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Stability Analysis for Prioritized Closed-Loop Inverse Kinematic Algorithms for Redundant Robotic Systems

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***Abstract*— A wide number of robotic applications makes use of priority-based kinematic control algorithms for redundant systems. Starting from the classical applications in position control of manipulators, the kinematic-based approaches were lately applied to, e.g., visual servoing or multi-robot coordination control. The basic approach consists in the definition of several tasks properly combined in priority. A rigorous stability analysis that ensures the possibility to effectively achieve the defined tasks, however, is missing. In this paper, by resorting to a Lyapunov-based stability discussion for the prioritized inverse kinematics algorithms, an effective condition is given to verify that the tasks are properly matched; moreover, minimum bound for the control gains are determined.**

I. Introduction

A robotic system is kinematically redundant when it possesses more Degrees Of Freedom (DOFs) than those required to execute a given task. Task-priority redundancy resolution techniques [1], [2], [3] were proposed in that it allows the specification of a primary task which is fulfilled with higher priority with respect to a secondary task. Extensions of the algorithm proposed in [2] to a large number of tasks is given in [4]. Within the same framework, the work presented in [5] investigates the use of a proper weighted pseudo-inverse. In [6] the null-space projector is used together with a projection based on the transpose of the Jacobian and the stability analysis is presented for the two-task case.

An alternative approach is the augmented Jacobian [7]. In this case, the secondary task is added to the primary task so as to obtain a square Jacobian matrix which can be inverted. The main drawback of this technique is that new singularities may arise in configurations in which the primary Jacobian is still full rank. Those singularities, named *algorithmic singularities*, occur when the additional task causes conflict with the primary task.

In the framework of task priority, in [8] a singularity robust solution is proposed for two tasks. As noticed in [9], [10], where the authors propose a solution to achieve visual servoing by properly sequencing tasks, its generalization to a generic number of tasks is not trivial. In [11] a task-priority solution is used to implement a behavioral control for robotic systems.

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Most of the papers devoted at implementing priority-based approaches for robotic systems recognize the possible occurrence of undesired behaviors due to the non-commutativity of the null-space projectors. A rigorous stability analysis, however, still is missing to better understand the conditions that ensure convergence of the defined tasks. In this paper, a Lyapunov-based stability discussion is given that allows to determine a minimum bound for the control gains; moreover, a simple condition is given to verify that the tasks are properly matched. An n dimensional planar manipulator is used as case study to illustrate the obtained results.

II. Mathematical Background

By defining as $\boldsymbol{\sigma} \in \mathbb{R}^m$ the task variable to be controlled and as $\boldsymbol{q} \in \mathbb{R}^n$ the vector of the system configuration, it is:

$$\boldsymbol{\sigma} = \boldsymbol{f}(\boldsymbol{q}) \tag{1}$$

with the corresponding differential relationship:

$$\dot{\boldsymbol{\sigma}} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q}) \dot{\boldsymbol{q}}, \tag{2}$$

where $\boldsymbol{J}(\boldsymbol{q}) \in \mathbb{R}^{m \times n}$ is the configuration-dependent task Jacobian matrix and $\dot{\boldsymbol{q}} \in \mathbb{R}^n$ is the system velocity. Notice that n depends on the specific robotic system considered, in case of an industrial manipulator n is generally $n = 6$ and $\boldsymbol{q}$ is the vector of joint positions. For a differential mobile robot $n = 3$, and the term *system configuration* simply refers to the robot position/orientation. For a multi-robot system n is related to the number of robots, in case of a full actuated underwater vehicle $n = 6$, finally, an anthropomorphic robots can reach very large value of n.

Motion references $\boldsymbol{q}_{\text{des}}(t)$ for the robotic system starting from desired values $\boldsymbol{\sigma}_{\text{des}}(t)$ of the task function are usually generated by inverting the (locally linear) mapping (2) [12]. A typical requirement is to pursue minimum-norm velocity, leading to the least-squares solution (dependencies in the Jacobian are dropped out to increase readability):

$$\dot{\boldsymbol{q}}_{\text{des}} = \boldsymbol{J}^{\dagger} \dot{\boldsymbol{\sigma}}_{\text{des}} = \boldsymbol{J}^{\text{T}} \left(\boldsymbol{J} \boldsymbol{J}^{\text{T}} \right)^{-1} \dot{\boldsymbol{\sigma}}_{\text{des}}. \tag{3}$$

In order to avoid the well known problem of numerical drift, a Closed Loop Inverse Kinematics (CLIK) version of the algorithm is usually implemented [8], namely,

$$\dot{\boldsymbol{q}}_{\text{des}} = \boldsymbol{J}^{\dagger} \left(\dot{\boldsymbol{\sigma}}_{\text{des}} + \boldsymbol{\Lambda} \widetilde{\boldsymbol{\sigma}} \right), \tag{4}$$

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where $\boldsymbol{\Lambda}$ is a suitable positive-definite matrix of gains and $\widetilde{\boldsymbol{\sigma}}$ is the task error defined as

$$\widetilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}_{\mathrm{des}} - \boldsymbol{\sigma}.$$

In case of system redundancy, i.e., if $n > m$, the classic general solution contains a null projector operator [1]:

$$\dot{\boldsymbol{q}}_{\mathrm{des}} = \boldsymbol{J}^{\dagger}\left(\dot{\boldsymbol{\sigma}}_{\mathrm{des}} + \boldsymbol{\Lambda}\widetilde{\boldsymbol{\sigma}}\right) + \left(\boldsymbol{I}_n - \boldsymbol{J}^{\dagger}\boldsymbol{J}\right)\dot{\boldsymbol{q}}_{\mathrm{null}}, \tag{5}$$

where $\boldsymbol{I}_n$ is the $(n \times n)$ Identity matrix and the vector $\dot{\boldsymbol{q}}_{\mathrm{null}} \in \mathbb{R}^n$ is an arbitrary system velocity vector. It can be recognized that the operator $\left(\boldsymbol{I}_n - \boldsymbol{J}^{\dagger}\boldsymbol{J}\right)$ projects a generic velocity vector in the null space of the Jacobian matrix. This corresponds to generate a motion of the robotic system that does not affect that of the given task; this is usually named as *internal* motion inheriting its meaning from the original application of these techniques where the primary task was the end-effector position of a manipulator.

For highly redundant systems, multiple tasks can be arranged in priority in order to try to fulfill most of them, hopefully all of them, simultaneously. Let us consider, for sake of simplicity, 3 tasks, that will be denoted with the subscript a, b and c, respectively:

$$\begin{aligned} \boldsymbol{\sigma}_{\mathrm{a}} &= \boldsymbol{f}_{\mathrm{a}}(\boldsymbol{q}) \\ \boldsymbol{\sigma}_{\mathrm{b}} &= \boldsymbol{f}_{\mathrm{b}}(\boldsymbol{q}) \\ \boldsymbol{\sigma}_{\mathrm{c}} &= \boldsymbol{f}_{\mathrm{c}}(\boldsymbol{q}) \end{aligned}$$

where $\boldsymbol{\sigma}_{\mathrm{a}} \in \mathbb{R}^{m_a}$, $\boldsymbol{\sigma}_{\mathrm{b}} \in \mathbb{R}^{m_b}$ and $\boldsymbol{\sigma}_{\mathrm{c}} \in \mathbb{R}^{m_c}$. For each of the tasks a corresponding Jacobian matrix can be defined, in detail $\boldsymbol{J}_{\mathrm{a}} \in \mathbb{R}^{m_a \times n}$, $\boldsymbol{J}_{\mathrm{b}} \in \mathbb{R}^{m_b \times n}$ and $\boldsymbol{J}_{\mathrm{c}} \in \mathbb{R}^{m_c \times n}$. Let us further define the corresponding null space projectors as

$$\begin{aligned} \boldsymbol{N}_{\mathrm{a}} &= \left(\boldsymbol{I}_n - \boldsymbol{J}_{\mathrm{a}}^{\dagger}\boldsymbol{J}_{\mathrm{a}}\right) \\ \boldsymbol{N}_{\mathrm{b}} &= \left(\boldsymbol{I}_n - \boldsymbol{J}_{\mathrm{b}}^{\dagger}\boldsymbol{J}_{\mathrm{b}}\right). \end{aligned}$$

A generalization of the singularity-robust task priority inverse kinematic solution proposed in [8] leads to the following equation [11]:

$$\begin{aligned} \dot{\boldsymbol{q}}_{\mathrm{des}} &= \boldsymbol{J}_{\mathrm{a}}^{\dagger}\boldsymbol{\Lambda}_{\mathrm{a}}\widetilde{\boldsymbol{\sigma}}_{\mathrm{a}} + \boldsymbol{N}_{\mathrm{a}}\left(\boldsymbol{J}_{\mathrm{b}}^{\dagger}\boldsymbol{\Lambda}_{\mathrm{b}}\widetilde{\boldsymbol{\sigma}}_{\mathrm{b}} + \boldsymbol{N}_{\mathrm{b}}\boldsymbol{J}_{\mathrm{c}}^{\dagger}\boldsymbol{\Lambda}_{\mathrm{c}}\widetilde{\boldsymbol{\sigma}}_{\mathrm{c}}\right) \quad (6) \\ &= \boldsymbol{J}_{\mathrm{a}}^{\dagger}\boldsymbol{\Lambda}_{\mathrm{a}}\widetilde{\boldsymbol{\sigma}}_{\mathrm{a}} + \boldsymbol{N}_{\mathrm{a}}\boldsymbol{J}_{\mathrm{b}}^{\dagger}\boldsymbol{\Lambda}_{\mathrm{b}}\widetilde{\boldsymbol{\sigma}}_{\mathrm{b}} + \boldsymbol{N}_{\mathrm{a}}\boldsymbol{N}_{\mathrm{b}}\boldsymbol{J}_{\mathrm{c}}^{\dagger}\boldsymbol{\Lambda}_{\mathrm{c}}\widetilde{\boldsymbol{\sigma}}_{\mathrm{c}} \end{aligned}$$

where a regulation problem has been considered and the priority of the tasks follows the alphabetical order. This algorithm has a clear geometrical interpretation: the tasks are separately inverted by the use of the pseudoinverse of the corresponding Jacobian; the velocities associated with the lower priority task are further projected in the null space of the sole higher task.

However, as noticed in [9] and [10], the null space projectors are not commutative and the solution in eq. (6) may lead to undesired behaviors as detailed in the next Section. A *correct* projection is considered where the generic task is not projected onto the null space of the sole higher priority task but onto the null space of the task achieved by considering the augmented Jacobian of all the higher priority ones. For the 3 tasks example, thus, by defining:

$$\boldsymbol{J}_{\mathrm{ab}} = \begin{bmatrix} \boldsymbol{J}_{\mathrm{a}} \\ \boldsymbol{J}_{\mathrm{b}} \end{bmatrix}, \quad \boldsymbol{N}_{\mathrm{ab}} = \left(\boldsymbol{I}_n - \boldsymbol{J}_{\mathrm{ab}}^{\dagger}\boldsymbol{J}_{\mathrm{ab}}\right) \tag{7}$$

the desired velocities are

$$\dot{\boldsymbol{q}}_{\mathrm{des}} = \boldsymbol{J}_{\mathrm{a}}^{\dagger}\boldsymbol{\Lambda}_{\mathrm{a}}\widetilde{\boldsymbol{\sigma}}_{\mathrm{a}} + \boldsymbol{N}_{\mathrm{a}}\boldsymbol{J}_{\mathrm{b}}^{\dagger}\boldsymbol{\Lambda}_{\mathrm{b}}\widetilde{\boldsymbol{\sigma}}_{\mathrm{b}} + \boldsymbol{N}_{\mathrm{ab}}\boldsymbol{J}_{\mathrm{c}}^{\dagger}\boldsymbol{\Lambda}_{\mathrm{c}}\widetilde{\boldsymbol{\sigma}}_{\mathrm{c}} \tag{8}$$

It is worth noticing that the solution in eq. (8) looses the geometrical interpretation of the solution in eq. (6) and strongly couples all the tasks. On the other hand, the approach in eq. (6) may lead to undesired behavior. It is of interest, thus, to understand the stability of the two solutions to better design the inverse kinematic solution. Moreover, even with the solution in eq. (8) it is useful to derive a condition among the tasks' Jacobians that guarantees the absence of conflicting requirements among them.

A. Definitions

To make easier the following readings the two approaches investigated will be denote with a descriptive name. In particular, the approach in eq. (6) will be denoted as *successive projections method* while the approach in eq. (8) will be denoted as *augmented projections method*.

Applying basic geometric similarities, some definitions concerning the relationships between two tasks will also be given in this Section.

Given two generic tasks, denoted with the lower scripts x and y, they will be defined as *orthogonal* if:

$$\boldsymbol{J}_{\mathrm{x}}\boldsymbol{J}_{\mathrm{y}}^{\dagger} = \boldsymbol{O}_{m_x \times m_y} \tag{9}$$

where $\boldsymbol{O}_{m_x \times m_y}$ is the $(m_x \times m_y)$ null matrix. The two tasks will be defined as *dependent* if

$$\rho(\boldsymbol{J}_{\mathrm{x}}^{\dagger}) + \rho(\boldsymbol{J}_{\mathrm{y}}^{\dagger}) > \rho\left(\boldsymbol{J}_{\mathrm{x}}^{\dagger} \bigcup \boldsymbol{J}_{\mathrm{y}}^{\dagger}\right). \tag{10}$$

where $\rho(\cdot)$ denotes the rank of the matrix. Finally, they will be defined as *independent* if

$$\rho(\boldsymbol{J}_{\mathrm{x}}^{\dagger}) + \rho(\boldsymbol{J}_{\mathrm{y}}^{\dagger}) = \rho\left(\boldsymbol{J}_{\mathrm{x}}^{\dagger} \bigcup \boldsymbol{J}_{\mathrm{y}}^{\dagger}\right) \tag{11}$$

and they are not orthogonal.

It is worth noticing that the three conditions of orthogonality, dependency and independency given may be verified by resorting to the transpose of the corresponding Jacobians instead of the pseudoinverse, in fact, they share the same span. Thus, the independency condition becomes:

$$\rho(\boldsymbol{J}_{\mathrm{x}}^{\mathrm{T}}) + \rho(\boldsymbol{J}_{\mathrm{y}}^{\mathrm{T}}) = \rho\left(\boldsymbol{J}_{\mathrm{x}}^{\mathrm{T}} \bigcup \boldsymbol{J}_{\mathrm{y}}^{\mathrm{T}}\right). \tag{12}$$

Moreover, the orthogonality condition between two tasks is equivalent to a existence of a $\pi/2$ angle between the subspaces $\boldsymbol{J}_{\mathrm{x}}^{\mathrm{T}}$ and $\boldsymbol{J}_{\mathrm{y}}^{\mathrm{T}}$ [13], the dependency condition corresponds to a null angle and the independency condition to an angle strictly included in the range $]0, \pi/2[$.

In the following section it will be demonstrated that these definitions, and condition in eq. (12), play an important role in the eventual convergence of the task errors. Moreover, the independency condition can be easily verified on the symbolic definition of the Jacobians; it is not necessary, thus, to resort to numerical investigation of the matrices.

III. Stability Analysis

For sake of simplicity the stability analysis will be first discussed with respect to solely three tasks, its generalization to a generic number of task will then be provided.

Let us define $\tilde{\boldsymbol{\sigma}} \in \mathbb{R}^{m_a+m_b+m_c}$ as

$$\tilde{\boldsymbol{\sigma}} = [\, \tilde{\boldsymbol{\sigma}}_{\mathrm{a}}^{\mathrm{T}} \quad \tilde{\boldsymbol{\sigma}}_{\mathrm{b}}^{\mathrm{T}} \quad \tilde{\boldsymbol{\sigma}}_{\mathrm{c}}^{\mathrm{T}} \,]^{\mathrm{T}}, \tag{13}$$

that is the stacked vector of tasks' errors. A possible Lyapunov function candidate is given by

$$V = \frac{1}{2}\tilde{\boldsymbol{\sigma}}^{\mathrm{T}}\tilde{\boldsymbol{\sigma}} \tag{14}$$

whose time derivative is

$$\dot{V} = \tilde{\boldsymbol{\sigma}}^{\mathrm{T}}\dot{\tilde{\boldsymbol{\sigma}}} = \tilde{\boldsymbol{\sigma}}^{\mathrm{T}}\left(\dot{\boldsymbol{\sigma}}_{\mathrm{des}} - \dot{\boldsymbol{\sigma}}\right) \tag{15}$$

that, assuming a regulation problem, yelds

$$\dot{V} = -\tilde{\boldsymbol{\sigma}}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{J}_{\mathrm{a}} \\ \boldsymbol{J}_{\mathrm{b}} \\ \boldsymbol{J}_{\mathrm{c}} \end{bmatrix} \dot{\boldsymbol{q}} \tag{16}$$

that, substituting the system velocity in eq. (6) or in eq. (8) into eq. (16), can be rearranged as

$$\begin{aligned} \dot{V} &= -\tilde{\boldsymbol{\sigma}}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{\Lambda}_{\mathrm{a}} & \boldsymbol{O}_{m_a,m_b} & \boldsymbol{O}_{m_a,m_c} \\ \boldsymbol{J}_{\mathrm{b}}\boldsymbol{J}_a^{\dagger}\boldsymbol{\Lambda}_{\mathrm{a}} & \boldsymbol{J}_{\mathrm{b}}\boldsymbol{N}_{\mathrm{a}}\boldsymbol{J}_b^{\dagger}\boldsymbol{\Lambda}_{\mathrm{b}} & \boldsymbol{J}_{\mathrm{b}}\overline{\boldsymbol{N}}\boldsymbol{J}_c^{\dagger}\boldsymbol{\Lambda}_{\mathrm{c}} \\ \boldsymbol{J}_{\mathrm{c}}\boldsymbol{J}_a^{\dagger}\boldsymbol{\Lambda}_{\mathrm{a}} & \boldsymbol{J}_{\mathrm{c}}\boldsymbol{N}_{\mathrm{a}}\boldsymbol{J}_b^{\dagger}\boldsymbol{\Lambda}_{\mathrm{b}} & \boldsymbol{J}_{\mathrm{c}}\overline{\boldsymbol{N}}\boldsymbol{J}_c^{\dagger}\boldsymbol{\Lambda}_{\mathrm{c}} \end{bmatrix} \tilde{\boldsymbol{\sigma}} \\ &= -\tilde{\boldsymbol{\sigma}}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{O}_{m_a,m_b} & \boldsymbol{O}_{m_a,m_c} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} & \boldsymbol{M}_{23} \\ \boldsymbol{M}_{31} & \boldsymbol{M}_{32} & \boldsymbol{M}_{33} \end{bmatrix} \tilde{\boldsymbol{\sigma}} \\ &= -\tilde{\boldsymbol{\sigma}}^{\mathrm{T}}\boldsymbol{M}\tilde{\boldsymbol{\sigma}} \end{aligned} \tag{17}$$

where

$$\overline{\boldsymbol{N}} = \begin{cases} \boldsymbol{N}_{\mathrm{a}}\boldsymbol{N}_{\mathrm{b}} & \text{for eq. (6)} \\ \boldsymbol{N}_{\mathrm{ab}} & \text{for eq. (8).} \end{cases}$$

The sign of $\dot{V}$ in eq. (17) is not determined and need to be further analyzed. The matrix $\boldsymbol{M}$ were decomposed into sub-matrices $\boldsymbol{M}_{ij}$ of proper dimensions. In general, all the sub-matrices are different from the null matrix except for the elements corresponding to the first m_a rows and columns ranging from m_a+1 to $m_a+m_b+m_c$, i.e., the sub-matrices $\boldsymbol{M}_{12}$ and $\boldsymbol{M}_{13}$, that are null by construction.

In wath follows, the assumption that the Jacobians are full rank will be made.

A necessary condition for $\boldsymbol{M}$ to be positive definite is that all the sub-matrices on the main diagonal are positive definite (see the Appendix). The $(m_a \times m_a)$ sub-matrix $\boldsymbol{M}_{11}$ is obviously positive definite as long as the gain matrix $\boldsymbol{\Lambda}_{\mathrm{a}} > \boldsymbol{O}$. The $(m_b \times m_b)$ sub-matrix $\boldsymbol{M}_{22}$ is positive definite if tasks a and b are independent, i.e., if condition in eq. (12) holds and if the gain matrix $\boldsymbol{\Lambda}_{\mathrm{b}} > \boldsymbol{O}$. The $(m_c \times m_c)$ sub-matrix $\boldsymbol{M}_{33}$ is positive definite for the augmented projection method, eq. (8), if the task c is independent to the augmented Jacobian obtained by stacking tasks a and b and if the gain matrix $\boldsymbol{\Lambda}_{\mathrm{c}} > \boldsymbol{O}$. For the successive projection method, eq. (6), the sign of this sub-matrix depends also on the angle among the subspaces and the independency is not sufficient to prove its positive definitiveness, however, a sufficient condition is that an orthogonality relationship between two of the three tasks holds.

Given the sub-matrices $\boldsymbol{M}_{ii}$ positive definite, a sufficient condition for $\boldsymbol{M}$ to be positive definite is given by its eventual lower triangular form (see Appendix), thus, it is of interest to verify this condition.

The sign of the sub-matrices holding to the lower triangle are not determinant for the overall identification of the sign of $\boldsymbol{M}$. For sake of completeness, however, it is worth noticing that the sub-matrices $\boldsymbol{M}_{21}$ and $\boldsymbol{M}_{31}$ are null if the tasks b and c, respectively, and a are orthogonal, otherwise they are not determined in sign. The sub-matrix $\boldsymbol{M}_{32}$ is not null if c and b are independent in the null of a, otherwise it is not determined in sign.

The sub-matrix $\boldsymbol{M}_{23}$ with $\overline{\boldsymbol{N}} = \boldsymbol{N}_{\mathrm{ab}}$, i.e., for the augmented projection method, is always null by construction since the first matrix multiplication $\boldsymbol{J}_{\mathrm{b}}\boldsymbol{N}_{\mathrm{ab}} = \boldsymbol{O}$. The use of successive projection method, i.e., with $\overline{\boldsymbol{N}} = \boldsymbol{N}_{\mathrm{a}}\boldsymbol{N}_{\mathrm{b}}$ does not guarantee that this sub-matrix is null, in particular, $\boldsymbol{M}_{23} = \boldsymbol{O}$ only if two successive tasks are orthogonal to one other, i.e., a is orthogonal to b and/or b is orthogonal to c.

Overall, the successive projection method, eq. (6), leads to a positive definite $\boldsymbol{M}$, and thus, according to eq. (17) to a strictly negative Lyapunov function, if there exists an orthogonality condition between successive tasks while an independency condition with respect to them is sufficient for the remaining task. For the augmented projection method, eq. (8), the condition necessary and sufficient is that an independency condition in eq. (12) holds between the second and first tasks and between the third task and the augmented Jacobian obtained stacking the first two tasks.

Extension to N tasks

So far the 3-task case has been discussed. The generalization to N tasks leads to the conclusion that, implementing the augmented projection method, the matrix $\boldsymbol{M}$ is always a lower block-triangular matrix. The sub-matrices on the main diagonal, however, are positive definite only if the condition (12) holds between each task and the Jacobian obtained by stacking all the higher-priority tasks. In the latter case, a choice of positive definite matrix gains leads to a negative definite Lyapunov function and thus to the convergence of all the task errors to zero.

When the successive projection method is implemented, however, the matrix $\boldsymbol{M}$ is not anymore guaranteed both to be a lower block-triangular matrix and to exhibit positive definite sub-matrices on the diagonal. In such a case the matrix $\boldsymbol{M}$ does not exhibit evident properties concerning its definiteness and general stability conclusions can not be made.

In conclusion, the successive projection method is stable when two tasks are considered and they are at least independent, in case of three tasks it is required that there exists an orthogonality condition between two successive tasks while an independency condition with respect to them is sufficient for the remaining task. For more than three tasks no simple property exists. The augmented projection method on the other hand, can be used with the desired number of tasks as long as the independency condition (12) holds when an additional task is considered with respect to Jacobian obtained by stacking all the higher-priority tasks.

IV. Case Study

A number of case studies may be considered to verify the presented results such as, e.g., industrial robotics or multi-robot systems; in this section an hyper-redundant planar manipulator will be considered. This specific robot allows to appreciate the practical meaning of condition (12).

Let us consider an hyper-redundant n-link planar manipulator with revolute joint positions $\boldsymbol{q} \in \mathbb{R}^n$, let us assume $n >> 1$. The i th manipulator's link has length l_i.

A natural control objective is the position of the end effector $\boldsymbol{x}_{\text{ee}} \in \mathbb{R}^2$:

$$\boldsymbol{\sigma}_{\text{a}} = \boldsymbol{x}_{\text{ee}}(\boldsymbol{q}) = \begin{bmatrix} \sum_{i=1}^{n} l_i \cos\left(\sum_{j=1}^{i} q_j\right) \\ \sum_{i=1}^{n} l_i \sin\left(\sum_{j=1}^{i} q_j\right) \end{bmatrix}$$

whose Jacobian $\boldsymbol{J}_{\text{a}} \in \mathbb{R}^{2\times n}$ can be computed as

$$\boldsymbol{J}_{\text{a}} = \begin{bmatrix} \cdots & -\sum_{i=k}^{n} l_i \sin\left(\sum_{j=1}^{i} q_j\right) & \cdots \\ \cdots & \underbrace{\sum_{i=k}^{n} l_i \cos\left(\sum_{j=1}^{i} q_j\right)}_{k \text{ column}} & \cdots \end{bmatrix}$$

where k is a generic column of the matrix. The rank of $\boldsymbol{J}_{\text{a}}$ is always $\rho(\boldsymbol{J}_{\text{a}}) = 2$ except when the manipulator reaches a singular configuration given by an alignment of all the n joint positions. In the following the assumption that this situation does not occur will be made.

An additional task may be the end-effector orientation

$$\boldsymbol{\sigma}_{\text{b}} = \sum_{i=1}^{n} q_i$$

whose Jacobian $\boldsymbol{J}_{\text{b}} \in \mathbb{R}^{1\times n}$ is

$$\boldsymbol{J}_{\text{b}} = [\cdots \quad 1 \quad \cdots]$$

whose rank is always full: $\rho(\boldsymbol{J}_{\text{b}}) = 1$.

The two defined tasks for this simple case are arranged in priority according to:

priority	task's description	task's dim.
1	end-effector position	2
2	end-effector orientation	1

Thus, according to the results presented in this paper, it is possible to analytically verify the appropriateness of this additional task checking condition in eq. (12). This can be done resorting to symbolic or numerical instruments and, in this case, leads to the *evident* conclusion that the two tasks are indeed independent and thus compatible as long as the manipulator is not in a singular configuration and $n > 2$. If only two tasks are considered, both the approaches can be used and lead to an error convergent to zero.

Given the large number of degrees of freedom it is possible to add additional tasks. For instance, let us consider that the position of an intermediate position corresponding to a joint $\underline{n} < n$ needs to be controlled. This might the case, for instance, of a planar macro-micro manipulator. Let us assume that this task is added as last-priority task:

priority	task's description	task's dim.
1	end-effector position	2
2	end-effector orientation	1
3	robot intermediate position	2

Its task function is

$$\boldsymbol{\sigma}_{\mathrm{c}} = \begin{bmatrix} \sum_{i=1}^{\underline{n}} l_i \cos\left(\sum_{j=1}^{i} q_j\right) \\ \sum_{i=1}^{\underline{n}} l_i \sin\left(\sum_{j=1}^{i} q_j\right) \end{bmatrix}$$

with corresponding Jacobian $\boldsymbol{J}_{\mathrm{c}} \in \mathbb{R}^{2\times n}$

$$\boldsymbol{J}_{\mathrm{c}} = \begin{bmatrix} \cdots & -\sum_{i=k}^{\underline{n}} l_i \sin\left(\sum_{j=1}^{i} q_j\right) & \cdots 0 & \cdots & 0 \\ \cdots & \sum_{i=k}^{\underline{n}} l_i \cos\left(\sum_{j=1}^{i} q_j\right) & \underbrace{\cdots 0 \quad \cdots \quad 0}_{n-\underline{n}} \end{bmatrix}.$$

where the last $n-\underline{n}$ columns are null and the index k refers to a value smaller than $n-\underline{n}$.

The condition in eq. (12) between the last task and the augmented Jacobian obtained by stacking the previous tasks in priority gives the condition

$$\rho\,(\,\boldsymbol{J}_{\mathrm{c}}^{\mathrm{T}}\,) + \rho\,([\,\boldsymbol{J}_{\mathrm{a}}^{\mathrm{T}} \quad \boldsymbol{J}_{\mathrm{b}}^{\mathrm{T}}\,]) = \rho\,([\,\boldsymbol{J}_{\mathrm{a}}^{\mathrm{T}} \quad \boldsymbol{J}_{\mathrm{b}}^{\mathrm{T}} \quad \boldsymbol{J}_{\mathrm{c}}^{\mathrm{T}}\,])$$

that is verified with $n \geq 5$ and $\underline{n} \leq n-2$. Since the condition of independency holds, but not the orthogonality one among any of the tasks, it is possible to assert the stability for the augmented projection method but no conclusion can be made for the successive projection method.

V. Conclusions

Use of kinematic control is of great importance in robotics applications. In recent years, complex structures as humanoids with large degrees of freedom were used as realistic test-beds. Priority-based inverse kinematic algorithms were often used to achieve several control objectives simultaneously. A rigorous stability analysis, proving what kind of list of tasks may be fulfilled simultaneously, has been presented in this paper. In particular, the successive projection method turns out to be stable only if up to three tasks are defined and there is at least an orthogonality condition between two successive tasks while an independency condition with respect to them is sufficient for the remaining task. The augmented projection method on the other hand, can be used with the desired number of tasks as long as the independency condition holds between each task and the Jacobian obtained by stacking all the higher-priority tasks.

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Appendix

Given the matrix $\boldsymbol{M}$ with the structure given in eq. (17)

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{O}_{m_a,m_b} & \boldsymbol{O}_{m_a,m_c} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} & \boldsymbol{M}_{23} \\ \boldsymbol{M}_{31} & \boldsymbol{M}_{32} & \boldsymbol{M}_{33} \end{bmatrix}$$

a necessary condition to claim its positive definiteness is that the sub-matrices on the main diagonal $\boldsymbol{M}_{ii}$ are positive definite; moreover, a sufficient condition is that $\boldsymbol{M}$ is lower block-triangular. These conditions will give corresponding constraints on the matrix gains $\boldsymbol{\Lambda}_{\mathrm{i}}$ and on the Jacobians.

Let us define the vector $\boldsymbol{\zeta} \in \mathbb{R}^{m_a+m_b+m_c}$ as

$$\boldsymbol{\zeta} = [\,\boldsymbol{\zeta}_{ma}^{\mathrm{T}} \quad \boldsymbol{\zeta}_{mb}^{\mathrm{T}} \quad \boldsymbol{\zeta}_{mc}^{\mathrm{T}}\,]^{\mathrm{T}}$$

where $\boldsymbol{\zeta}_{ma}$, $\boldsymbol{\zeta}_{mb}$ and $\boldsymbol{\zeta}_{mc}$ are non-null arbitrary vectors of dimension m_a, m_b and m_c, respectively. The necessary condition follows immediately from the definition of positive definite matrix applied to $\boldsymbol{M}$:

$$\boldsymbol{\zeta}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{\zeta} > 0 \qquad \forall\boldsymbol{\zeta} \neq \boldsymbol{0} \tag{18}$$

selecting $\boldsymbol{\zeta}_{ma} = \boldsymbol{0}_{ma}$ and $\boldsymbol{\zeta}_{mb} = \boldsymbol{0}_{mb}$. This immediately gives the necessary condition that $\boldsymbol{M}_{33}$ needs to be positive definite. Extension to the remaining sub-matrices on the main diagonal is trivial.

The sufficient condition gives the following structure for the matrix $\boldsymbol{M}$

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{O}_{m_a,m_b} & \boldsymbol{O}_{m_a,m_c} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} & \boldsymbol{O}_{m_b,m_c} \\ \boldsymbol{M}_{31} & \boldsymbol{M}_{32} & \boldsymbol{M}_{33} \end{bmatrix}$$

and applying the definition (18):

$$\begin{aligned} \boldsymbol{\zeta}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{\zeta} &= \boldsymbol{\zeta}_{ma}^{\mathrm{T}}\boldsymbol{M}_{11}\boldsymbol{\zeta}_{ma} + \boldsymbol{\zeta}_{mb}^{\mathrm{T}}\boldsymbol{M}_{22}\boldsymbol{\zeta}_{mb} + \\ &\quad + \boldsymbol{\zeta}_{mc}^{\mathrm{T}}\boldsymbol{M}_{33}\boldsymbol{\zeta}_{mc} + \boldsymbol{\zeta}_{mb}^{\mathrm{T}}\boldsymbol{M}_{21}\boldsymbol{\zeta}_{ma} + \\ &\quad + \boldsymbol{\zeta}_{mc}^{\mathrm{T}}\boldsymbol{M}_{31}\boldsymbol{\zeta}_{ma} + \boldsymbol{\zeta}_{mc}^{\mathrm{T}}\boldsymbol{M}_{32}\boldsymbol{\zeta}_{mb} \end{aligned}$$

that can be underestimated as

$$\begin{aligned} \boldsymbol{\zeta}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{\zeta} &\geq \underline{\lambda}_{11}\,\underline{\lambda}_{\mathrm{a}}\zeta_{ma}^2 + \underline{\lambda}_{22}\,\underline{\lambda}_{\mathrm{b}}\zeta_{mb}^2 + \underline{\lambda}_{33}\,\underline{\lambda}_{\mathrm{c}}\zeta_{mc}^2 + \\ &\quad -\overline{\lambda}_{21}\,\overline{\lambda}_{\mathrm{a}}\,\|\boldsymbol{\zeta}_{mb}\|\,\|\boldsymbol{\zeta}_{ma}\| + \\ &\quad -\overline{\lambda}_{31}\,\overline{\lambda}_{\mathrm{a}}\,\|\boldsymbol{\zeta}_{mc}\|\,\|\boldsymbol{\zeta}_{ma}\| + \\ &\quad -\overline{\lambda}_{32}\,\overline{\lambda}_{\mathrm{b}}\,\|\boldsymbol{\zeta}_{mc}\|\,\|\boldsymbol{\zeta}_{mb}\|, \end{aligned}$$

where the underline (upperline) denotes the smallest (largest) singular value of the corresponding (sub)matrix. It is convenient to rewrite the relation above resorting to the matrix formalism:

$$\boldsymbol{\zeta}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{\zeta} \geq \frac{1}{2}\begin{bmatrix} \|\boldsymbol{\zeta}_{ma}\| \\ \|\boldsymbol{\zeta}_{mb}\| \\ \|\boldsymbol{\zeta}_{mc}\| \end{bmatrix}^{\mathrm{T}} \boldsymbol{P} \begin{bmatrix} \|\boldsymbol{\zeta}_{ma}\| \\ \|\boldsymbol{\zeta}_{mb}\| \\ \|\boldsymbol{\zeta}_{mc}\| \end{bmatrix}$$

where $\boldsymbol{P} \in \mathbb{R}^{3\times3}$ is an Hermitian matrix defined as

$$\boldsymbol{P} = \begin{bmatrix} 2\underline{\lambda}_{11}\,\underline{\lambda}_{\mathrm{a}} & -\overline{\lambda}_{21}\,\overline{\lambda}_{\mathrm{a}} & -\overline{\lambda}_{31}\,\overline{\lambda}_{\mathrm{a}} \\ -\overline{\lambda}_{21}\,\overline{\lambda}_{\mathrm{a}} & 2\underline{\lambda}_{22}\,\underline{\lambda}_{\mathrm{b}} & -\overline{\lambda}_{32}\,\overline{\lambda}_{\mathrm{b}} \\ -\overline{\lambda}_{31}\,\overline{\lambda}_{\mathrm{a}} & -\overline{\lambda}_{32}\,\overline{\lambda}_{\mathrm{b}} & 2\underline{\lambda}_{33}\,\underline{\lambda}_{\mathrm{c}} \end{bmatrix}$$

that, having the scalar diagonal elements positive, is positive definite if

$$2\,|p_{ij}| \leq p_{ii} + p_{jj} \qquad \forall i, j$$

In the following, for sake of simplicity we will assume $\underline{\lambda}_{\mathrm{a}} = \overline{\lambda}_{\mathrm{a}} = \lambda_{\mathrm{a}}$, $\underline{\lambda}_{\mathrm{b}} = \overline{\lambda}_{\mathrm{b}} = \lambda_{\mathrm{b}}$ and $\underline{\lambda}_{\mathrm{c}} = \lambda_{\mathrm{c}}$. Notice that this correspond to select, e.g., $\boldsymbol{\Lambda}_{\mathrm{a}} = \lambda_{\mathrm{a}}\boldsymbol{I}_{ma\times ma}$. After some basic manipulations, it is possible to extract the conditions on the gains:

$$\begin{aligned} \lambda_{\mathrm{a}} &> 0 \\ \lambda_{\mathrm{b}} &> \max\left\{0, \frac{\overline{\lambda}_{21} - \underline{\lambda}_{11}}{\underline{\lambda}_{22}}\lambda_{\mathrm{a}}\right\} \\ \lambda_{\mathrm{c}} &> \max\left\{0, \frac{\overline{\lambda}_{31} - \underline{\lambda}_{11}}{\underline{\lambda}_{33}}\lambda_{\mathrm{a}}, \frac{\overline{\lambda}_{32} - \underline{\lambda}_{22}}{\underline{\lambda}_{33}}\lambda_{\mathrm{b}}\right\}. \end{aligned}$$

It is worth noticing that a proper selection of the gains always allows to impose the positive definiteness to the matrix $\boldsymbol{M}$.

The extension to N tasks can be achieved iterating the discussion above and it is omitted for brevity.

An alternative sufficient condition is that $\boldsymbol{M}$ is upper block-triangular, this implies that the matrix $\boldsymbol{M}$ exhibits the following structure

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{O}_{m_a,m_b} & \boldsymbol{O}_{m_a,m_c} \\ \boldsymbol{O}_{m_b,m_a} & \boldsymbol{M}_{22} & \boldsymbol{M}_{23} \\ \boldsymbol{O}_{m_c,m_a} & \boldsymbol{O}_{m_c,m_b} & \boldsymbol{M}_{33} \end{bmatrix}$$

In Section III the conditions for having the sub-matrices $\boldsymbol{M}_{21}$, $\boldsymbol{M}_{31}$ and $\boldsymbol{M}_{32}$ null are reported: the sub-matrices $\boldsymbol{M}_{21}$ and $\boldsymbol{M}_{31}$ are null if the tasks b and c are orthogonal with respect to the task a. The sub-matrix $\boldsymbol{M}_{32}$ is null if c and b are orthogonal in the null of a. Moreover, when using eq. (8) the sub-matrix $\boldsymbol{M}_{23}$ is always null by construction. Using both the approaches in eq. (6) or in eq. (8) this alternative sufficient condition provides more conservatives constraints with respect to the requirement to have a lower block-triangular matrix.