

Inverse Agreement Algorithms with Application to Swarm Dispersion for Multiple Nonholonomic Agents

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Abstract—We propose an inverse agreement control strategy for multiple nonholonomic agents that forces the team members to disperse in the workspace in a distributed manner. Both the cases of an unbounded and a circular bounded workspace are considered. In the first case, we show that the closed loop system reaches a configuration in which the minimum distance between any pair of agents is larger than a specific lower bound. It is proved that this lower bound coincides with the agents' sensing radius. In the case of a bounded workspace, the control law is modified to force the agents to remain within the workspace boundary throughout the closed loop system evolution. Moreover the proposed control guarantees collision avoidance between the team members. The results are supported through relevant computer simulations.

I. INTRODUCTION

The emerging use of multi-robot and multi-vehicle systems in various applications has recently raised the need for control designs that force a team of multiple vehicles/robots (from now on called “agents”) to achieve various goals. As the number of agents increases, centralized control designs fail to guarantee robustness and are harder to implement than decentralized approaches, which also provide a reduce in the computational complexity of the overall feedback scheme.

Among the various objectives that the control design aims to impose on the agents, convergence of the multi-agent team to a common configuration, known as the agreement problem, is a specification that has been extensively pursued recently. Many control laws that achieve distributed agreement have been presented, e.g., [1],[16], [6],[12],[18]. Furthermore, the application of motion models of large populations of animals/insects (swarms) in nature to multi-agent systems is also a field of extensive research activity in the last few years. Relevant results include algorithms for swarm aggregation [9] and flocking [15],[20].

This paper presents a control methodology for nonholonomic swarm dispersion which can be considered as an inverse agreement problem. Each agent follows a flow, whose inverse leads the team to agreement [6]. The control is distributed, in the sense that each agent has only knowledge of the states of agents located within its sensing zone at each

time instant. The sensing zone is assumed to be a circular area around each agent. The application of this inverse agreement strategy is dispersion of the nonholonomic agents, i.e., convergence to a configuration where the minimum distance between the swarm members is bounded from below by a *controllable* lower bound. It is shown that this bound coincides with the radius of the sensing zone of the agents. The results are then extended to the case of a circular bounded workspace. Applications of the proposed dispersion algorithm include coverage control [7], [4], and optimal placement of a multi-robot team in small areas [11],[2]. However, we also show that inverse agreement algorithms can be used to provide solutions to various problems in multi-agent control. This is a topic of probable related research directions. We note that the results of this paper were shown to hold for the case of single integrator point agents [5].

The rest of the paper is organized as follows: Section II presents the system and problem in hand. Tools from nonsmooth and matrix theory used in the paper are reviewed in Section III. The swarm dispersion methodology is presented in Section IV, while the case of a bounded workspace is considered in Section V. Simulations supporting the theory are included in Section VI. Section VII summarizes the results of this paper and indicates current research efforts.

II. SYSTEM AND PROBLEM DESCRIPTION

Consider N nonholonomic circular agents operating in a workspace $W \subset \mathbb{R}^2$. Let $q_i = [x_i, y_i]^T \in \mathbb{R}^2$ be the position of agent i . All agents have a common radius r . However, the proposed framework is easily applied to agents with varying, not necessarily common, radii. The configuration space is spanned by $q = [q_1^T, \dots, q_N^T]^T$. Each agent has an orientation θ_i with respect to the global coordinate frame. The orientation vector of the agents is $\theta = [\theta_1 \dots \theta_N]^T$. The configuration of each agent is $p_i = [q_i^T \ \theta_i]^T \in \mathbb{R}^2 \times (-\pi, \pi]$. Agent motion is described by the kinematics:

$$\begin{aligned} \dot{x}_i &= u_i \cos \theta_i \\ \dot{y}_i &= u_i \sin \theta_i, \quad i \in \mathcal{N} = [1, \dots, N] \\ \dot{\theta}_i &= \omega_i \end{aligned} \quad (1)$$

where u_i, ω_i denote the translational and rotational velocity (control inputs) of agent i , respectively.

Each agent can only sense agents whose center is found within a circle of radius d , common for all agents, around the center of the agent. This circle is called the *sensing zone* of agent i and d its *sensing radius*. We require that d is larger than the sum of the radii of any two agents, i.e. $d > 2r$. We denote by N_i the subset including the

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agents that i can sense at each time instant, i.e., $N_i = \{j \in \mathcal{N}, j \neq i : \|q_i - q_j\| \leq d\}$. The objective is dispersion of the team members in a distributed manner. We assume that a large number of agents is gathered in a workspace close to each other. The goal is to design control laws that force the agents to converge to sufficiently large distances between them. Specifically, we equip each agent with a repulsive potential field with respect to each other agent within its sensing zone. The main result states that the closed loop system converges to a configuration where the sensing zone of each agent is empty, i.e. every agent is located at a distance no less than d from every other agent. In addition, the control law is designed to guarantee collision avoidance between the team members, in the sense that the discs that represent the agent never overlap throughout the closed loop system evolution. The stability analysis is performed first for a bounded and then an unbounded workspace.

A dispersion potential between agents i and j is given by

$$\gamma_{ij}(\beta_{ij}) = \begin{cases} \frac{1}{2}\beta_{ij}, & 0 \leq \beta_{ij} \leq c^2 \\ \phi(\beta_{ij}), & c^2 \leq \beta_{ij} \leq d^2 \\ h, & d^2 \leq \beta_{ij} \end{cases}$$

where $\beta_{ij} = \|q_i - q_j\|^2 - 4r^2$. The positive scalars c, d, h and the function ϕ are chosen in such a way so that γ_{ij} is everywhere C^1 . In this paper, we choose the polynomial function $\phi(x) = a_2x^2 + a_1x + a_0$. The parameters of $\phi(x)$ satisfy the differentiability requirement for γ_{ij} , provided that they fulfil the following relations: $a_2 = \frac{1}{4(c^2-d^2)}, a_1 = \frac{d^2}{2(d^2-c^2)}, a_0 = \frac{c^4}{4(c^2-d^2)}, h = \frac{d^2+c^2}{4}$. The gradient and the partial derivative of γ_{ij} are computed by $\nabla \gamma_{ij} = 2\rho_{ij}D_{ij}q$ and $\frac{\partial \gamma_{ij}}{\partial q_i} = 2\rho_{ij}(D_{ij})_i q$ where $\rho_{ij} \triangleq \frac{\partial \gamma_{ij}}{\partial \beta_{ij}}$ and the matrices $D_{ij}, (D_{ij})_i$, for $i < j$ are given by $D_{ij} = \tilde{D}_{ij} \otimes I_2$, where $(\tilde{D}_{ij})_{ii} = (\tilde{D}_{ij})_{jj} = 1, (\tilde{D}_{ij})_{ij} = (\tilde{D}_{ij})_{ji} = -1$ and $(\tilde{D}_{ij})_{kl} = 0$ for $k, l \neq i, j$, and $(D_{ij})_i = [O_{1 \times (i-1)} \quad 1 \quad O_{1 \times (j-i-1)} \quad -1 \quad O_{1 \times (N-j)}] \otimes I_2$ where \otimes denotes the standard Kronecker product between two matrices [10]. The definition of the matrices $D_{ij}, (D_{ij})_i$, for $i > j$ is straightforward. It can easily be shown that $\rho_{ij} > 0$ for $0 < \beta_{ij} < d^2$ and $\rho_{ij} = 0$ for $\beta_{ij} \geq d^2$.

III. MATHEMATICAL PRELIMINARIES

We review tools from matrix theory and nonsmooth analysis that we use in the analysis of the following paragraphs.

A. Tools from Matrix Theory

For an undirected graph $\mathcal{G} = (V, E)$ with n vertices denote by V its set of vertices and by E its set of edges. If there is an edge connecting two vertices i, j , i.e. $(i, j) \in E$, then i, j are called *adjacent*. A *path* of length r from a vertex i to a vertex j is a sequence of $r + 1$ distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. If there is a path between any two vertices of the graph \mathcal{G} , then \mathcal{G} is called *connected*. The *undirected graph* $\mathcal{G} = (V, E)$ corresponding to a real symmetric $n \times n$ matrix

M is a graph with n vertices indexed by $1, \dots, n$ such that there is an edge between vertices $i, j \in V$ if and only if $M_{ij} \neq 0$, i.e. $(i, j) \in E \Leftrightarrow M_{ij} \neq 0$. A $n \times n$ real symmetric matrix with non-positive off-diagonal elements and zero row sums is called a *symmetric Metzler matrix*. All the eigenvalues of a symmetric Metzler matrix are non-negative and zero is a trivial eigenvalue [13]. Zero is a simple eigenvalue of a symmetric Metzler matrix if and only if the corresponding undirected graph is connected. The trivial corresponding eigenvector is the vector of ones, $\vec{\mathbf{1}}$ [14].

B. Tools from Nonsmooth Analysis

For a differential equation with discontinuous right-hand side we have the following definition:

Definition 1: [8] In the case of a finite dimensional state-space, the vector function $x(\cdot)$ is called a *Filippov solution* of $\dot{x} = f(x)$, where f is measurable and essentially locally bounded, if it is absolutely continuous and $\dot{x} \in K[f](x)$ almost everywhere where $K[f](x) \equiv \overline{\text{co}}\{\lim_{x_i \rightarrow x} f(x_i) | x_i \notin N_0\}$ and N_0 is a set of measure zero that contains the set of points where f is not differentiable.

The following chain rule provides a calculus for the time derivative of the energy function in the nonsmooth case:

Theorem 1: [19] Let x be a Filippov solution to $\dot{x} = f(x)$ on an interval containing t and $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Lipschitz and regular function. Then $V(x(t))$ is absolutely continuous, $(d/dt)V(x(t))$ exists almost everywhere and

$$\frac{d}{dt}V(x(t)) \in \text{a.e. } \dot{V}(x) := \bigcap_{\xi \in \partial V(x(t))} \xi^T K[f](x(t))$$

where ‘‘a.e.’’ stands for ‘‘almost everywhere’’.

In this theorem, ∂V is *Clarke’s generalized gradient*. The definition of ∂V and of the *regularity* of a function are found in [3]. In this paper, the Lyapunov function V is smooth and hence regular, while ∂V is a singleton which is equal to the usual gradient everywhere: $\partial V(x) = \{\nabla V(x)\} \forall x$. We use the following nonsmooth version of LaSalle’s principle:

Theorem 2: [19] Let Ω be a compact set such that every Filippov solution to $\dot{x} = f(x), x(0) = x(t_0)$ starting in Ω is unique and remains in Ω for all $t \geq t_0$. Let $V : \Omega \rightarrow \mathbb{R}$ be a time independent regular function such that $v \leq 0, \forall v \in \dot{V}$ (if \dot{V} is the empty set then this is trivially satisfied). Define $S = \{x \in \Omega | 0 \in \dot{V}\}$. Then every trajectory in Ω converges to the largest invariant set, M , in the closure of S .

IV. SWARM DISPERSION WITH COLLISION AVOIDANCE

Consider $V = \sum_i \sum_{j \neq i} \frac{1}{\gamma_{ij}}$ as a candidate Lyapunov function for the closed loop system. Since V is smooth in the domain of interest, its generalized gradient is a singleton and coincides with its usual gradient. We can compute $\nabla V = \sum_i \sum_{j \neq i} \left(-\frac{1}{\gamma_{ij}^2}\right) \nabla \gamma_{ij} = -\sum_i \sum_{j \neq i} \frac{2\rho_{ij}}{\gamma_{ij}^2} D_{ij}q = -2(R_1 \otimes I_2)q$, where the matrix R_1 is given by

$$(R_1)_{ij} = \begin{cases} \sum_{j \neq i} \frac{\rho_{ij}}{\gamma_{ij}^2} + \sum_{j \neq i} \frac{\rho_{ji}}{\gamma_{ji}^2}, & i = j \\ -\frac{\rho_{ij}}{\gamma_{ij}^2} - \frac{\rho_{ji}}{\gamma_{ji}^2}, & i \neq j \end{cases}$$

Then, $\frac{\rho_{ij}}{\gamma_{ij}^2} = \frac{\rho_{ji}}{\gamma_{ji}^2} \Rightarrow R_1 = 2R_2$ where

$$(R_2)_{ij} = \begin{cases} \sum_{j \neq i} \frac{\rho_{ij}}{\gamma_{ij}^2}, & i = j \\ -\frac{\rho_{ij}}{\gamma_{ij}^2}, & i \neq j \end{cases}$$

and hence $\nabla V = -4(R_2 \otimes I_2)q$. Denote the stack vector $q = [x, y]^T$ into the coefficients that correspond to the x, y directions of the agents respectively, while the notation $(a)_i$ denotes the i -th element of the vector a . We propose the following control law for each agent i :

$$u_i = \text{sgn} \{r_{xi} \cos \theta_i + r_{yi} \sin \theta_i\} \cdot (r_{xi}^2 + r_{yi}^2) \quad (2)$$

$$\omega_i = -(\theta_i - \arctan 2(r_{yi}, r_{xi})) \quad (3)$$

where $r_{xi} = (R_2x)_i, r_{yi} = (R_2y)_i$. We point out that since $\rho_{ij} = 0$ for $\beta_{ij} \geq d^2$, the proposed control law satisfies the agents' limited sensing radius, since each agent i has to take into account only agents within its sensing zone in order to implement the control laws (2),(3). In particular, we have $r_{xi} = (R_2x)_i = \sum_{j \neq i} \frac{2\rho_{ij}}{\gamma_{ij}^2} (x_i - x_j) = \sum_{j \in N_i} \frac{2\rho_{ij}}{\gamma_{ij}^2} (x_i - x_j)$, $r_{yi} = (R_2y)_i = \sum_{j \neq i} \frac{2\rho_{ij}}{\gamma_{ij}^2} (y_i - y_j) = \sum_{j \in N_i} \frac{2\rho_{ij}}{\gamma_{ij}^2} (y_i - y_j)$, and hence agent i must be aware of the relative positions only of those agents belonging to N_i at each time instant. The generalized time derivative of V is given by:

$$\begin{aligned} \dot{\tilde{V}} &= (\nabla V)^T \cdot K \begin{bmatrix} u_1 \cos \theta_1 \\ u_1 \sin \theta_1 \\ \vdots \\ u_N \cos \theta_N \\ u_N \sin \theta_N \end{bmatrix} \subset \\ &\subset -4((R_2 \otimes I_2)q)^T K \begin{bmatrix} u_1 \cos \theta_1 \\ u_1 \sin \theta_1 \\ \vdots \\ u_N \cos \theta_N \\ u_N \sin \theta_N \end{bmatrix} \subset \\ &\subset -4(R_2x)^T \begin{bmatrix} K[u_1] \cos \theta_1 \\ \vdots \\ K[u_N] \cos \theta_N \end{bmatrix} - \\ &-4(R_2y)^T \begin{bmatrix} K[u_1] \sin \theta_1 \\ \vdots \\ K[u_N] \sin \theta_N \end{bmatrix} \subset \\ &\subset -\sum_{i \in \mathcal{N}} \{4K[u_i](r_{xi} \cos \theta_i + r_{yi} \sin \theta_i)\} \end{aligned}$$

where we used Theorem 1.3, [17] to calculate the inclusions of the Filippov set. Since $K[\text{sgn}(x)]x = \{|x|\}$ ([17], Theorem 1.7), the choice of controls (2),(3) yields

$$\dot{\tilde{V}} = -\sum_i \{4|r_{xi} \cos \theta_i + r_{yi} \sin \theta_i|(r_{xi}^2 + r_{yi}^2)\} \leq 0 \quad (4)$$

so that the generalized derivative of V reduces to a singleton.

The first result establishes collision avoidance:

Lemma 1: Consider the system of multiple kinematic agents (1) driven by the control law (2),(3) and starting from a feasible set of initial positions $\mathcal{I}(q) =$

$\{q \mid \|q_i - q_j\| > 2r, \forall i, j \in \mathcal{N}, i \neq j\}$. Then the set $\mathcal{I}(q)$ is invariant for the trajectories of the closed loop system.

Proof: For every initial condition $q(0) \in \mathcal{I}(q)$, $\dot{\tilde{V}}$ remains non-positive for all $t \geq 0$, by virtue of (4). Hence $V(q(t)) \leq V(q(0)) < \infty$ for all $t \geq 0$. Since $V \rightarrow \infty$ if and only if $\|q_i - q_j\| \rightarrow 2r$ for at least one pair $i, j \in \mathcal{N}$, we conclude that $q(t) \in \mathcal{I}(q)$, for all $t \geq 0$. \diamond

By virtue of Lemma 1, collision avoidance is guaranteed. The control design however is also directly related to the final configurations of the nonholonomic swarm members. The main result of this section is summarized in the following:

Theorem 2: Consider the system of multiple nonholonomic agents (1) driven by the control law (2),(3) and starting from a set of initial positions $\mathcal{I}(q) \cap \mathcal{F}(q)$ where $\mathcal{I}(q) = \{q \mid \|q_i - q_j\| > 2r, \forall i, j \in \mathcal{N}, i \neq j\}$ was defined in Lemma 1 and $\mathcal{F}(q) = \{q \mid \|q_i - q_j\| < (N-1)d^*, \forall i, j \in \mathcal{N}, i \neq j\}$ where $d^* > d$ is chosen arbitrarily. Then the agents reach a static configuration (i.e. all agents eventually stop) which satisfies $\|q_i - q_j\| \geq d, \forall i, j \in \mathcal{N}, i \neq j$, with zero orientations.

Proof: Since the set of initial conditions is contained in $\mathcal{I}(q)$, we have $\|q_i(t) - q_j(t)\| > 2r \Rightarrow q_i(t) \neq q_j(t)$, for all $i, j \in \mathcal{N}, i \neq j$, and all $t \geq 0$, by virtue of Lemma 1. Pick V as a candidate Lyapunov function. The generalized time derivative of V is reduced to a singleton within $\mathcal{I}(q)$, and is given by (4): $\dot{\tilde{V}} = -\sum_i \{4|r_{xi} \cos \theta_i + r_{yi} \sin \theta_i|(r_{xi}^2 + r_{yi}^2)\} \leq 0$.

The boundedness of solutions of the closed-loop system is checked based on the relative positions of the swarm members. Pick $d^* > d$. Since $\rho_{ij} = 0$ whenever $\beta_{ij} > d$, the set $\|q_i - q_j\| \leq (N-1)d^*$ for all $i, j \in \mathcal{N}$ is positively invariant for the trajectories of the closed-loop system. By virtue of Lemma 1, $\mathcal{I}(q) \cap \mathcal{F}(q)$ is also positively invariant. Since this set is closed and bounded, we can apply Theorem 2, to deduce that the agents converge to the largest invariant subset of the set $S = \{(r_{xi} = r_{yi} = 0) \vee (r_{xi} \cos \theta_i + r_{yi} \sin \theta_i = 0), \forall i \in \mathcal{N}\}$.

However, for each $i \in \mathcal{N}$, we have $|\omega_i| = \frac{\pi}{2}$ whenever $r_{xi} \cos \theta_i + r_{yi} \sin \theta_i = 0$, due to (3). In particular, (3) renders the surface $r_{xi} \cos \theta_i + r_{yi} \sin \theta_i = 0$ non-invariant for agent i , whenever i is not located at the desired equilibrium, namely when $r_{xi} = r_{yi} = 0$. Hence the largest invariant set S_v contained in S is $S \supset S_v = \{r_{xi} = r_{yi} = 0, \forall i \in \mathcal{N}\}$. Within S_v , we have $r_{xi} = r_{yi} = 0, \forall i \in \mathcal{N}$ and hence $u_i = 0$, for all $i \in \mathcal{N}$, i.e., all agents eventually stop. Moreover, within S_v , we have $\dot{\theta}_i = -\theta_i$ for all i , and hence the orientations of all agents tend to zero. Note now $r_{xi} = r_{yi} = 0, \forall i \in \mathcal{N}$ is equivalent to $r_{xi} = r_{yi} = 0, \forall i \in \mathcal{N} \Rightarrow (R_2 \otimes I_2)q = 0$. Hence S_v can be written as $S_v = \{q \mid (R_2 \otimes I_2)q = 0\}$. We next show that S_v can only be invariant when it coincides with the set $S_0 = \{q \mid \rho_{ij} = 0, \forall i, j \in \mathcal{N}, i \neq j\}$. Clearly, S_0 is a subset of S_v which is invariant for the trajectories of the closed-loop system. Suppose now that $\rho_{ij} > 0$ for some pairs of the team members. We denote the undirected graph corresponding to R_2 by $\mathcal{G}(R_2)$. The assumption that $\rho_{ij} > 0$ for some pairs

$i, j \in \mathcal{N}, i \neq j$ guarantees that $\mathcal{G}(R_2)$ has at least one edge. The graph $\mathcal{G}(R_2)$ can now be decomposed into its connected components. Please note that since the graph is undirected, no vertex can belong to two different components simultaneously. Ignoring the connected components containing only one vertex (i.e. vertices k for which $\rho_{kj} = 0$ for all $j \neq k$), and rearranging the agent indices accordingly, equation $(R_2 \otimes I_2)q = 0$ can be decomposed into different equations, each of which corresponds to a different connected component of $\mathcal{G}(R_2)$. Specifically for the connected component containing agents/vertices $\{i_1, i_2, \dots, i_l\}, i_j \in \mathcal{N}, j = 1, \dots, l$ with $l \leq n$ we have $(\tilde{R}_2 \otimes I_2)\tilde{q} = 0$, where $\tilde{q} = [q_{i_1}^T \dots q_{i_l}^T]^T$ and the $l \times l$ matrix \tilde{R}_2 has the same form as R_2 taking into account the set of agents $\{i_1, i_2, \dots, i_l\}$. By denoting \tilde{x}, \tilde{y} the stack vectors of \tilde{q} in the x, y directions, we have $(\tilde{R}_2 \otimes I_2)\tilde{q} = 0 \Rightarrow \tilde{R}_2\tilde{x} = \tilde{R}_2\tilde{y} = 0$. Matrix \tilde{R}_2 has zero row sums and non-positive off-diagonal elements, i.e., it is a symmetric Metzler matrix. Thus, the eigenvalues of \tilde{R}_2 are nonnegative. Furthermore, since \tilde{R}_2 corresponds to a connected graph (a connected component of $\mathcal{G}(R_2)$), zero is a simple eigenvalue of \tilde{R}_2 with corresponding eigenvector the vector of ones, $\mathbf{1}$. Hence equations $\tilde{R}_2\tilde{x} = \tilde{R}_2\tilde{y} = 0$ guarantee that both \tilde{x}, \tilde{y} are eigenvectors of \tilde{R}_2 belonging to $\text{span}\{\mathbf{1}\}$. Thus, all elements of \tilde{q} attain the same value, implying that all agents converge to a common configuration at steady state. However this is impossible, since, due to the invariance of $\mathcal{I}(q)$, no trajectory of the closed-loop system starting from $\mathcal{I}(q)$ can ever leave this set, i.e., $\|q_i(t) - q_j(t)\| > 2r$, and hence $q_i(t) \neq q_j(t)$, for all $i, j \in \mathcal{N}, i \neq j$, and all $t \geq 0$. We conclude that the largest invariant subset of S is S_0 . Since $\rho_{ij} = 0$ only for $\|q_i - q_j\| \geq d$, the proof is complete. \diamond

Thus, the closed-loop system converges to a configuration where each agent is located at a distance no less than d from every other agent. This reveals an important geometric property: since any pair of agents is located at least at a distance d from each other, each agent occupies a disc of radius $d/2$ in which no other agent is present. In other words, the agents are dispersed to n disjoint discs of radius $d/2$.

V. THE BOUNDED WORKSPACE CASE

We proposed a dispersion algorithm for multiple non-holonomic agents in an unbounded workspace. In practical applications such as coverage control and sensor deployment the problem is to modify the controller in order to take into account the workspace boundary. In this paper, we consider the case of a cyclic workspace W_0 of radius R_{W_0} . However, the proposed design is applicable to any convex workspace. In order to take into account the agent radii, we define a virtual workspace W in the interior of W_0 that has the same center as W_0 and radius $R_W = R_{W_0} - r < R_{W_0}$ (see Figure 1). By proving that the center of each agent q_i remains within W for all time it is also guaranteed that the discs representing the agents remain within the real workspace W_0 . Hence we consider the virtual workspace W in the sequel.

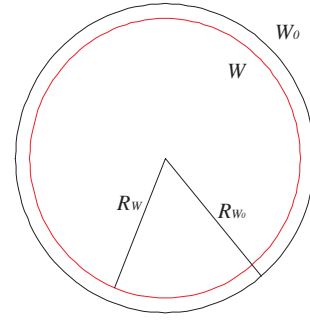


Fig. 1. The virtual boundary W is a smaller circle in the interior of the real boundary W_0 with the same center and radius equal to $R_W = R_{W_0} - r$.

A potential similar to the inter-agent dispersion potential is used for the agent-boundary repulsion. Copying with the sensing capabilities of the agents, the repulsive potential of each agent with respect to the workspace boundary is defined as

$$\gamma_{ib}(\beta_{ib}) = \begin{cases} \frac{1}{2}\beta_{ib}, & 0 \leq \beta_{ib} \leq c_b^2 \\ \varphi_b(\beta_{ib}), & c_b^2 \leq \beta_{ib} \leq d_b^2 \\ h_b, & d_b^2 \leq \beta_{ib} \end{cases}$$

where $\beta_{ib} = \|q_i - q_{i,\min}\|^2$, $d_b < d$ and $q_{i,\min} = \arg \min_{q \in \partial W} \|q_i - q\|^2$. Note that $q_{i,\min}$ is continuous for all i due to the convexity of W . The positive scalars h_b, c_b and the function φ_b are defined in such a way so that γ_{ib} is rendered everywhere continuously differentiable. Each agent has to have knowledge of the workspace boundary only when located at a distance smaller than d_b from it.

Denote $\rho_{ib} = \frac{\partial \gamma_{ib}}{\partial \beta_{ib}}$. We have $\rho_{ib} = 0$ for $\beta_{ib} > d_b^2$ and $\rho_{ib} > 0$ for $\beta_{ib} \leq d_b^2$. Denote by q_{\min} the stack vector of all q_{\min}^i and by x_{\min}, y_{\min} its coefficients in the x, y directions. Similarly to the case of an unbounded workspace, we use $V_b = \sum_i \sum_{j \neq i} \frac{1}{\gamma_{ij}} + \sum_i \frac{1}{\gamma_{ib}}$ as a candidate

Lyapunov function. Since V_b is smooth in the domain of interest, ∂V_b is a singleton and coincides with its usual gradient. The latter is calculated by $\nabla V_b = -4(R_3 \otimes I_2)q + 4(R_4 \otimes I_2)q_{\min}$ where $R_3 = R_2 + \text{diag}\left\{\frac{\rho_{1b}}{\gamma_{1b}^2}, \dots, \frac{\rho_{Nb}}{\gamma_{Nb}^2}\right\}$ and $R_4 = \text{diag}\left\{\frac{\rho_{1b}}{\gamma_{1b}^2}, \dots, \frac{\rho_{Nb}}{\gamma_{Nb}^2}\right\}$. We propose the following control law for each agent i of the nonholonomic swarm

$$u_i = \text{sgn}\{r_{xi}^b \cos \theta_i + r_{yi}^b \sin \theta_i\} \cdot ((r_{xi}^b)^2 + (r_{yi}^b)^2) \quad (5)$$

$$\omega_i = -(\theta_i - \arctan 2(r_{yi}^b, r_{xi}^b)) \quad (6)$$

where $r_{xi}^b = (R_3 x)_i - (R_4 x_{\min})_i$, $r_{yi}^b = (R_3 y)_i - (R_4 y_{\min})_i$.

The generalized time derivative of V_b is now given by

$$\dot{V}_b = (\nabla V_b)^T \cdot K \begin{bmatrix} u_1 \cos \theta_1 \\ u_1 \sin \theta_1 \\ \vdots \\ u_N \cos \theta_N \\ u_N \sin \theta_N \end{bmatrix} \subset$$

$$\begin{aligned}
& -4((R_3 \otimes I_2)q - (R_4 \otimes I_2)q_{\min})^T \begin{bmatrix} K[u_1] \cos \theta_1 \\ K[u_1] \sin \theta_1 \\ \vdots \\ K[u_N] \cos \theta_N \\ K[u_N] \sin \theta_N \end{bmatrix} \\
& \subset -4(R_3x - R_4x_{\min})^T \begin{bmatrix} K[u_1] \cos \theta_1 \\ \vdots \\ K[u_N] \cos \theta_N \\ K[u_1] \sin \theta_1 \\ \vdots \\ K[u_N] \sin \theta_N \end{bmatrix} \\
& -4(R_3y - R_4y_{\min})^T \begin{bmatrix} K[u_1] \cos \theta_1 \\ \vdots \\ K[u_N] \cos \theta_N \\ K[u_1] \sin \theta_1 \\ \vdots \\ K[u_N] \sin \theta_N \end{bmatrix} \\
& \subset -\sum_i \{4K[u_i] (r_{xi}^b \cos \theta_i + r_{yi}^b \sin \theta_i)\}
\end{aligned}$$

The choice of control laws (5),(6) now yields

$$\dot{V}_b = -\sum_i \left\{ 4|r_{xi}^b \cos \theta_i + r_{yi}^b \sin \theta_i| \left((r_{xi}^b)^2 + (r_{yi}^b)^2 \right) \right\} \quad (7)$$

so that \dot{V}_b reduces again to a singleton. The first result guarantees that the interior of the workspace is invariant:

Lemma 3: Consider (1) driven by (5),(6) and starting from the set of initial conditions $\mathcal{I}(q) \cap \mathcal{J}(q)$ where $\mathcal{J}(q) = \{q | q_i \in \text{int}(W) \triangleq W \setminus \partial W, \forall i \in \mathcal{N}\}$ is the interior of W and $\mathcal{I}(q)$ was defined previously. Then $\mathcal{I}(q) \cap \mathcal{J}(q)$ is invariant for the trajectories of the closed-loop system.

Proof: $\mathcal{I}(q)$ is invariant by Lemma 1. For every initial condition $q(0) \in \mathcal{I}(q) \cap \mathcal{J}(q)$, \dot{V}_b remains non-positive for all $t \geq 0$, by virtue of (7). Hence $V_b(q(t)) \leq V_b(q(0)) < \infty$ for all $t \geq 0$. Since $V_b \rightarrow \infty$ whenever $q_i \rightarrow q_{i,\min}$ for at least one agent $i \in \mathcal{N}$, and the latter implies $q_i \rightarrow \partial W$, we conclude that $q(t) \in \mathcal{J}(q)$, for all $t \geq 0$. \diamond

Thus, if agents start within the interior of the virtual workspace, they are forced to remain within it. Collision avoidance, i.e., Lemma 1 also holds. We now show that agents reach a configuration where $u_i = 0$, $\theta_i = 0$ for all i :

Corollary 4: Consider the system of multiple nonholonomic agents (1) driven by (5),(6) and starting from the set of initial conditions $\mathcal{I}(q) \cap \mathcal{J}(q)$. Then the system reaches a configuration in which $u_i = 0$ and $\theta_i = 0$ for all $i \in \mathcal{N}$.

Proof: $\mathcal{J}(q)$ is closed and bounded for the trajectories of the closed-loop system, by Lemma 3, and (7) guarantees that \dot{V}_b is negative semidefinite. By Theorem 2, the trajectories of the closed-loop system reach the largest invariant subset of the set $S^b = \{(r_{xi}^b = r_{yi}^b = 0) \vee (r_{xi}^b \cos \theta_i + r_{yi}^b \sin \theta_i = 0), \forall i \in \mathcal{N}\}$. Using the same arguments as in the proof of Theorem 2, we deduce that the largest invariant set S_v^b contained in S^b is $S^b \supset S_v^b = \{r_{xi}^b = r_{yi}^b = 0, \forall i \in \mathcal{N}\}$. Within S_v^b , we have $r_{xi}^b = r_{yi}^b = 0, \forall i \in \mathcal{N}$ and hence $u_i = 0$, for all $i \in \mathcal{N}$. Within S_v^b , we also have $\dot{\theta}_i = -\theta_i$ for all i , and hence the orientations of all agents tend to zero. \diamond

Note that $r_{xi}^b = r_{yi}^b = 0, \forall i \in \mathcal{N} \Rightarrow (R_3 \otimes I_2)q - (R_4 \otimes I_2)q_{\min} = 0$. We now show that the control law is related to the final relative positions of the agents in a manner similar to the unbounded case. From the proof of Corollary

4 we derive that the system converges to the largest invariant subset of the set S_v^b . Please note that the result of Lemma 3 holds for arbitrarily small c_b, d_b . For $c_b, d_b \rightarrow 0$, we have that either $q_i \rightarrow q_{i,\min}$, or $\rho_{ib} \rightarrow 0$, for those agents that do not satisfy the condition $q_i \rightarrow q_{i,\min}$. Thus, in this case

$$\begin{aligned}
& (R_3 \otimes I_2)q - (R_4 \otimes I_2)q_{\min} = \\
& = (R_2 \otimes I_2)q - (R_4 \otimes I_2)(q - q_{\min}) \\
& = (R_2 \otimes I_2)q - \left(\left(\text{diag} \left\{ \frac{\rho_{1b}}{\gamma_{1b}^2}, \dots, \frac{\rho_{Nb}}{\gamma_{Nb}^2} \right\} \right) \otimes I_2 \right) (q - q_{\min}) \\
& = (R_2 \otimes I_2)q - \\
& \quad - \left[\frac{\rho_{1b}}{\gamma_{1b}^2} (q_1 - q_{1,\min}) \quad \dots \quad \frac{\rho_{Nb}}{\gamma_{Nb}^2} (q_N - q_{N,\min}) \right]^T \\
& = (R_2 \otimes I_2)q
\end{aligned}$$

since for each $i \in \mathcal{N}$, we have either $q_i \rightarrow q_{i,\min}$, or $\rho_{ib} \rightarrow 0$, for $c_b, d_b \rightarrow 0$ as discussed above. Thus, S_v^b coincides with the set S_v of the proof of Theorem 2. As proved in that Theorem, the largest invariant subset within S_v is the set $S_0 = \{q | \rho_{ij} = 0, \forall i, j \in \mathcal{N}, i \neq j\} = \{q | \|q_i - q_j\| \geq d, \forall i, j \in \mathcal{N}, i \neq j\}$. Hence the system reaches a configuration in which all agents remain within the workspace bounds and each agent is located at a distance no less than d from every other agent in the group, *provided that such configuration exists within the workspace bounds*.

This result is formally stated in the next Theorem:

Theorem 5: Consider the system (1) driven by the controls (5),(6) and starting from $\mathcal{I}(q) \cap \mathcal{J}(q)$. Assume that the set $B(q) = \{q \in \text{int}(W) | \|q_i - q_j\| \geq d, \forall i, j \in \mathcal{N}, i \neq j\}$ is nonempty. Then the system reaches a configuration in which all agents remain in the interior of the workspace, and $\|q_i - q_j\| \geq d, \forall i, j \in \mathcal{N}, i \neq j$, with zero orientations.

Remark 1: Similarly to the unbounded case, $B(q)$ being non-empty corresponds to a case where each agent occupies a $d/2$ -disc at steady state. If $B(q)$ is empty, the workspace is not large enough to fulfill the above geometric condition, and the system converges to a configuration that minimizes the cost function V_b , respecting the constraint of Lemma 3 that the agents are forced to remain within the workspace boundary. Thus, some of the $d/2$ -discs may overlap.

Remark 2: Since the centers of agents remain within the virtual workspace boundary, the actual discs that represent the agents remain within the real workspace boundary ∂W_0 .

Remark 3: The results can be extended to the case where the workspace is an arbitrary convex region.

VI. SIMULATIONS

To support the results presented in the previous paragraphs, we provide a series of computer simulations.

In the first simulation, six nonholonomic agents navigate under the controls (2),(3). Screenshots I-III in Figure 2 show the evolution of the closed-loop system. The agents are located at their initial positions in the first screenshot. The sensing radius d is equal to 0.0894 in this simulation. The collision avoidance objective is fulfilled, due to the proposed control design. The agents disperse in the workspace and eventually stop in screenshot III. Screenshot IV depicts the final positions of agents. Each agent occupies a disc of radius $d/2$. These discs are visualized in Screenshot IV by the

large discs whose center is the center of corresponding agent. By virtue of Theorem 2, the large discs are disjoint. In the

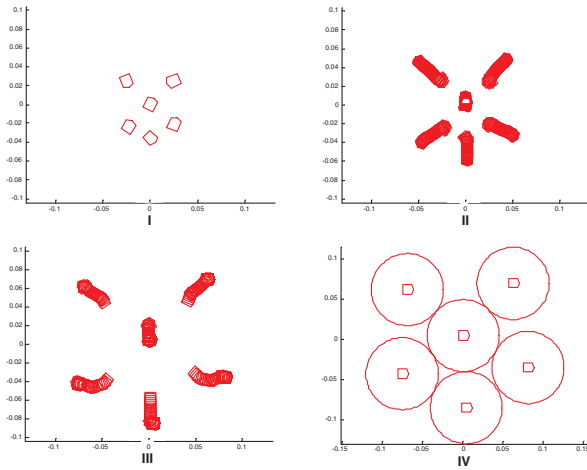


Fig. 2. Swarm dispersion for six nonholonomic agents. The agents disperse in the workspace and eventually occupy six disjoint discs of radius $d/2$.

second simulation (Figure 3), agents navigate in a bounded workspace under (5),(6). We have again $d = 0.0894$ and the workspace radius is $R_W = 15d$. Agents start from an initial condition where they are aggregated near the workspace center. Some agents approach the workspace boundary and remain within it due to the repulsive potential. Collision avoidance is fulfilled throughout the closed-loop system evolution. The workspace is large enough to allow the agents to occupy six disjoint discs of radius $d/2$ at steady state, i.e. the set B of Theorem 5 is nonempty. This is depicted in the last screenshot of Figure 3.

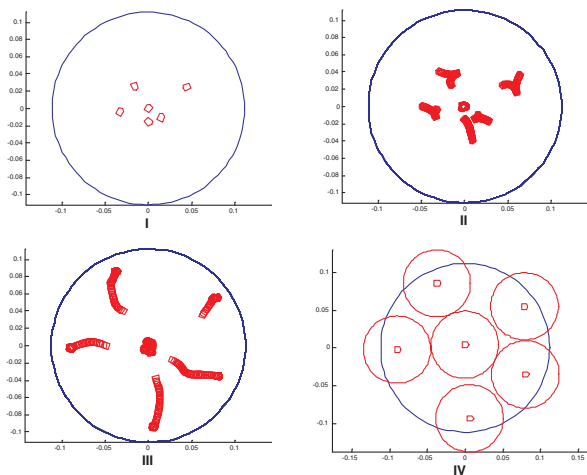


Fig. 3. Swarm dispersion for six nonholonomic agents in a bounded workspace. The workspace is large enough to allow the agents to occupy six disjoint discs of radius $d/2$ at steady state. Agents are forced to remain within the workspace boundary.

VII. CONCLUSIONS

We proposed a distributed inverse agreement strategy for multiple nonholonomic agents that forces the agents to

disperse in the workspace. Both the cases of unbounded and circular workspace were treated. In the first case, we showed that the closed-loop system reaches a configuration in which the minimum distance between any pair of agents is larger than a specific lower bound. It was proven that this bound coincides with the agents' sensing radius. In the case of a bounded workspace, the control law was redefined in order to force the agents to remain within the workspace boundary throughout the closed loop system evolution. The results were supported through computer simulations. Current research involves exploring the relation of the sensing radius, the number of agents and the radius of the workspace with the emptiness of the set B of Theorem 5.

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