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Resolving the Problem of Non-integrability of Nullspace Velocities for Compliance Control of Redundant Manipulators by using Semi-definite Lyapunov functions

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***Abstract*— In this paper a compliance control law for kinematically redundant manipulators is proposed. The controller contains a Cartesian compliance part and a nullspace compliance part which are complemented by a power-conserving decoupling term. The approach deliberately avoids inertia shaping in order to obtain a control law which does not require the measurement of external forces and becomes less sensitive with respect to model uncertainties. While the controller formulation explicitly uses nullspace velocity coordinates, no integration of these velocities is required. Except for the kinematic singularities of the manipulator's Jacobian matrix, no further algorithmic singularities are introduced. Asymptotic stability of the closed-loop system is shown by utilizing semi-definite Lyapunov functions. Finally, a short planar simulation study is presented which validates the effectiveness of the approach.**

I. INTRODUCTION

It is widely accepted in the robotics community that kinematically redundant manipulators have many advantages compared to non-redundant systems from a practical point of view. Kinematically redundancy usually leads to a larger dexterous workspace and provides the control engineer with some freedom for incorporating other important subtasks like optimizations of the joint configuration or obstacle avoidance while performing a given main task related to the end-effector motion [1].

The operational space formulation [2] provides a powerful framework for designing Cartesian controllers both for non-redundant and for redundant manipulators. It aims at achieving a decoupled linear dynamics for the end-effector motion similar to the feedback linearization approach from nonlinear control theory. For controlling the nullspace motion a projection matrix based on the *dynamically consistent* pseudoinverse of the Jacobian matrix is applied.

Park [3] proposed a dynamics formulation in which the Cartesian coordinates are augmented by appropriate nullspace velocities. This was done by extending the Jacobian matrix such that the extended matrix becomes quadratic and non-singular. Thereby, also the dynamically consistent pseudoinverse from the operational space formulation was incorporated. The controllers derived from this approach usually also contain an exactly decoupled Cartesian error dynamics [3], [4], [5].

Baillieul [6] proposed the use of an extended task space in which the Cartesian coordinates are augmented by some additional coordinates which describe the nullspace motion. This, however, introduces in general also new algorithmic singularities due to the particular choice of the new task coordinates. Also a decoupling between the Cartesian motion and the nullspace motion can be achieved only via shaping of the Cartesian inertia matrix.

In this paper the focus is put on the implementation of a compliance control law. Inertia shaping is avoided for two reasons: Firstly, the inertia shaping would require feedback of the external forces. While the external forces actuated at the end-effector can often be directly measured via an additional force/torque sensor mounted at the tip of the robot, the forces exerted on the robot's structure can usually not be measured. Secondly, inertia shaping requires a precise model also in the non-redundant case and therefore sometimes is difficult to be implemented. Avoiding inertia shaping thus allows to implement simpler control laws. However, while the implementation of the controller gets simpler, the stability analysis gets more involved, because the Cartesian dynamics cannot be analyzed independently from the nullspace motion any more [7].

The controller design from this paper is based on an augmentation of the Jacobian matrix, which follows closely the formulation proposed by Park [4], [3]. After a short discussion of the problems in designing a Cartesian compliance controller without inertia shaping in Section II, the proposed approach is presented in Section III. First, the model is reformulated in Section III-A by introducing some nullspace velocity coordinates which complement the Cartesian velocities. Then, the control law is formulated in Section III-B and after a short discussion about the controller parametrization in Section III-C its stability properties are analyzed in Section III-D. A planar simulation study is presented in Section IV. Finally, in Section V the paper is concluded with a short summary.

II. CARTESIAN COMPLIANCE CONTROL

The dynamical model of a robot with n joints can be written as

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) \quad = \quad \boldsymbol{\tau} + \boldsymbol{\tau}_{ext} \ , \qquad (1)$$

with the symmetric and positive definite inertia matrix $\boldsymbol{M}(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$, the centrifugal and Coriolis terms $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \in \mathbb{R}^n$, and the gravity torques $\boldsymbol{g}(\boldsymbol{q}) \in \mathbb{R}^n$. The state variables of the system are the n generalized[1] coordinates $\boldsymbol{q} \in \mathbb{R}^n$ and the corresponding generalized velocities $\dot{\boldsymbol{q}} \in \mathbb{R}^n$. The generalized forces[2] $\boldsymbol{\tau} \in \mathbb{R}^n$ are the control inputs and $\boldsymbol{\tau}_{ext} \in \mathbb{R}^n$ denotes the external generalized forces acting from the environment on the robot. In this model the matrix $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$ can always be chosen such that $\dot{\boldsymbol{M}}(\boldsymbol{q}) = \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})^T$ holds, which is equivalent to the skew symmetry of the matrix $\dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$. It is well known that this property corresponds to the passivity of the system with respect to the input $\boldsymbol{\tau}$ and the output $\dot{\boldsymbol{q}}$.
Furthermore, for the design of the Cartesian compliance a set of $m < n$ Cartesian coordinates are defined via the mapping $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{q}) \in \mathbb{R}^m$. Accordingly, the mapping from joint velocities to Cartesian velocities can be written with the Jacobian matrix $\boldsymbol{J}(\boldsymbol{q}) \in \mathbb{R}^{m\times n}$ as

$$\dot{\boldsymbol{x}} \;=\; \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \qquad \boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} \; . \tag{2}$$

The control goal is now to achieve a given compliance behavior specified by a symmetric and positive definite Cartesian stiffness matrix $\boldsymbol{K} \in \mathbb{R}^{m\times m}$ and a positive definite damping matrix $\boldsymbol{D} \in \mathbb{R}^{m\times m}$ at a constant Cartesian virtual equilibrium position $\boldsymbol{x}_d$. As already mentioned in the introduction, we avoid additional inertia shaping, because this would require feedback of the external generalized forces $\boldsymbol{\tau}_{ext}$ and usually also requires a precise model of the inertia matrix.
In an area where the manipulator's Jacobian matrix has full row-rank, the desired compliance relation can be implemented with a feedback law of the form

$$\boldsymbol{\tau} \;=\; \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{\tau}_c \; , \tag{3}$$

$$\boldsymbol{\tau}_c \;=\; -\boldsymbol{J}(\boldsymbol{q})^T \left(\boldsymbol{K}(\boldsymbol{x}-\boldsymbol{x}_d) + \boldsymbol{D}\dot{\boldsymbol{x}}\right) \; , \tag{4}$$

leading to the Cartesian dynamics

$$\boldsymbol{\Lambda}(\boldsymbol{q})\ddot{\boldsymbol{x}} + \boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{D}\dot{\boldsymbol{x}} + \boldsymbol{K}(\boldsymbol{x}-\boldsymbol{x}_d) = \boldsymbol{\Lambda}(\boldsymbol{q})\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{\tau}_{ext} \; , \tag{5}$$

with

$$\boldsymbol{\Lambda}(\boldsymbol{q}) \;=\; (\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{J}(\boldsymbol{q})^T)^{-1} \; ,$$

$$\boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}) \;=\; \boldsymbol{\Lambda}(\boldsymbol{q}) \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}^{-1}(\boldsymbol{q})\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \dot{\boldsymbol{J}}(\boldsymbol{q})\right)\dot{\boldsymbol{q}} \; ,$$

describing the *Cartesian behavior* but not the nullspace behavior. Notice that for this controller the stability properties of the Cartesian dynamics can not be analyzed independently from the nullspace dynamics, because (5) depends on $\boldsymbol{q}$ and not only on the Cartesian coordinates $\boldsymbol{x}$. This situation would be different for a control approach in which in addition to the compliance implementation also inertia shaping/decoupling would be included in the Cartesian control action.
In the next section the Cartesian control law (4) will be augmented by an additional nullspace compliance as well as a power-conserving decoupling between the Cartesian dynamics and the nullspace dynamics.
Since (4) is specified with respect to some local Cartesian coordinates $\boldsymbol{x}$, the chosen orientation representation in these Cartesian coordinates clearly affects the singularities of the Jacobian matrix and therefore is a relevant issue in practice. Alternatively, one can also use a singularity-free stiffness implementation as the ones proposed by Zhang and Fasse [8] or Natale [9].

[1]positions for prismatic joints and angles for rotational joints
[2]forces for prismatic joints and torques for rotational joints

III. NULLSPACE COMPLIANCE CONTROL

The goal for the design of the nullspace controller is to achieve a compliance behavior specified by positive nullspace stiffness and damping factors $k_n > 0$ and $d_n > 0$. Since the control law does not include inertia shaping, there will always be dynamic couplings between the Cartesian dynamics and the nullspace dynamics. Despite these dynamical couplings it is required that the static stiffness resulting from the Cartesian compliance should not be affected by the additional nullspace control action.

A. Model Reformulation

Before starting with the design of the complete control law, the equations of motion will be reformulated in such a way that the Cartesian dynamics and the nullspace dynamics can be distinguished more easily. Baillieul [6] introduced some additional task coordinates for describing the complete dynamics in task coordinates. This, however, leads in general to additional singularities depending on the particular choice of the task coordinates. Therefore, we follow here instead the approach of Park [4] in which $n-m$ additional nullspace velocity coordinates $\boldsymbol{v}_n = \boldsymbol{N}(\boldsymbol{q})\dot{\boldsymbol{q}}$ are used. The matrix $\boldsymbol{N}(\boldsymbol{q})$ must be designed such that the *extended* Jacobian matrix $\boldsymbol{J}_N(\boldsymbol{q})$ according to

$$\begin{pmatrix}\dot{\boldsymbol{x}}\\ \boldsymbol{v}_n\end{pmatrix} = \boldsymbol{J}_N(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{pmatrix}\boldsymbol{J}(\boldsymbol{q})\\ \boldsymbol{N}(\boldsymbol{q})\end{pmatrix}\dot{\boldsymbol{q}} \; . \tag{6}$$

is non-singular. Notice that these nullspace velocities $\boldsymbol{v}_n$ in general are non-integrable, meaning that there do not exist compatible nullspace coordinates $\boldsymbol{n}(\boldsymbol{q})$ such that $\boldsymbol{N}(\boldsymbol{q}) = \partial\boldsymbol{n}(\boldsymbol{q})/\partial\boldsymbol{q}$. This clearly is an obstacle for designing nullspace compliance controllers. This problem will be overcome in the analysis of this paper by utilizing a stability theorem based on *semi-definite* Lyapunov functions.
One particular choice for $\boldsymbol{N}(\boldsymbol{q})$ which uses the manipulator's inertia matrix as a metric is given by

$$\boldsymbol{N}(\boldsymbol{q}) = \left(\boldsymbol{Z}(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{Z}(\boldsymbol{q})^T\right)^{-1}\boldsymbol{Z}(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q}) \; , \tag{7}$$

where $\boldsymbol{Z}(\boldsymbol{q})$ is a full row-rank nullspace base matrix of $\boldsymbol{J}(\boldsymbol{q})$ fulfilling the condition $\boldsymbol{J}(\boldsymbol{q})\boldsymbol{Z}(\boldsymbol{q})^T = \boldsymbol{0}$ [4]. Such a matrix $\boldsymbol{Z}(\boldsymbol{q})$ can be found, e.g., numerically based on a singular value decomposition of the Jacobian or even in analytic form using the techniques described in [10], [11]. One can show that by this choice the extended Jacobian $\boldsymbol{J}_N(\boldsymbol{q})$ is non-singular indeed and the inverse is given by

$$\boldsymbol{J}_N(\boldsymbol{q})^{-1} \;=\; \begin{bmatrix}\boldsymbol{J}^{M+}(\boldsymbol{q}) & \boldsymbol{Z}(\boldsymbol{q})^T\end{bmatrix} \; , \tag{8}$$

where $\boldsymbol{J}^{M+}(\boldsymbol{q})$ denotes the *dynamically consistent weighted pseudo-inverse* defined as

$$\boldsymbol{J}^{M+}(\boldsymbol{q}) = \boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{J}(\boldsymbol{q})^T(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{J}(\boldsymbol{q})^T)^{-1} .$$

The joint velocity $\dot{\boldsymbol{q}}$ can thus be computed from the Cartesian velocity $\dot{\boldsymbol{x}}$ and the nullspace velocity $\boldsymbol{v}_n$ via

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{M+}(\boldsymbol{q})\dot{\boldsymbol{x}} + \boldsymbol{Z}(\boldsymbol{q})^T\boldsymbol{v}_n . \quad (9)$$

From this it is straightforward to rewrite (1) in the extended velocity coordinates as

$$\boldsymbol{\Lambda}_N(\boldsymbol{q})\begin{pmatrix}\ddot{\boldsymbol{x}}\\ \dot{\boldsymbol{v}}_n\end{pmatrix} + \boldsymbol{\mu}_N(\boldsymbol{q},\dot{\boldsymbol{q}})\begin{pmatrix}\dot{\boldsymbol{x}}\\ \boldsymbol{v}_n\end{pmatrix} = \boldsymbol{J}_N(\boldsymbol{q})^{-T}(-\boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{\tau} + \boldsymbol{\tau}_{ext}) , \quad (10)$$

with the matrices $\boldsymbol{\Lambda}_N(\boldsymbol{q})$ and $\boldsymbol{\mu}_N(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)$ given by

$$\begin{aligned}
\boldsymbol{\Lambda}_N(\boldsymbol{q}) &= \boldsymbol{J}_N(\boldsymbol{q})^{-T}\boldsymbol{M}(\boldsymbol{q})\boldsymbol{J}_N(\boldsymbol{q})^{-1} ,\\
&= \begin{bmatrix}\boldsymbol{\Lambda}_x(\boldsymbol{q}) & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{\Lambda}_n(\boldsymbol{q})\end{bmatrix} ,\\
\boldsymbol{\Lambda}_x(\boldsymbol{q}) &= (\boldsymbol{J}(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{J}(\boldsymbol{q})^T)^{-1} ,\\
\boldsymbol{\Lambda}_n(\boldsymbol{q}) &= \boldsymbol{Z}(\boldsymbol{q})\boldsymbol{M}(\boldsymbol{q})\boldsymbol{Z}(\boldsymbol{q})^T ,
\end{aligned}$$

and (omitting dependence on $\boldsymbol{q}$)

$$\begin{aligned}
\boldsymbol{\mu}_N(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n) &= \begin{bmatrix}\boldsymbol{\mu}_x(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n) & \boldsymbol{\mu}_{xn}(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)\\ \boldsymbol{\mu}_{nx}(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n) & \boldsymbol{\mu}_n(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)\end{bmatrix} ,\\
\boldsymbol{\mu}_x(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n) &= \boldsymbol{\Lambda}_x\left(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \dot{\boldsymbol{J}}\right)\boldsymbol{J}^{M+} ,\\
\boldsymbol{\mu}_{xn}(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n) &= \boldsymbol{\Lambda}_x\left(\boldsymbol{J}\boldsymbol{M}^{-1}\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \dot{\boldsymbol{J}}\right)\boldsymbol{Z}^T ,\\
\boldsymbol{\mu}_{nx}(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n) &= -\boldsymbol{\mu}_{xn}(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)^T ,\\
\boldsymbol{\mu}_n(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n) &= \boldsymbol{\Lambda}_n\left(\boldsymbol{N}\boldsymbol{M}^{-1}\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) - \dot{\boldsymbol{N}}\right)\boldsymbol{Z}^T .
\end{aligned}$$

Notice that the particular choice in (7) for $\boldsymbol{N}(\boldsymbol{q})$ led to a block-diagonal matrix $\boldsymbol{\Lambda}_N(\boldsymbol{q})$. Based on this the property $\boldsymbol{\mu}_{nx}(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n) = -\boldsymbol{\mu}_{xn}(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)^T$ follows directly from the skew symmetry property $\dot{\boldsymbol{\Lambda}}_N(\boldsymbol{q}) = \boldsymbol{\mu}_N(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n) + \boldsymbol{\mu}_N(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)^T$ which is due to $\dot{\boldsymbol{M}}(\boldsymbol{q}) = \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})^T$. A direct proof of this property using the above expressions, however, requires a lengthy derivation. From now on we will use the variables $(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)$ as the state variables instead of $(\boldsymbol{q},\dot{\boldsymbol{q}})$. The relation between these two sets of coordinates is given by (6) and (9). Furthermore, it will be convenient to rewrite the external generalized forces $\boldsymbol{\tau}_{ext}$ as the sum of some Cartesian forces $\boldsymbol{F}_x$ and some nullspace forces $\boldsymbol{F}_n$ which are related to $\boldsymbol{\tau}_{ext}$ via

$$\boldsymbol{\tau}_{ext} = \boldsymbol{J}(\boldsymbol{q})^T\boldsymbol{F}_x + \boldsymbol{N}(\boldsymbol{q})^T\boldsymbol{F}_n . \quad (11)$$

B. Controller Design

In the following a compliance control law for the model (9)-(10) is proposed in which a nullspace compliance term $\boldsymbol{\tau}_n$ and a *power-conserving* decoupling term $\boldsymbol{\tau}_d$ are added to (4), see also Figure 1. The term $\boldsymbol{\tau}_n$ is designed simply by projecting the output torque of a joint level stiffness onto the complement of the range of $\boldsymbol{J}(\boldsymbol{q})^T$ and adding a damping term for the nullspace velocities $\boldsymbol{v}_n$:

$$\boldsymbol{\tau}_n = -\boldsymbol{N}(\boldsymbol{q})^T\boldsymbol{Z}(\boldsymbol{q})k_n\boldsymbol{e}_q - \boldsymbol{N}(\boldsymbol{q})^T d_n\boldsymbol{v}_n .$$

Herein, $\boldsymbol{e}_q := \boldsymbol{q} - \boldsymbol{q}_d$ denotes the deviation of the generalized coordinates from a desired virtual equilibrium configuration $\boldsymbol{q}_d$. In the stability analysis the use of such a projection $\boldsymbol{N}(\boldsymbol{q})^T\boldsymbol{Z}(\boldsymbol{q})$ is often problematic, because in general one cannot find a potential function for the complete stiffness term $-\boldsymbol{N}(\boldsymbol{q})^T\boldsymbol{Z}(\boldsymbol{q})k_n(\boldsymbol{q} - \boldsymbol{q}_d)$. In the analysis below, this problem will be resolved by utilizing some results from the stability theory using semi-definite Lyapunov functions.
The role of the term $\boldsymbol{\tau}_d$ is to eliminate undesired coupling

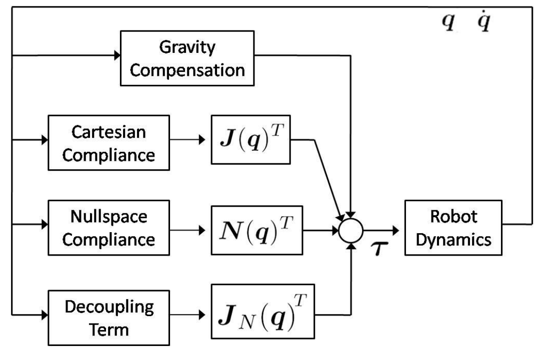

Fig. 1. Control structure.

terms in the dynamical equations for $\dot{\boldsymbol{x}}$ and $\boldsymbol{v}_n$. While the inertia matrix $\boldsymbol{\Lambda}_N(\boldsymbol{q})$ is already in block-diagonal form due to the particular choice of $\boldsymbol{N}(\boldsymbol{q})$, the matrix $\boldsymbol{\mu}_N(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)$ is still fully occupied. Therefore, the corresponding coupling terms in $\boldsymbol{\mu}_N(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)$ are compensated by the feedback

$$\boldsymbol{\tau}_d = \boldsymbol{J}(\boldsymbol{q})^T\boldsymbol{\mu}_{xn}(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)\boldsymbol{v}_n + \boldsymbol{N}(\boldsymbol{q})^T\boldsymbol{\mu}_{nx}(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)\dot{\boldsymbol{x}} .$$

Notice that this is a power-conserving feedback in the sense that the transmitted power $P_d = \boldsymbol{\tau}_d^T\dot{\boldsymbol{q}}$ is always zero due to $\boldsymbol{\mu}_{xn}(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n) = -\boldsymbol{\mu}_{nx}(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)^T$. While in practice these terms admittedly are of minor importance, this term $\boldsymbol{\tau}_d$ is required for the stability analysis.
Summarizing all the different controller actions, we finally have a control law of the form

$$\begin{aligned}
\boldsymbol{\tau} &= \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{\tau}_c + \boldsymbol{\tau}_n + \boldsymbol{\tau}_d , \quad (12)\\
\boldsymbol{\tau}_c &= -\boldsymbol{J}(\boldsymbol{q})^T\left(\boldsymbol{K}(\boldsymbol{f}(\boldsymbol{q}) - \boldsymbol{x}_d) + \boldsymbol{D}\dot{\boldsymbol{x}}\right) ,\\
\boldsymbol{\tau}_n &= -\boldsymbol{N}(\boldsymbol{q})^T\left(k_n\boldsymbol{Z}(\boldsymbol{q})\boldsymbol{e}_q + d_n\boldsymbol{v}_n\right) ,\\
\boldsymbol{\tau}_d &= \boldsymbol{J}(\boldsymbol{q})^T\boldsymbol{\mu}_{xn}(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)\boldsymbol{v}_n + \boldsymbol{N}(\boldsymbol{q})^T\boldsymbol{\mu}_{nx}(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)\dot{\boldsymbol{x}} .
\end{aligned}$$

Accordingly, the closed-loop dynamics can be derived from (9)-(10) with (11) and (12) as

$$\begin{aligned}
\boldsymbol{\Lambda}_x(\boldsymbol{q})\ddot{\boldsymbol{x}} + (\boldsymbol{\mu}_x(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n) + \boldsymbol{D})\dot{\boldsymbol{x}} + \boldsymbol{K}(\boldsymbol{f}(\boldsymbol{q}) - \boldsymbol{x}_d) &= \boldsymbol{F}_x ,\\
\boldsymbol{\Lambda}_n(\boldsymbol{q})\dot{\boldsymbol{v}}_n + (\boldsymbol{\mu}_n(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n) + d_n)\boldsymbol{v}_n + k_n\boldsymbol{Z}(\boldsymbol{q})\boldsymbol{e}_q &= \boldsymbol{F}_n .
\end{aligned}$$

C. Controller Discussion

First of all, one may wonder why the controller depends on a virtual Cartesian equilibrium position $\boldsymbol{x}_d$ as well as on a virtual joint space equilibrium configuration $\boldsymbol{q}_d$ (via $\boldsymbol{e}_q$). In practice it is indeed the best solution if $\boldsymbol{x}_d$ and $\boldsymbol{q}_d$ are chosen in a *compatible* way, i.e. such that $\boldsymbol{f}(\boldsymbol{q}_d) = \boldsymbol{x}_d$ holds. Given only a Cartesian equilibrium position, one can use an appropriate optimization criterion, like e.g. collision avoidance or optimization of a manipulability measure, for determining an optimal pose $\boldsymbol{q}_d$ compatible to $\boldsymbol{x}_d$ in a higher planning level. In case that these values are not chosen in a compatible way, the controller still achieves convergence of the joint configuration $\boldsymbol{q}$ to the set of local constraint minima $\boldsymbol{q}^* = \min_{\boldsymbol{f}(\boldsymbol{q})=\boldsymbol{x}_d} ||\boldsymbol{q}-\boldsymbol{q}_d||_2$, while for $\boldsymbol{f}(\boldsymbol{q}_d)=\boldsymbol{x}_d$ the equilibrium is asymptotically stable (see Proposition 1 in Section III-D below).

Notice that the nullspace compliance term $\boldsymbol{\tau}_n$ for itself is not really a new contribution. The term $\boldsymbol{N}(\boldsymbol{q})^T\boldsymbol{Z}(\boldsymbol{q})$ occurring in the expression for $\boldsymbol{\tau}_n$ is nothing else than a dynamically consistent nullspace projection matrix based on $\boldsymbol{Z}(\boldsymbol{q})$. The main motivation of this work indeed was to find a way how to extend a projection based nullspace compliance control law (without inertia shaping) in order to give a formal proof of asymptotic stability. The inclusion of the power-conserving feedback $\boldsymbol{\tau}_d$, however, deserves a closer examination. Instead of exactly decoupling the Cartesian dynamics and the nullspace dynamics, it merely eliminates the outer-diagonal parts of $\boldsymbol{\mu}_N(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)$. While its computation requires $\boldsymbol{J}(\boldsymbol{q})$, $\dot{\boldsymbol{J}}(\boldsymbol{q})$, and $\boldsymbol{Z}(\boldsymbol{q})$, the time derivative of the nullspace base matrix is not needed. This is an important feature because the computation of $\boldsymbol{Z}(\boldsymbol{q})$ (e.g. if done by a numerical method like a singular value decomposition of $\boldsymbol{J}(\boldsymbol{q})$) does not require the elements of $\boldsymbol{Z}(\boldsymbol{q})$ continuous. In the next section it will be shown that the incorporation of the term $\boldsymbol{\tau}_d$ enables the proof of asymptotic stability of the closed-loop system.

D. Stability Analysis

The stability properties of the closed-loop system can be summarized as follows

Proposition 1: Consider the system (9),(10) with the control law (12). The matrices $\boldsymbol{K},\boldsymbol{D} \in \mathbb{R}^{m\times m}$ are assumed to be symmetric and positive definite and k_n and d_n are positive controller gains. Then the closed-loop system is strict output passive with respect to the input $\boldsymbol{F}_x$ and the output $\dot{\boldsymbol{x}}$. Suppose also that the Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ has full-row rank in the considered workspace and consequently $\boldsymbol{J}_N(\boldsymbol{q})$ is non-singular. If the virtual equilibrium position $\boldsymbol{q}_d$ is compatible with the virtual Cartesian equilibrium position $\boldsymbol{x}_d$ such that $\boldsymbol{f}(\boldsymbol{q}_d)=\boldsymbol{x}_d$ holds, then the equilibrium point $(\boldsymbol{q}=\boldsymbol{q}_d, \dot{\boldsymbol{q}}=\boldsymbol{0})$ is asymptotically stable for the case of free motion, i.e. for $\boldsymbol{\tau}_{ext}=\boldsymbol{0}$.

Proof: The passivity statement can be easily proven by considering the positive semi-definite function

$$S = \frac{1}{2}\dot{\boldsymbol{x}}^T\boldsymbol{\Lambda}_x(\boldsymbol{q})\dot{\boldsymbol{x}} + \frac{1}{2}(\boldsymbol{f}(\boldsymbol{q})-\boldsymbol{x}_d)^T\boldsymbol{K}(\boldsymbol{f}(\boldsymbol{q})-\boldsymbol{x}_d)$$

as a storage function. Using the property $\dot{\boldsymbol{\Lambda}}_x(\boldsymbol{q}) = \boldsymbol{\mu}_x(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)+\boldsymbol{\mu}_x(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)^T$ one can show that the time derivative of this function along a solution of the closed-loop system is given by

$$\dot{S} = -\dot{\boldsymbol{x}}^T\boldsymbol{D}\dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}^T\boldsymbol{F}_x \ ,$$

from which the passivity property of the Proposition follows immediately.

The proof of asymptotic stability will be based on the following two theorems concerning the stability analysis with semi-definite Lyapunov functions [12]. The used notation of *conditional stability* is clarified in the Appendix.

Theorem 1: [12] Consider a system of the form $\dot{\boldsymbol{z}}=\boldsymbol{f}(\boldsymbol{z})$, $\boldsymbol{z}\in\mathbb{R}^n$, with equilibrium point $\boldsymbol{z}^*$. Let $V(\boldsymbol{z})$ be a C^1 positive semi-definite function which has a negative semi-definite time derivative along the solutions of the system, i.e.

$$\dot{V}(\boldsymbol{z}) = \frac{\partial V(\boldsymbol{z})}{\partial \boldsymbol{z}}\boldsymbol{f}(\boldsymbol{z}) \leq 0 \ . \tag{13}$$

Let $\mathcal{A}$ be the largest positively invariant set contained in $\{\boldsymbol{z}\in\mathbb{R}^n | V(\boldsymbol{z})=0\}$. If $\boldsymbol{z}^*$ is asymptotically stable conditionally to $\mathcal{A}$, then it is a stable equilibrium of $\dot{\boldsymbol{z}}=\boldsymbol{f}(\boldsymbol{z})$.

In [13] a further extension of this theorem for passive systems is given. The particular case of strict output passivity[3] is stated in the following theorem.

Theorem 2: [13] Let the system

$$\begin{aligned} \dot{\boldsymbol{z}} &= \boldsymbol{g}_1(\boldsymbol{z}) + \boldsymbol{g}_2(\boldsymbol{z})\boldsymbol{u} \ , \\ \boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{z}) \end{aligned}$$

with state $\boldsymbol{z}\in\mathbb{R}^n$, input $\boldsymbol{u}\in\mathbb{R}^m$, and output $\boldsymbol{y}\in\mathbb{R}^m$ be strictly output passive for the output $\boldsymbol{y}=\boldsymbol{h}(\boldsymbol{z})$. Let further be $\mathcal{A}$ the largest positively invariant set contained in $\{\boldsymbol{z}\in\mathbb{R}^n|\boldsymbol{h}(\boldsymbol{z})=\boldsymbol{0}\}$. If the equilibrium $\boldsymbol{z}^*$ is asymptotically stable conditionally to $\mathcal{A}$, then it is asymptotically stable for $\boldsymbol{u}=\boldsymbol{0}$.

The passivity property required in this theorem was already shown above. From the closed-loop system one can easily see that for the case of free motion the largest positively invariant set contained in $(\boldsymbol{q},\dot{\boldsymbol{x}}=\boldsymbol{0},\boldsymbol{v}_n)$ is given by $\mathcal{A}=\{(\boldsymbol{q},\dot{\boldsymbol{x}}=\boldsymbol{0},\boldsymbol{v}_n)|\boldsymbol{f}(\boldsymbol{q})=\boldsymbol{x}_d\}$. It remains to prove that the equilibrium $(\boldsymbol{q}=\boldsymbol{q}_d,\dot{\boldsymbol{x}}=\boldsymbol{0},\boldsymbol{v}_n=\boldsymbol{0})$ is asymptotically stable conditionally to the set $\mathcal{A}$. Consider, therefore, the Lyapunov function candidate

$$V_A(\boldsymbol{e}_q,\boldsymbol{v}_n) = \frac{1}{2}\boldsymbol{v}_n^T\boldsymbol{\Lambda}_n(\boldsymbol{q})\boldsymbol{v}_n + \frac{k_n}{2}\boldsymbol{e}_q^T\boldsymbol{e}_q$$

which is positive definite in $\mathcal{A}$ and only positive semi-definite in the whole set $(\boldsymbol{q},\dot{\boldsymbol{q}})$. Utilizing $\dot{\boldsymbol{\Lambda}}_n(\boldsymbol{q}) = \boldsymbol{\mu}_n(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)+\boldsymbol{\mu}_n(\boldsymbol{q},\dot{\boldsymbol{x}},\boldsymbol{v}_n)^T$ one can compute the time derivative of this function as (note that $\boldsymbol{\tau}_{ext}=\boldsymbol{0}$)

$$\dot{V}_A(\boldsymbol{e}_q,\boldsymbol{v}_n) \quad = \quad -d_n\boldsymbol{v}_n^T\boldsymbol{v}_n - \boldsymbol{v}_n^T k_n \boldsymbol{Z}(\boldsymbol{q})\boldsymbol{e}_q + k_n\boldsymbol{e}_q^T\dot{\boldsymbol{q}} \ .$$

In the set $\mathcal{A}$ the relation (9) reduces to $\dot{\boldsymbol{q}}=\boldsymbol{Z}(\boldsymbol{q})^T\boldsymbol{v}_n$, such that $\dot{V}_A(\boldsymbol{e}_q,\boldsymbol{v}_n)$ simplifies to

$$\dot{V}_A(\boldsymbol{e}_q,\boldsymbol{v}_n) = -d_n\boldsymbol{v}_n^T\boldsymbol{v}_n \leq 0$$

which ensures stability conditionally to $\mathcal{A}$. In order to show asymptotic stability one can refer to LaSalle's invariance principle. According to this the state converges to the largest positively invariant set contained in $\{(\boldsymbol{q},\dot{\boldsymbol{x}}=\boldsymbol{0},\boldsymbol{v}_n=\boldsymbol{0})|\boldsymbol{f}(\boldsymbol{q})=\boldsymbol{x}_d\}$. By observing the closed-loop system equations one can see that this set is given by $\{(\boldsymbol{q},\dot{\boldsymbol{x}}=\boldsymbol{0},\boldsymbol{v}_n=$

[3]A system $\dot{\boldsymbol{z}}=\boldsymbol{f}(\boldsymbol{z},\boldsymbol{u})$ with input $\boldsymbol{u}$ and output $\boldsymbol{y}$ is said to be strictly output passive if there exists a non-negative function $S(\boldsymbol{z})$ and an $\epsilon>0$ such that $S(\boldsymbol{z}(t))-S(\boldsymbol{z}(0)) \leq \int_0^t(\boldsymbol{y}(s)^T\boldsymbol{u}(s)-\epsilon||\boldsymbol{y}(s)||^2)ds$ holds [13] for all $t>0$.

$\mathbf{0})|\boldsymbol{f}(\boldsymbol{q}) = \boldsymbol{x}_d,\ \boldsymbol{Z}(\boldsymbol{q})\boldsymbol{e}_q = \mathbf{0}\}$. Since $\boldsymbol{Z}(\boldsymbol{q})$ is a full row-rank nullspace base matrix, the point $\boldsymbol{q} = \boldsymbol{q}_d$ is an isolated point in this set. Consequently, the system is asymptotically stable conditionally to the set $\mathcal{A}$. By applying the theorem above also shows the asymptotic stability of the closed-loop system. ■

IV. SIMULATION

For evaluation of the proposed nullspace compliance control law a simulation of a planar four degrees-of-freedom robot was performed. Figure 2 shows a sketch of the model in the starting configuration for the simulations. As Cartesian coordinates the position coordinates of the end-effector have been chosen. Therefore, one has $m = 2$, and consequently a degree of redundancy of $r = n - m = 2$. The computation of $\boldsymbol{Z}(\boldsymbol{q})$ (as well as all the other dynamics and kinematics components) was performed symbolically in Maple and exported as a C-code function.
In a first simulation a step of 10cm in the x-direction of the

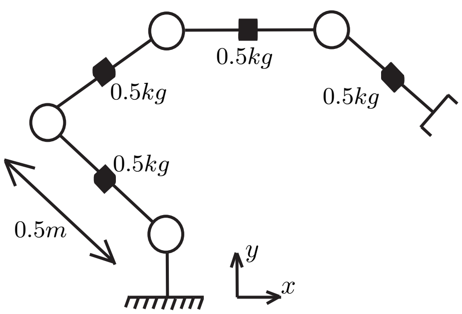

Fig. 2. Simulation model in the start configuration.

Cartesian virtual equilibrium position $\boldsymbol{x}_d$ was commanded with a compatible choice of $\boldsymbol{q}_d$, i.e. such that $\boldsymbol{f}(\boldsymbol{q}_d) = \boldsymbol{x}_d$ holds. The Cartesian errors in $x-$ and $y-$direction are shown in Figure 3. Moreover, Figure 4 shows the Euclidean norm of the joint configuration error $\boldsymbol{e}_q$. In this simulation $\boldsymbol{q}_d$ was chosen compatible to $\boldsymbol{x}_d$. Therefore, the joint error

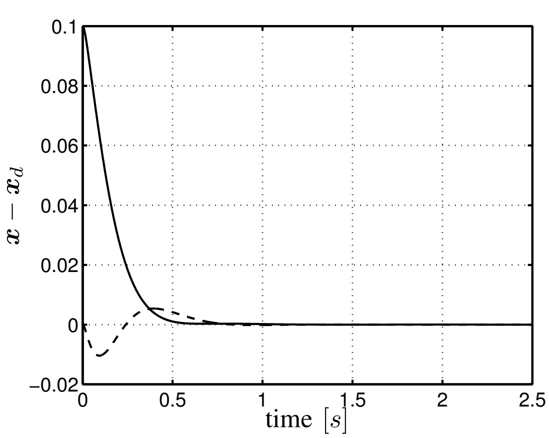

Fig. 3. Step response for the Cartesian coordinates in $x-$ (solid line) and $y-$direction (dashed line).

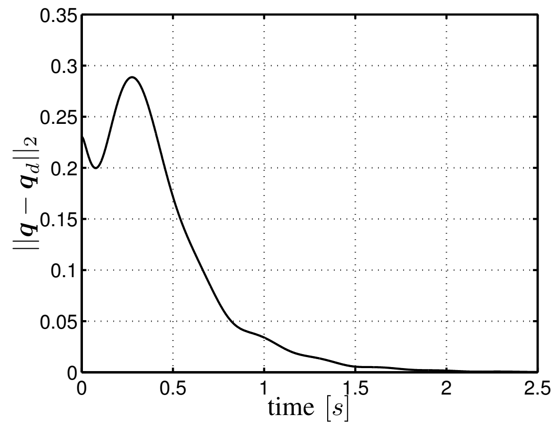

Fig. 4. Joint configuration error for the Cartesian step response.

converges to zero after convergence of the Cartesian error. In a second simulation the role of the decoupling term $\boldsymbol{\tau}_d$ is analyzed more closely. Therefore, a step in $\boldsymbol{q}_d$, with fixed $\boldsymbol{x}_d$, is commanded. The proposed controller is compared to a controller of the form $\boldsymbol{\tau} = \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{\tau}_c + \boldsymbol{\tau}_n$, with $\boldsymbol{\tau}_c$ and $\boldsymbol{\tau}_n$ given in (12), but without $\boldsymbol{\tau}_d$. The Cartesian error is shown in Figure 5. While there is no error at all for the proposed controller (solid line), the error for the simplified controller (dashed and dotted lines) stems from the excitation of the Cartesian dynamics via the centrifugal and Coriolis term $\boldsymbol{\mu}_{xn}(\boldsymbol{q}, \dot{\boldsymbol{x}}, \boldsymbol{v}_n)\boldsymbol{v}_n$. But notice that there still is a coupling between the Cartesian dynamics and the nullspace dynamics due to the dependence of $\boldsymbol{\Lambda}_x(\boldsymbol{q})$ and $\boldsymbol{\mu}_x(\boldsymbol{q}, \dot{\boldsymbol{x}}, \boldsymbol{v}_n)$ on $\boldsymbol{q}$ and $\dot{\boldsymbol{x}}$. This can be seen in Figure 6, where the deviation of the eigenvalues λ_i $(i = 1, 2)$ of the Cartesian inertia matrix $\boldsymbol{\Lambda}_x(\boldsymbol{q})$ from their initial values are depicted. Finally, the joint error for this simulation is shown in Figure 7. Since $\boldsymbol{q}_d$ is not chosen compatible to $\boldsymbol{x}_d$ in this case, the joint error does not converge to zero, but it converges to a constraint local minimum $\boldsymbol{q}^* = \min_{\boldsymbol{f}(\boldsymbol{q}) = \boldsymbol{x}_d} ||\boldsymbol{q} - \boldsymbol{q}_d||_2$.

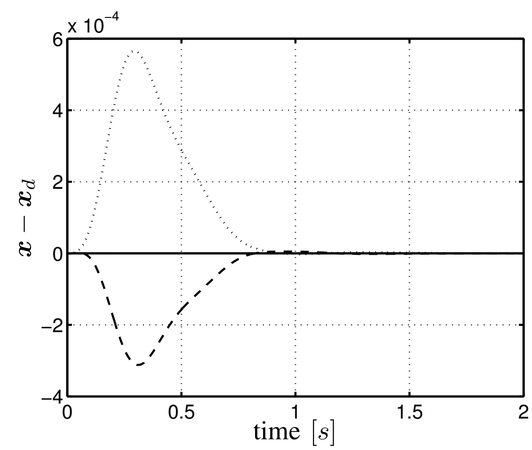

Fig. 5. Cartesian coordinates for the step in $\boldsymbol{q}_d$, with fixed $\boldsymbol{x}_d$ The solid lines (equal to zero) show the simulation result with the proposed controller, while the dashed ($x-$coordinate) and dotted ($y-$coordinate) line show the result for the controller without $\boldsymbol{\tau}_d$.

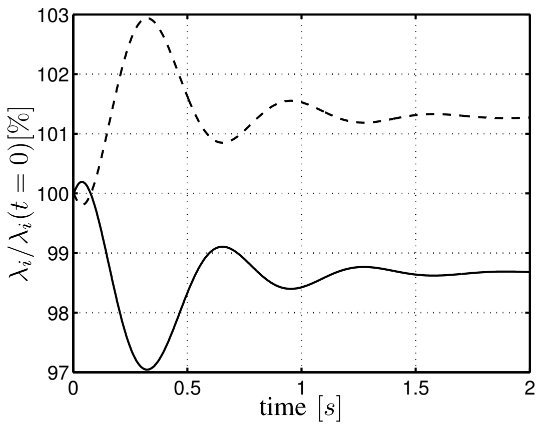

Fig. 6. Change of the eigenvalues of the Cartesian inertia matrix for the step response in $\boldsymbol{q}_d$ for the proposed controller.

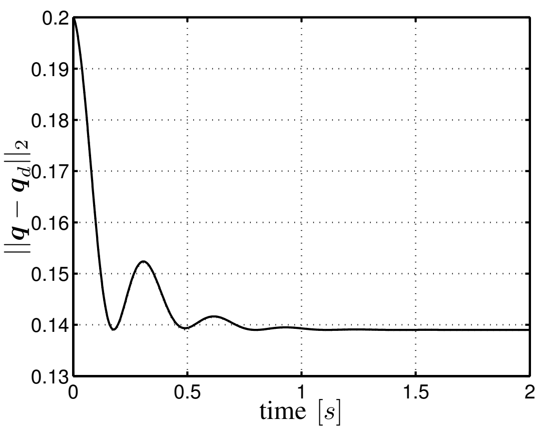

Fig. 7. Joint configuration error for the proposed controller.

V. SUMMARY

The main contribution of this paper is the formulation of a compliance control law which allows to implement a Cartesian compliance and a nullspace compliance without requiring inertia shaping. A power-conserving decoupling term was needed in order to eliminate some undesired centrifugal and Coriolis terms in the equations of motion. Using this partial decoupling it is possible to prove asymptotic stability by applying a result from the stability theory with semi-definite Lyapunov functions.

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APPENDIX

The following definitions taken from [14] summarize the notion of *conditional stability*, which is utilized for the stability proof in Section III-D. Consider, therefore, a time-invariant system of the form

$$\dot{\boldsymbol{z}} = \boldsymbol{f}(\boldsymbol{z}) \tag{14}$$

with state $\boldsymbol{z} \in \mathbb{R}^n$. Assume that the point $\boldsymbol{z}_s$ is a stationary point of (14), i.e. $\boldsymbol{f}(\boldsymbol{z}_s) = \boldsymbol{0}$. Suppose that the solution $\boldsymbol{z}(t)$ to (14) with initial state $\boldsymbol{z}(0) = \boldsymbol{z}_0$ exists for all $t > 0$. For the conditional stability all requirements of the stability definitions must hold only for those initial conditions which lie in a particular set $\mathcal{A} \subset \mathbb{R}^n$. The notion of conditional stability is therefore weaker than the usual (Lyapunov) stability.

Definition 1: A stationary point $\boldsymbol{z}_s$ of (14) is said to be stable conditionally to $\mathcal{A} \subset \mathbb{R}^n$, if $\boldsymbol{z}_s \in \mathcal{A}$ and for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for any initial condition $\boldsymbol{z}_0 \in \mathcal{A}$ the following implication holds:

$$||\boldsymbol{z}_0 - \boldsymbol{z}_s|| < \delta(\epsilon) \Rightarrow ||\boldsymbol{z}(t) - \boldsymbol{z}_s|| < \epsilon \ , \quad \forall t \geq 0 \ . \tag{15}$$

Definition 2: A stationary point $\boldsymbol{z}_s$ of (14) is said to be attractive conditionally to $\mathcal{A} \subset \mathbb{R}^n$, if $\boldsymbol{z}_s \in \mathcal{A}$ and there exists an $\eta(\boldsymbol{z}_s) > 0$ such that for any initial condition $\boldsymbol{z}_0 \in \mathcal{A}$ the following implication holds:

$$||\boldsymbol{z}_0 - \boldsymbol{z}_s|| < \eta(\boldsymbol{z}_s) \Rightarrow \lim_{t\rightarrow\infty} \boldsymbol{z}(t) = \boldsymbol{z}_s \ . \tag{16}$$

Definition 3: A stationary point $\boldsymbol{z}_s$ of (14) is said to be asymptotically stable conditionally to $\mathcal{A} \subset \mathbb{R}^n$ if it is both stable and attractive conditionally to $\mathcal{A}$.

Definition 4: The stationary point $\boldsymbol{z}_s$ of (14) is said to be globally asymptotically stable conditionally to $\mathcal{A} \subset \mathbb{R}^n$ if it is asymptotically stable conditionally to $\mathcal{A}$ and $\eta(\boldsymbol{z}_s) = +\infty$.

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