respectively, without using the limit operators. We list the notations related to paths that will be used in the paper in the following paragraph.

A path is said to be *bounded*, if and only if, its image is bounded. Otherwise, the path is *unbounded*. We can concatenate paths, or conversely subdivide a path into subpaths, say: $\boldsymbol{\zeta} = \boldsymbol{\zeta}_1\boldsymbol{\zeta}_2...\boldsymbol{\zeta}_m$, such that the end point of $\boldsymbol{\zeta}_k$ approaches the starting point of $\boldsymbol{\zeta}_{k+1}$ for any positive integer $k < m$. We define the measurement of a path $\boldsymbol{\zeta}$ with respect to a measurement f as $F(\boldsymbol{\zeta}) \equiv \sup f(\boldsymbol{\zeta}((0,1)))$, i.e. the greatest measurement among all formations in the image of $\boldsymbol{\zeta}$ Again, F may be referred as $F_\vee$ or $F_\wedge$ when f is specified as $f_\vee$ or $f_\wedge$, respectively. We say that $\boldsymbol{\zeta}$ is *non-increasing*, if and only if, $f(\boldsymbol{\zeta})$ is non-increasing and $F(\boldsymbol{\zeta}) = f(\boldsymbol{\zeta}(0))$. While $\boldsymbol{\zeta}$ is non-decreasing, if and only if, $f(\boldsymbol{\zeta})$ is *non-decreasing* and $F(\boldsymbol{\zeta}) = f(\boldsymbol{\zeta}(1))$. The reverse of a non-increasing path $\boldsymbol{\zeta}$ is defined as $\bar{\boldsymbol{\zeta}}$ such that $\bar{\boldsymbol{\zeta}}(t) = \boldsymbol{\zeta}(1-t)$ is *non-decreasing* and vice versa. Given paths $\boldsymbol{\zeta}$ and $\boldsymbol{\xi}$ that connect the same pair of end points, $\boldsymbol{\zeta}$ is said to *dominate* $\boldsymbol{\xi}$, if and only if,

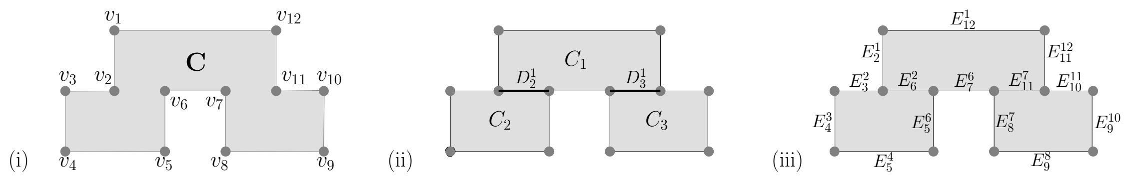

Fig. 2. (i) an example of a two dimensional configuration space $\mathbf{C}$ (shown as a shaded region) where $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_{12}$ are its 0-faces. (ii) and (iii) the structure induced from $\mathbf{C} = C_1 \cup C_2 \cup C_3$. In this situation, observe that D_2^1 and D_3^1 are exactly E_6^2 and E_{11}^7, respectively.

$F(\boldsymbol{\zeta}) \leq F(\boldsymbol{\xi})$. A path $\boldsymbol{\zeta}$ on $\mathbf{X} \subseteq \mathbb{R}^{\eta\phi}$ is called *optimal* on $\mathbf{X}$, if and only if, for every path $\boldsymbol{\xi}$ on $\mathbf{X}$ that dominates $\boldsymbol{\zeta}$, the path $\boldsymbol{\zeta}$ also dominates $\boldsymbol{\xi}$.

Since a path $\boldsymbol{\zeta}$ that starts at $\boldsymbol{x} \in \mathcal{C}$ such that $F(\boldsymbol{\zeta}) = \delta$ corresponds to a motion that starts from an initial formation $\boldsymbol{x}$ such that the formation measurement is kept below δ during the entire motion, the existence of some unbounded path $\boldsymbol{\zeta}$ from $\boldsymbol{x}$ such that $F(\boldsymbol{\zeta}) = f(\boldsymbol{x})$ implies that we cannot guarantee to cage the object using $\boldsymbol{x}$ as an initial formation. (the dispersion at the initial formation is already large enough for the object to escape). Let $\Gamma_{\mathbf{X}}(\boldsymbol{x})$ be the set of all unbounded paths on $\mathbf{X}$ that begins at $\boldsymbol{x}$, and $f^*(\boldsymbol{x})$ be the least measurement among the unbounded paths from $\boldsymbol{x}$ i.e. $f^*(\boldsymbol{x}) \equiv \inf\{F(\Gamma_{\mathcal{C}}(\boldsymbol{x}))\}$.

Definition 1: The set of all *non-caging*, and the set of all *caging* formations in $\mathcal{C}$ with respect to f are:

- $\mathcal{C}^- \equiv \{\boldsymbol{x} \in \mathcal{C} : f^*(\boldsymbol{x}) = f(\boldsymbol{x})\}$, and
- $\mathcal{C}^+ \equiv \{\boldsymbol{x} \in \mathcal{C} : f^*(\boldsymbol{x}) > f(\boldsymbol{x})\}$, respectively,

such that $f^*(\boldsymbol{x})$ is the *critical measurement* when $\boldsymbol{x} \in \mathcal{C}^+$. Observe from the definition that (i) $f^*(\boldsymbol{x}) \geq f(\boldsymbol{x})$ for any $\boldsymbol{x} \in \mathcal{C}$, and the compliment of $\mathcal{C}^-$ is $\mathcal{C}^+$, (ii) $\mathcal{C}^+$ must be bounded. If $\mathcal{C}^+$ is unbounded, then there exists an unbounded path from $\boldsymbol{x}$ that lies entirely in $\mathcal{C}^+$. Let $\boldsymbol{y}$ be the point on this path with $f(\boldsymbol{y})$ equal to the measurement of this path. We then have $f^*(\boldsymbol{y}) = f(\boldsymbol{y})$ contradicting that $\boldsymbol{y}$ is in $\mathcal{C}^+$. This fact allows us to crop the workspace $\mathcal{W}$ with a sufficiently large η-cube $\mathcal{B}$ such that every formation outside and on the boundary of $\mathcal{B}$ is non-caging (the size of $\mathcal{B}$ depends on the measurement). Let us denote the compact version of the configuration space by $\mathbf{C} \equiv \mathbf{W}^\phi$ where $\mathbf{W} \equiv \mathcal{W} \cap \mathcal{B}$, and let $\Gamma'_{\mathbf{X}}(\boldsymbol{x})$ be the set of all paths on $\mathbf{X}$ from $\boldsymbol{x}$ to some non-caging formation. It is possible to evaluate the critical measurement for an initial formation $\boldsymbol{x} \in \mathbf{C}$ by identifying a path with least measurement in $\Gamma'_{\mathbf{C}}(\boldsymbol{x})$ instead of $\Gamma_{\mathcal{C}}(\boldsymbol{x})$.

Lemma 1: $f^*(\boldsymbol{x}) = f'(\boldsymbol{x}) \equiv \inf\{F(\Gamma'_{\mathbf{C}}(\boldsymbol{x}))\}, \forall \boldsymbol{x} \in \mathbf{C}$.

Proof: Suppose that $f'(\boldsymbol{x}) < f^*(\boldsymbol{x})$, there must be a path $\boldsymbol{\zeta}$ in $\Gamma'_{\mathbf{C}}(\boldsymbol{x})$ from $\boldsymbol{x}$ to $\boldsymbol{y} \in \mathcal{C}^-$ such that $F(\boldsymbol{\zeta}) < f^*(\boldsymbol{x})$. However, there exists an unbounded path $\boldsymbol{\xi}$ from $\boldsymbol{y}$ with $F(\boldsymbol{\xi}) = f(\boldsymbol{y})$. The path $\boldsymbol{\zeta\xi}$ is in $\Gamma_{\mathcal{C}}(\boldsymbol{x})$ but $F(\boldsymbol{\zeta\xi}) = F(\boldsymbol{\zeta}) < f^*(\boldsymbol{x})$, this is a contradiction.

In case that $f'(\boldsymbol{x}) > f^*(\boldsymbol{x})$, there must be an unbounded path $\boldsymbol{\zeta}$ in $\Gamma_{\mathcal{C}}(\boldsymbol{x})$ from $\boldsymbol{x}$ such that $F(\boldsymbol{\zeta}) < f'(\boldsymbol{x})$. Since $\mathcal{C}^+$ is bounded, some non-caging formation $\boldsymbol{y}$ must lie on the path $\boldsymbol{\zeta}$. Therefore, $\boldsymbol{\zeta} = \boldsymbol{\alpha\beta}$ such that $\boldsymbol{\beta}$ is an unbounded path from $\boldsymbol{y}$. Since $\boldsymbol{y}$ is non-caging, it follows that $\boldsymbol{\alpha} \in \Gamma'_{\mathbf{C}}(\boldsymbol{x})$ but $F(\boldsymbol{\alpha}) = F(\boldsymbol{\zeta}) < f'(\boldsymbol{x})$, this is a contradiction. ■

Let $\sim$ be a binary relationship on $\mathcal{C}^+$ such that $\boldsymbol{x} \sim \boldsymbol{y}$, if and only if, $f^*(\boldsymbol{x}) = f^*(\boldsymbol{y})$ and $\boldsymbol{x}, \boldsymbol{y}$ are connected by a path $\boldsymbol{\zeta}$ with $F(\boldsymbol{\zeta}) < f^*(\boldsymbol{x})$. Obviously, $\sim$ forms an equivalent relation on $\mathcal{C}^+$, partitioning $\mathcal{C}^+$ into disjoint subsets such that every formation in the same subset has the same critical measurement. All of such disjoint subsets are all distinct *caging sets with respect to* f in the system.

In the next section, we propose the caging roadmap [10], overlaid on $\mathbf{C}$. This caging roadmap not only helps us characterize all caging sets and determine their critical measurements but also holds sufficient information for identifying which caging set contains a given formation.

III. Roadmaps For Object Caging

A caging roadmap consists of dimensionless components in $\mathbf{C}$, called nodes, and their paths on $\mathbf{C}$ that satisfies:

- *accessibility and departability:* each point (finger formation) in $\mathbf{C}$ connects to a node in the roadmap by some non-increasing path in $\mathbf{C}$,
- *connectivity:* for each path $\boldsymbol{\zeta}$ on $\mathbf{C}$ connecting a pair of nodes, there exists a path $\boldsymbol{\zeta}'$ on the roadmap that dominates $\boldsymbol{\zeta}$.

Let $\mathbf{M} \equiv \mathbf{V} \cup \mathbf{E}$ where *Accessibility and departability* ensures that every point in the configuration can access some node in $\mathbf{V}$ with a non-increasing path on $\mathbf{C}$. Together with *connectivity*, the following fact holds:

Proposition 1: For any path $\boldsymbol{\zeta}$ on $\mathbf{C}$, it is possible to construct a path $\boldsymbol{\zeta}' \equiv \boldsymbol{\alpha\beta\gamma}$ that dominates $\boldsymbol{\zeta}$ such that:

- $\boldsymbol{\alpha}(1) \in \mathbf{M}$, $\boldsymbol{\alpha}$ is non-increasing,
- $\boldsymbol{\beta}$ lies on $\mathbf{M}$,
- $\boldsymbol{\gamma}(0) \in \mathbf{M}$, $\boldsymbol{\gamma}$ is non-decreasing.

Proof: By *accessibility and departability*, there exists a non-increasing path $\boldsymbol{\alpha}$ from $\boldsymbol{\zeta}(0)$ to $\boldsymbol{u} \in \mathbf{V}$ and a non-increasing path $\bar{\boldsymbol{\gamma}}$ from $\boldsymbol{\zeta}(1)$ to $\boldsymbol{v} \in \mathbf{V}$. By *connectivity*, $\boldsymbol{\beta}$ on $\mathbf{M}$ that dominates $\bar{\boldsymbol{\alpha}}\boldsymbol{\zeta}\bar{\boldsymbol{\gamma}}$ exists. This means that: $F(\boldsymbol{\beta}) \leq F(\bar{\boldsymbol{\alpha}}\boldsymbol{\zeta}\bar{\boldsymbol{\gamma}})$. Since $F(\bar{\boldsymbol{\alpha}}) = f(\boldsymbol{\zeta}(0)) \leq F(\boldsymbol{\zeta})$ and $F(\bar{\boldsymbol{\gamma}}) = f(\boldsymbol{\zeta}(1)) \leq F(\boldsymbol{\zeta})$, it follows that $F(\boldsymbol{\beta}) \leq F(\boldsymbol{\zeta})$. Therefore, $\boldsymbol{\zeta}' \equiv \boldsymbol{\alpha\beta\gamma}$ dominates $\boldsymbol{\zeta}$. ■

When $\boldsymbol{\zeta}$ is a path on $\mathbf{C}$ from a node to some non-caging formation, the path $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ vanish. The latter also vanishes since the non-caging formation is connected to a node by a non-increasing path and a formation that connects to a non-caging formation by a non-increasing path must be non-caging, implied from the definition. Therefore:

Corollary 1: $f^*(\boldsymbol{x}) = \inf\{F(\Gamma'_{\mathbf{M}}(\boldsymbol{x}))\}, \forall \boldsymbol{x} \in \mathbf{V}$.

Our approach to constructing caging roadmaps $\mathbf{M}_\vee \equiv \mathbf{V}_\vee \cup \mathbf{E}_\vee$ and $\mathbf{M}_\wedge \equiv \mathbf{V}_\wedge \cup \mathbf{E}_\wedge$ for general $f_\vee$ and $f_\wedge$, respectively, operates on a subcover $\{C_1, C_2, ...\}$ of the configuration space (i.e. $\mathbf{C} = \bigcup_i C_i$, see Fig. 2(i), for example) such that $C_1, C_2, ...$ are convex $(\eta\phi)$-faces. We refer to the faces belonging to $C_1, C_2, ...$ as: (a) $\boldsymbol{v}_1, \boldsymbol{v}_2, ...$, the 0-faces, which are also 0-faces of $\mathbf{C}$ (see Fig. 2(i)), (b) D_i^j, the $(\eta\phi - 1)$-face which is an intersection of adjacent C_i and C_j (see Fig. 2(ii)), and (c) E_i^j, the 1-face which links a pair of 0-faces: v_i and v_j (see Fig. 2(iii)). In those faces, we identify nodes (i.e. the local minimum points of f) and paths linking between node pairs. Yet, the strategy employed depends on the applied measurement.

A. Convex Measurement

When the measurement function is convex, say $f_\vee$, it can be implied from Jensen's inequality that:

Lemma 2: For any point $\boldsymbol{x}$ in a convex set $C \subseteq \mathbb{R}^{\eta\phi}$ and any minimum point $\boldsymbol{u}$ of $f_\vee$ in C ($f_\vee(\boldsymbol{u}) = \inf f_\vee(C)$), the path $\boldsymbol{\zeta}(t) = (1-t)\boldsymbol{x} + t\boldsymbol{u}$ lies on C and is non-increasing.

Proof: Since C is convex and $\boldsymbol{\zeta}$ is a straight line; thus, lies in C. Suppose that $f_\vee(\boldsymbol{\zeta}(t_0)) < f_\vee(\boldsymbol{\zeta}(t_1))$ for some $0 \leq t_0 < t_1 \leq 1$. But it can be implied from Jensen's inequality that the measurement at $\boldsymbol{\zeta}(t_1)$ between $\boldsymbol{\zeta}(t_0)$ and $\boldsymbol{\zeta}(1) = \boldsymbol{u}$, is not greater than $f_\vee(\boldsymbol{\zeta}(t_0))$, i.e. $f_\vee(\boldsymbol{\zeta}(t_1)) \leq f_\vee(\boldsymbol{\zeta}(t_0))$, because $f_\vee(\boldsymbol{u}) \leq f_\vee(\boldsymbol{\zeta}(t_0))$, a contradiction. ∎

Given a minimum point $\boldsymbol{u}_i$ in C_i, $f_\vee(\boldsymbol{u}_i) = \inf f_\vee(C_i)$, it follows from the previous lemma that:

Corollary 2: $\mathbf{V}_\vee \equiv \{\boldsymbol{u}_1, \boldsymbol{u}_2, ...\}$ satisfies *accessibility and departability*.

Observe that an image of every path on $C_i \cup C_j$ from $\boldsymbol{u}_i$ to $\boldsymbol{u}_j$ must contain at least one point in D_i^j or the path is not continuous. Let us denote by $\boldsymbol{w}_i^j$ a minimum point of $f_\vee$ in D_i^j, $f_\vee(\boldsymbol{w}_i^j) = \inf f_\vee(D_i^j)$. Clearly, measurement of paths passing $C_i \cap C_j$ cannot be lower than that of $\boldsymbol{w}_i^j$. By Lemma 2, there exists a non-increasing path $\boldsymbol{\zeta}$ from $\boldsymbol{w}_i^j$ to $\boldsymbol{u}_i$ and a non-increasing path $\boldsymbol{\xi}$ from $\boldsymbol{w}_i^j$ to $\boldsymbol{u}_j$ since $\boldsymbol{w}_i^j$ is in both C_i and C_j. Concatenating these paths yields an optimal path $\boldsymbol{\varepsilon}_i^j \equiv \bar{\boldsymbol{\zeta}}\boldsymbol{\xi}$ on $C_i \cup C_j$ because $F_\vee(\boldsymbol{\varepsilon}_i^j) = F_\vee(\bar{\boldsymbol{\zeta}}) = F_\vee(\boldsymbol{\xi}) = f_\vee(\boldsymbol{w}_i^j)$. These optimal paths serve as building blocks for constructing a path connecting any two nodes in $\mathbf{V}$. Given a path in $\mathbf{C}$, it is possible subdivide it into subpaths, each of which passes some D_i^j and is in the union of adjacent convex sets, say $C_i \cup C_j$. We replace each subpath passing D_i^j with $\boldsymbol{\varepsilon}_i^j$ so that the new path results in a concatenation of the optimal paths. Obviously the new path dominates the old one. Consequently, we define $\mathbf{E}_\vee \equiv \bigcup_{i,j} \boldsymbol{\varepsilon}_i^j([0,1])$ so that:

Lemma 3: $\mathbf{M}_\vee$ satisfies *connectivity*.

It follows from Corollary 2 and Lemma 3 that:

Theorem 1: $\mathbf{M}_\vee$ is a caging roadmap.

B. Concave Measurement

Unlike convex measurement, minimum points of concave measurement $f_\wedge$ tends to be at 0-faces of the convex faces in the subcover, see Fig. 3 for example. Let us denote the δ-superlevel set of the concave function $f_\wedge$ by $\mathcal{S}(\delta) \equiv \{\boldsymbol{x} \in \mathbb{R}^{\eta\phi} : f_\wedge(\boldsymbol{x}) > \delta\}$, $\mathcal{S}(\delta)$ is convex for any δ, see [9].

Lemma 4: Any point $\boldsymbol{x}$ in a convex set $C \subseteq \mathbb{R}^{\eta\phi}$ is connected to some extreme point of C by a non-increasing path on C.

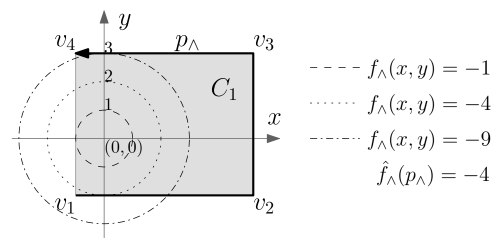

Fig. 3. In this example, C_1 is in the subcover of a two-dimensional $\mathbf{C}$, and the concave measurement is defined as $f_\wedge(x,y) = -(x^2+y^2)$. C_1 has four extreme points $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$, and $\boldsymbol{v}_4$. Observe that one possible optimal path on C_1 from v_1 to v_4 is a path $\boldsymbol{\zeta}_\wedge$ that moves along the straight line segments $\overline{\boldsymbol{v_1 v_2}}$, $\overline{\boldsymbol{v_2 v_3}}$ and then $\overline{\boldsymbol{v_3 v_4}}$.

Proof: Let H be a support hyperplane of $\mathcal{S}(f_\wedge(x))$ at $\boldsymbol{x}$. We denote H^+ the closed halfspace that does not contain $\mathcal{S}(f_\wedge(x))$ and is bounded by H. Clearly, some extreme point of C, say $\boldsymbol{v}$, is in H^+ otherwise $\boldsymbol{x}$ is not in the convex set C. We then have a straight line path on H^+ from $\boldsymbol{x}$ to the extreme point $\boldsymbol{v}$: $\boldsymbol{\zeta}(t) = (1-t)\boldsymbol{x} + t\boldsymbol{v}$; since C is convex. Suppose for a contradiction that $f_\vee(\zeta(t_0)) < f_\vee(\zeta(t_1))$ for some $0 \leq t_0 < t_1 \leq 1$. However, it can be implied from Jensen's inequality that the measurement at the point $\zeta(t_0)$, which is between $\zeta(t_1)$ and $\boldsymbol{x}$, is not less than $f_\wedge(\zeta(t_1))$, i.e. $f_\wedge(\zeta(t_0)) \geq f_\wedge(\zeta(t_1))$, because $\zeta(t_1) \in H^+$, $f_\wedge(\zeta(t_1)) \leq f_\wedge(\boldsymbol{x})$. This is a contradiction. ∎

When the convex set C in the previously stated lemma is a polytope, all of the extreme points of C corresponds to some of its 0-faces. Hence,

Corollary 3: $\mathbf{V}_\wedge \equiv \{\boldsymbol{v}_1, \boldsymbol{v}_2, ...\}$ satisfies *accessibility and departability*.

Let us return to the example in Fig. 3. It can be observed that for any pair of 0-faces of C_1 ($\boldsymbol{v_1}$ and $\boldsymbol{v_2}$, for instance) we can find an path that is optimal on C_1 and is entirely on 1-faces of C_1 (like $\boldsymbol{\zeta}_\wedge$). This motivates us to include, in $\mathbf{E}_\wedge$ all 1-faces of all convex faces in the subcover, $\mathbf{E}_\wedge \equiv \bigcup_{i,j} E_i^j$.

To show that $\mathbf{M}_\wedge$ satisfies *connectivity*, we construct a path $\boldsymbol{\zeta}^{(*)}$ that lies on the roadmap and dominates an arbitrary path $\boldsymbol{\zeta}^{(0)}$ on $\mathbf{C}$ connecting two 0-faces of $\mathbf{C}$. Prior to the construction of $\boldsymbol{\zeta}^{(*)}$, we gradually construct a new path $\boldsymbol{\zeta}^{(1)}$ based on $\boldsymbol{\zeta}^{(0)}$ such that $\boldsymbol{\zeta}^{(1)}$ dominates $\boldsymbol{\zeta}^{(0)}$ and the image of $\boldsymbol{\zeta}^{(1)}$ "gets closer" to the roadmap, and from $\boldsymbol{\zeta}^{(1)}$ we repeat the same process to evolve the path until we obtain a path $\boldsymbol{\zeta}^{(*)}$ which lies on the roadmap and dominates all of its predecessors $\boldsymbol{\zeta}^{(0)}, \boldsymbol{\zeta}^{(1)}, ...$ In each step in the evolution process, the evolving path gets closer to the roadmap because of the following fact.

Lemma 5: Let C, S be convex sets such that C is bounded, $\dim(C) > 1$ and S is open. For any path $\boldsymbol{\zeta}$ on $C \backslash S$ containing points in ∂C (affine boundary of C), some path $\boldsymbol{\zeta}'$ on $(\partial C) \backslash S$ also contains those points.

Proof: Let $\boldsymbol{o}$ be defined as follows: if $C \cap S \neq \varnothing$, $\boldsymbol{o}$ is an interior point of $C \cap S$; otherwise, $\boldsymbol{o}$ is an interior point of C but not in the image of $\boldsymbol{\zeta}$. We denote $\boldsymbol{\zeta}'(t) \in \partial C$ by a projection of $\boldsymbol{\zeta}(t)$ from $\boldsymbol{o}$, that is: $\boldsymbol{o}$, $\boldsymbol{\zeta}(t)$, and $\boldsymbol{\zeta}'(t)$ are co-linear points. Obviously, $\boldsymbol{\zeta}'$ is a path because C is convex. Since S is convex and $\boldsymbol{\zeta}(t)$ that lies between $\boldsymbol{\zeta}'(t)$ and $\boldsymbol{o}$ is

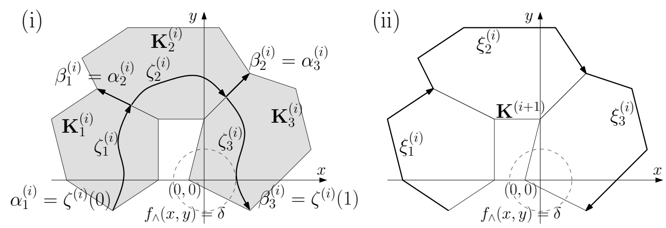

Fig. 4. In this example, we use the same concave measurement as in previous one, $f_\wedge(x,y)=-(x^2+y^2)$. Shown in (i) as shaded region is a two-dimensional $\mathbf{K}^{(i)}=\mathbf{K}_1^{(i)}\cup\mathbf{K}_2^{(i)}\cup\mathbf{K}_3^{(i)}$, there lies the path $\boldsymbol{\zeta}^{(i)}=\boldsymbol{\zeta}_1^{(i)}\boldsymbol{\zeta}_2^{(i)}\boldsymbol{\zeta}_3^{(i)}$ such that $F_\wedge(\boldsymbol{\zeta})=\delta$, (ii) the paths $\boldsymbol{\zeta}^{(i+1)}=\boldsymbol{\xi}_1^{(i)}\boldsymbol{\xi}_2^{(i)}\boldsymbol{\xi}_3^{(i)}$ such that $\boldsymbol{\xi}_k^{(i)}$ dominates $\bar{\boldsymbol{\alpha}}_k^{(i)}\boldsymbol{\zeta}_k^{(i)}\boldsymbol{\beta}_k^{(i)}$ for $k=1,2,3$ lies on $\mathbf{K}^{(i+1)}=\partial\mathbf{K}_1^{(i)}\cup\partial\mathbf{K}_2^{(i)}\cup\partial\mathbf{K}_3^{(i)}$ and dominates $\boldsymbol{\zeta}^{(i)}$.

not in S for any t, $\boldsymbol{\zeta}'$ lies on $(\partial C)\backslash S$. ∎

Since $\mathcal{S}(F_\wedge(\boldsymbol{\zeta}))$ is an open convex set for any path $\boldsymbol{\zeta}$ and by Lemma 5, it follows that:

Corollary 4: For a path $\boldsymbol{\zeta}$ on $C\subseteq\mathbb{R}^{\eta\phi}$ connecting points on ∂C, there exists a path $\boldsymbol{\zeta}'$ on ∂C that dominates $\boldsymbol{\zeta}$ and connecting the same pair of points.

In an evolution step, we evolve the path $\boldsymbol{\zeta}^{(i)}$ on a κ-dimensional structure $\mathbf{K}^{(i)}$ to $\boldsymbol{\zeta}^{(i+1)}$ on a $(\kappa-1)$-dimensional structure $\mathbf{K}^{(i+1)}$. Let $\mathbf{K}^{(i)}=\bigcup_j K_j^{(i)}$ where $K_j^{(i)}$ is a convex κ-face comprising a subcover of $\mathbf{K}^{(i)}$. It is possible to decompose the path $\boldsymbol{\zeta}^{(i)}$ into subpaths: $\boldsymbol{\zeta}^{(i)}=\boldsymbol{\zeta}_1^{(i)}\boldsymbol{\zeta}_2^{(i)}\boldsymbol{\zeta}_3^{(i)}...$; such that each $\boldsymbol{\zeta}_k^{(i)}$ is inside $K_{s(k)}^{(i)}$ (s represents a sequence of adjacent convex faces in the subcover that $\boldsymbol{\zeta}^{(i)}$ traverses through). Applying Lemma 4, on $\boldsymbol{\zeta}_k^{(i)}$, we connect the end points of $\boldsymbol{\zeta}_k^{(i)}$: say $\boldsymbol{\zeta}_k^{(i)}(0)$ and $\boldsymbol{\zeta}_k^{(i)}(1)$; with non-increasing paths, namely $\boldsymbol{\alpha}_k^{(i)}$ and $\boldsymbol{\beta}_k^{(i)}$, to some 0-faces of $K_{s(k)}^{(i)}$ (see Figure 4(i)). By Corollary 4, it is possible to find a path, say $\boldsymbol{\xi}_k^{(i)}$, that lies on $\partial K_{s(k)}^{(i)}$ and dominates $\bar{\boldsymbol{\alpha}}_k^{(i)}\boldsymbol{\zeta}_k^{(i)}\boldsymbol{\beta}_k^{(i)}$. Concatenate each subpaths constructed this way to obtain $\boldsymbol{\zeta}^{(i+1)}=\boldsymbol{\xi}_1^{(i)}\boldsymbol{\xi}_2^{(i)}\boldsymbol{\xi}_3^{(i)}...$ that dominates $\boldsymbol{\zeta}^{(i)}$. Observe that $\boldsymbol{\zeta}^{(i+1)}$ lies on a $(\kappa-1)$-dimensional structure $\mathbf{K}^{(i+1)}=\bigcup_k\partial K_{s(k)}^{(i)}$ (see Figure 4(ii)). This means that, in the next step, each subpaths of $\boldsymbol{\zeta}^{(i+1)}$ will lie on some convex $(\kappa-1)$-face in the subcover of $\mathbf{K}^{(i+1)}$. The path evolution process continues until we obtain a path that lies on a 1-dimensional structure. Since we intend to evolve the path initially on $\mathbf{K}^{(0)}=\mathbf{C}$ whose subcover is given by $\{C_1,C_2,...\}$, $\mathbf{E}_\wedge$ is defined to contain only 1-faces of $\mathbf{C}$, that is: $\mathbf{E}_\wedge\equiv\bigcup_{i,j}E_i^j$.

Lemma 6: $\mathbf{M}_\wedge$ satisfies *connectivity*.

Proof: Given a path $\boldsymbol{\zeta}^{(0)}$ connecting any two nodes. We apply the aforesaid path evolution process in such a way that, at i-th step of the process ($0\leq i<\eta\phi$), the subcover of $\mathbf{K}^{(i)}$ contains only $(\eta\phi-i)$-faces of $C_1,C_2,...$ where $\mathbf{K}^{(0)}=\mathbf{C}=\bigcup_i C_i$. After the process ends, we obtain a path $\boldsymbol{\zeta}^{(\eta\phi-1)}$ that dominates $\boldsymbol{\zeta}^{(0)}$ and lies on 1-faces of $C_1,C_2,...$ which are members of $\mathbf{E}_\wedge$. ∎

Theorem 2: $\mathbf{M}_\wedge$ is a caging roadmap.

IV. Algorithms

The process of characterizing all caging sets and determining all their critical values begins with constructing a caging roadmap of $\mathbf{C}$ for a given $\mathbf{W}$, an instance of a workspace and ϕ the number of fingers. Then, we compute all the critical measurements for each node and partition them into caging sets by running an algorithm resemble to *Dijkstra's shortest path* to propagate the critical measurements throughout the roadmap. Such process is explained in detail in the first two subsections, while the last one concerns how to determine whether a formation is in what caging set.

A. Roadmap Construction

Construction of both $\mathbf{M}_\vee$ and $\mathbf{M}_\wedge$ requires the subcover $\{C_1,C_2,...\}$ of $\mathbf{C}$ such that $C_1,C_2,...$ are convex and their 0-faces are 0-faces of $\mathbf{C}$. One can start from a subcover of $\mathbf{W}$ obtained by decomposing $\mathbf{W}$ into convex η-faces $W_1,W_2,...$ such that 0-faces of $W_1,W_2,...$ are 0-faces of $\mathbf{W}$ i.e. the decomposition does not introduce new vertices. Since $\mathbf{C}=\mathbf{W}^\phi$, it follows that $\{C_1,C_2,...\}\equiv\{W_1^{p(1)}\times W_2^{p(2)}\times...:\sum_j p(j)=\phi\}$. This choice of subcover is valid because cartesian products of convex sets produce a convex set. In the implementation, efficient algorithms for convex decomposition of $\mathbf{W}$ when dimension of $\mathbf{W}$ is two or three are available [11], [12].

Construction of roadmap based on the subcover of $\mathbf{C}$ then proceeds according to the definition of $\mathbf{M}_\vee$ and $\mathbf{M}_\wedge$ stated in the previous section. It can be observed that the amount of $0,1,(\eta\phi-1),\eta\phi$-faces of $\mathbf{C}$ implies that of nodes and edges of the roadmaps. We then let $X(z)=\sum_{i=0}^{\eta}x(i)z^i$ be the generating function of a complex structure $\mathbf{X}$ where $x(i)$ denotes the number of i-faces of $\mathbf{X}$. Analogously, we have $C(z)=\sum_{i=0}^{\eta\phi}c(i)z^i=W(z)^\phi=(\sum_{i=0}^{\eta}w(i)z^i)^\phi$, which means that:

- $|\mathbf{V}_\vee|=c(\eta\phi)=w(\eta)^\phi$,
- $|\mathbf{E}_\vee|=c(\eta\phi-1)=\phi w(\eta)^{\phi-1}w(\eta-1)$,
- $|\mathbf{E}_\wedge|=c(1)=\phi w(0)^{\phi-1}w(1)$,
- $|\mathbf{V}_\wedge|=c(0)=w(0)^\phi$,

The roadmap structure corresponds to a graph structure $(\mathcal{V},\mathcal{E})$ where $\mathcal{V}\equiv\{v_i\}$ and $\mathcal{E}\equiv\{e_i^j\}$ are sets containing the graph's nodes and edges such that e_i^j connects v_i and v_j. It is not necessary to compute the exact coordinates of the nodes nor the geometry of edges. What we need is just the minimal measurement required to traverse between end points connected by a path on each e_i^j, let us call this measurement the *cost* of edge e_i^j which denotes $c(e_i^j)$. This is because we are only interested in the critical measurements, not how the fingers or object escapes. From the previous section, the cost of an edge e_i^j is $F_\vee(\boldsymbol{\varepsilon}_i^j)=f_\vee(\boldsymbol{w}_i^j)$ or $\sup E_i^j$ in case of convex or concave measurement, respectively. We may naively reduce the problem of locating each minimum point to a convex optimization problem with convex or concave objective function (the measurement function) and the linear constraints (D_i^j or E_i^j). This requires to solve at most $|\mathcal{E}|$ of such problem instances.

B. Caging Sets and Critical Measurements

By Corollary 1, we propagate the critical measurements throughout the graph, starting from non-caging nodes on

$\partial(\mathcal{B}^\phi)$, see Section II, with the recurrence:

$$f^*(v_i) = \inf\{\sup\{c(e_i^j), v_j\} : e_i^j \in \mathcal{E}\}$$

The critical measurement propagation works like that in the Dijkstra's shortest path algorithm as in [7] i.e. pick a node that is not visited and has the known least measurement to a non-caging formation, visit it and set that least measurement to the critical measurement, update its adjacent nodes with new least measurement to a non-caging formation, repeat the process until all nodes are visited. An additional disjoint set data structure over the set $\mathcal{V}$ is required so as to characterize the caging sets. Initially, every node in $\mathcal{V}$ belongs to a set. As soon as node v_i is visited, v_i will be put in the same caging set as its adjacent node v_j if $f^*(v_i) = f^*(v_j)$ and the cost of the edge e_i^j is less than the critical measurement i.e. $c(e_i^j) < f^*(v_i)$. After the visit at each node, we perform disjoint set union in $O(\log^* |\mathcal{V}|)$ at most the number of the node's adjacent edges. Since each edge connects two nodes, this additional process requires $O(|\mathcal{E}| \log^* |\mathcal{V}|)$ running time.

C. Caging Set Query

Query a caging set given an arbitrary formation $\boldsymbol{x}$ in $\mathbf{C}$ answers the problem that such formation is either caging or non-caging and its critical measurement if it is caging. To do so, we identify a node in the roadmap that $\boldsymbol{x}$ connects to by a non-increasing path. This can be achieved via identifying some C_i containing $\boldsymbol{x}$. In case of convex measurement, the starting point is $\boldsymbol{u}_i$ by Lemma 2. In case of concave measurement, it is possible to pick any 0-face on H^+ as in Lemma 4 as a starting point. Since there exists a non-increasing path to the starting point, say the node $v \in \mathcal{V}$, the critical measurement at $\boldsymbol{x}$ is $f^*(\boldsymbol{x}) = \sup\{f(\boldsymbol{x}), f^*(v)\}$. To identify such C_i we need a point location algorithm preprocessed on $\mathbf{W}$. Then, we query ϕ times for each finger in the formation $\boldsymbol{x}$ to identify its containing convex sets among $W_1, W_2, \ldots$. The convex set C_i which results from the product of all such convex sets is the container of $\boldsymbol{x}$.

V. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

We proposed an approach designed to efficiently gather and retrieve information required to perform object caging via maintaining dispersion of the fingers. This approach generalizes our previously proposed ones in [5] and [7], extending the proposed algorithm to work with a system with arbitrary many ϕ fingers in any η dimensional workspace. The algorithm is based on constructing the caging roadmap Such roadmap consists of nodes and paths carefully chosen so as to incorporate all the distinct caging sets and facilitate the query task.

B. Future Works

We intend to implement the algorithm to gain more insight on the caging sets when the number of fingers are large. We suspected that caging with dispersion only have stable solutions (large caging sets, the difference between the minimal and the critical measurement) at an amount of fingers for a specific object. For example, three fingers are sufficient to cage a Y-shaped object on a plane if their dispersion (d_2) is kept under a certain critical value (the three fingers are at each concave section of the object). Adding more fingers to the system does not provide any significantly better solutions in this case. Observe that the caging sets of the system with two fingers is a subset of that with three, for example: two of the three fingers are at the same spot; and three finger solutions are parts of four finger solutions, and so on. When many fingers involve, the solutions are mostly degenerated and if the non-degenerated exist, they are likely to be unstable. Since in a system of more than two fingers, caging sets with respect to the convex and concave measurements are not all the caging sets. Roughly speaking, more fingers introduce more strategies in caging i.e. we can only either squeeze or stretch with two fingers; however, with an additional finger, we can do both at the same time. It remains open for further developing an algorithm to the larger set of cages as in [3].

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