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Dynamics and Control of Constrained Mechanical Systems in Terms of Reduced Quasi-Velocities

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***Abstract*—A unified formulation for deriving the equations of motion of constrained or unconstrained multi-body systems (MBS) in terms of (reduced) quasi–velocities is presented. In this formulation, the square–root of the mass matrix is used to transform disparate units into homogeneous units for all the quasi-velocities resulting the gauge invariance. We show that the square–root factorization of mass matrix and hence the quasi–velocities are not unique, rather they are related by unitary transformations. Subsequently, we show that a particular transformation leads to significant simplification of the dynamic modeling. The number of differentiations required to derive the equations of motion is reduced. This fact combined with the fact that the expression of the inverse of the mass matrix factorization can be given in a closed–form make the formulation suitable for symbolic manipulation or numerical computation. Moreover, in this formulation the equations of motion are decoupled from those of constrained force and each system has its own independent input (that is not attainable by other formulations). This allows the possibility to develop a simpler force control action that is totally independent from the motion control action. The structure of the formulation is also suitable for control purposes. Tracking control and regulation control of constrained multi-body systems based on a combination of feedbacks on the vectors of the quasi–velocity and the configuration variables (which may contain redundant variables) are presented.**

I. INTRODUCTION

Manipulators' dynamics are often described by the second-order nonlinear equations parameterized by a configuration-dependent inertia matrix and the nonlinear vector containing the Coriolis and centrifugal terms. Since these equations are the cornerstone for simulation and control of robotic manipulators, many researchers have attempted to develop efficient modelling techniques to derive the equations of motion of multi-body systems (MBSs) in novel forms. A unifying idea for most modeling techniques is to describe the equations of motion in terms of joint coordinates and their time–derivatives. However, this is not the only possibility. There also exist other techniques to describe the equations of motion in terms of *quasi–velocities*, i.e., a vector whose Euclidean norm is proportional to the square root of the system's kinetic energy, which can lead to simplification of these equations [1]–[10]. A recent survey on some of these techniques can be found in [9]. In short, the square–root factorization of mass matrix is used as a transformation to obtain the quasi–velocities, which are a linear combination of the velocity and the generalized coordinates [9].

It was shown by Kodistchek [1] that if the square–root factorization of the inertia matrix is integrable, then the robot dynamics can be significantly simplified. In such a case, transforming the generalized coordinates to quasi–coordinates by making use of the integrable factorization modifies the robot dynamics to a system of double integrator. Then, the cumbersome derivation of the Coriolis and centrifugal terms is not required. Rather than deriving the mass matrix of MBS first and then obtaining its factorization, Rodriguez *et al.* [5] derived the closed–form expressions of the mass matrix factorization of an MBS and its inverse directly from the link geometric and inertial parameters. This eliminates the need for the matrix inversion required to compute the forward dynamics.

The interesting question of when the factorization of the inertia matrix is integrable, i.e., the factorization being the Jacobian of some quasi–coordinates, was addressed independently in [4] and [3]. Using the notion that the inertia matrix defines a metric tensor on the configuration manifold, Spong [4] showed that the necessary and sufficient condition for the existence of an integrable factorization of the inertia matrix is that the metric tensor is a Euclidean metric tensor.[1] The concept of quasi–velocities has also been used for the set-point control of manipulators [6], [11]. However, the problem of the tracking control of manipulators using quasi–velocities feedback still remains unsolved owing to unintegrability of the quasi–velocities.

The main goal of this paper is to extend the concept of quasi–velocities for the efficient modeling of constrained MBSs for simulation, analysis, and control purposes. Taking advantage of the fact that square–root factorizations are invariant under unitary transformations, we find a particular transformation that greatly simplifies the Lagrangian equations of constrained MBS. The formulation appears to be in a compact form with minimum number of differentiation operations required to derive that the equations of motion is reduced. Moreover, the unitary transformation naturally and elegantly leads to decoupling of the equations of motion and those of the constrained forces in such a way that each system has its own control input. This gives the possibilities for an independent control design and analysis of a con-

[1] A manifold with a Euclidean metric is said to be "flat" and the curvature associated with it is identically zero [6].

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strained MBS that cannot be attained by other approaches [12]–[16]. We also present some properties of the quasi–velocity dynamic formulation that could be useful for control purposes. Unlike the previous derivation, the Coriolis term in our formulation is expressed explicitly in terms of (reduced) quasi–velocities that are not only simpler but also useful for control implementation. The dynamic model is used for developing tracking control of constrained MBSs based on a combination of feedbacks on the vector of reduced quasi–velocity and the vectors of configuration-variables, which can contain redundant variables. It is worth noting that, unlike the conventional schemes [13]–[15] for control of a constrained MBS, the quasi–velocity-based controller does not require a set of independent configuration-variables that is not always possible to find. Finally, various square–root factorizations of inertia matrix are reviewed.

II. Quasi-Variables Transformations

A. Square-Root Factorization of the Mass Matrix

Dynamics of a MBS with kinetic energy, T, and potential energy, P, obeys the standard *Euler–Lagrange* (EL) equations, which are given as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = f, \tag{1}$$

where $q \in \mathbb{R}^n$ is the vector of configuration-variables[2] used to define the configuration of the system, and f is the generalized forces acting on the system. The generalized forces $f = f_p + f_a$ contain all possible external forces including the conservative forces $f_p = -\partial P/\partial q$ owing to gravitational energy plus all active and dissipative forces represented by f_a. The system kinetic energy is in the following quadratic form:

$$T(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q}, \tag{2}$$

where the generalized inertia matrix $M(q)$ is *symmetric* and *positive definite* for all q. It is well known that any symmetric positive-definite matrix M can be decomposed as

$$M = WW^T, \tag{3}$$

where W is the *square root factorization* of M.

Considering the transformation

$$\bar{W} = WV,$$

where V is an orthogonal matrix, i.e., $VV^T = V^TV = I$, one can trivially verify that $\bar{W}\bar{W}^T = M$. Thus, we get the following remark

Remark 1: The square–root factorization (3) is not unique, rather they are related by unitary transformations.

Now, substituting (3) into (2) and then applying the EL formulation yields

$$\begin{aligned} f &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(WW^T\dot{q}\Big) - \frac{1}{2}\Big(\frac{\partial}{\partial q}\|W^T(q)\dot{q}\|^2\Big)^T \\ &= W\frac{\mathrm{d}}{\mathrm{d}t}\big(W^T\dot{q}\big) + \Big(\dot{W} - \frac{\partial\big(W^T(q)\dot{q}\big)^T}{\partial q}\Big)W^T\dot{q} \end{aligned} \tag{4}$$

[2]also known as generalized coordinates

Note that (4) is obtained using the property that for any vector field $a(q)$, we have

$$\frac{\partial}{\partial q}\|a(q)\|^2 = 2a^T\frac{\partial a}{\partial q}. \tag{5}$$

Define

$$v \triangleq W^T(q)\dot{q} \quad \text{and} \quad u \triangleq W^{-1}(q)f, \tag{6}$$

which are the so-called vectors of *quasi–velocity* and *quasi–force*, respectively. Since M is an invertible matrix, W^{-1} is well–defined and hence the reciprocals of relations (6) always exist. Pre-multiplying (4) by W^{-1} and the substituting (6) into the resultant equation, we arrive at the equations of mechanical systems expressed by the quasi–variables:

$$\dot{v} + \Gamma v = u, \tag{7a}$$

where

$$\Gamma \triangleq W^{-1}\Big(\dot{W} - \frac{\partial v^T}{\partial q}\Big) \tag{7b}$$

is the Coriolis term associated with the quasi–velocity.

B. Changing Coordinates by Unitary Transformations

Remark 1 states that the quasi–velocities (and also quasi–forces) can not be uniquely determined. Rather, the following variables:

$$\bar{v} = V^Tv \qquad \text{and} \qquad \bar{u} = V^Tu, \tag{8}$$

obtained by any unitary transformation V, are also valid choices for the new quasi–velocities and quasi–forces. Now we are interested to derive the equations of motion expressed by the new quasi–variables $\bar{v}$. To this end, using the reciprocal of (8), i.e., $v = V\bar{v}$ and $f = V\bar{f}$, into (7a) and then multiplying the resultant equation by V, we arrive at

$$\dot{\bar{v}} + V^T\dot{V}\bar{v} + V^T\Gamma V\bar{v} = \bar{u} \tag{9}$$

Analogous to the rotation transformation in the three-dimensional Euclidean space, consider matrix V as a transformation in the n-dimensional space. Then, it is known that the time–derivative of an orthogonal matrix V satisfies a differential equation of this form [17]

$$\dot{V} = -\Omega V, \tag{10}$$

where Ω is a *skew symmetric matrix* representing the angular rates in $n-D$ space [18]. It is worth noting that in the three-dimensional space, the angular rate matrix can be obtained from the vector of angular velocity by $\Omega = [\omega\times]$. For the n–dimensional case, the method for computing the elements of matrix Ω can be found in [17], [19], [20]. Finally, by replacing (10) in (9), we can show that the latter equation is equivalent to

$$\dot{\bar{v}} + \bar{\Gamma}\bar{v} = \bar{u}, \quad \text{where} \quad \bar{\Gamma} = V^T\big(\Gamma - \Omega\big)V. \tag{11}$$

C. Conservation of Kinetic Energy

The kinetic energy expressed by the quasi–velocities is trivially

$$T = \frac{1}{2}\|v\|^2. \tag{12}$$

In the absence of any external force, the principle of conservation of kinetic energy dictates that the kinetic energy of mechanical system is bound to be constant, i.e., $u = 0 \Longrightarrow \dot{T} = 0$. On the other hand, the zero-input response of a mechanical system is $\dot{v} = -\Gamma v$. Substituting the latter equation in the time-derivative of (12) gives

$$v^T \Gamma v = 0, \tag{13}$$

which is consistent with the earlier result reported by Jain *et al.* [6] that the Coriolis term associated with quasi–velocities does no mechanical work. Note that (13) is a necessary but not a sufficient condition for Γ to be a *skew-symmetric* matrix.

D. State-Space Model

It should be pointed out that although there is a one-to-one correspondence between velocity coordinate $\dot{q}$ and the quasi–velocity v, they are not synonymous. This is because the integration of the former variable leads to the generalized coordinate, while that of the latter variable does not always lead to a meaningful vector describing the configuration of the mechanical system.

Defining a matrix $R = W\Gamma$, we can calculate its element from (7b) as

$$R_{ij} = \sum_k \big(W_{ij,k} - W_{kj,i}\big)\dot{q}_k. \tag{14}$$

Now let us assume that $\dot{\xi} = v$. If ξ is a conservative field then it must be the gradient of a scalar function, and hence ξ is an explicit function of q, i.e., $\xi = \xi(q)$. In that case, (6) implies that $W^T(q)$ is actually a Jacobian as $W_{ij} = \xi_{j,i}$. Since the Jacobian is an invertible matrix, $\xi(q)$ must be an invertible function meaning that there is a one-to-one correspondence between ξ and q. Under this circumstance, ξ is called the vector of *quasi–coordinates*. It is well known that the existence of quasi–coordinates fundamentally simplifies the equations of motion [1]–[4]. It can be also seen from (7b) that if $\xi(q)$ exists and it is a smooth function, then the expression in the parenthesis of the right-hand side of (14) vanishes, i.e.,

$$W_{ij,k} - W_{kj,i} = \xi_{j,ik} - \xi_{j,ki} = 0,$$

because of the equality of mixed partials. Thus, $\Gamma \equiv 0$ and the equations of motion become a simple integrator system.

Technically speaking, a necessary and sufficient condition for the existence of the quasi–coordinates, ξ, is that the Riemannian manifold defined by the robot inertia matrix $M(q)$ be locally flat.[3] However, that has been proved to be a very stringent condition [3]. Nevertheless, vector $x^T = \begin{bmatrix} q^T & v^T \end{bmatrix}$ is sufficient to describe completely the states of MBSs. Hence, similar to [6], we look at the transformation only in the velocity space. That is, only the velocity coordinate is replaced with the quasi–velocity whereas the generalized coordinate remains. Setting (6) and (7a) in state space form gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} q \\ v \end{bmatrix} = \begin{bmatrix} W^{-T} \\ -\Gamma \end{bmatrix} v + \begin{bmatrix} 0 \\ I \end{bmatrix} u. \tag{15}$$

It is interesting to note that dynamics system (15) is in the form of the so-called *second-order kinematic model* of constrained mechanism, which appears in kinematics of nonholonomic systems. This is the manifestation of the fact that the integration of quasi–velocities, in general, does not lead to quasi–coordinates.

[3]By definition, a Riemannian manifold that is locally isometric to Euclidean manifold is called a locally flat manifold [4].

III. CONSTRAINED MBS

A. Equations of Motion

In this section, we extend the notion of the quasi–velocity for modeling of constrained mechanical systems where the coordinates are related by a set of m algebraic equations $\Phi(q) = 0$. The constraints can be written in the Pfaffian form as

$$A(q)\dot{q} = 0 \tag{16}$$

where Jacobian $A = \partial\Phi/\partial q \in \mathbb{R}^{m\times n}$ is not necessarily a full-rank matrix because of the possible redundant constraints. The EL equations of the constrained MBSs with kinetic energy T are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = f - A^T\lambda, \tag{17}$$

where $\lambda \in \mathbb{R}^m$ are the *generalized Lagrangian multipliers*.

Using any form of the square–root factorizations in a development similar to (6)-(7), we can show that (17) is equivalent to

$$\dot{v} + \Gamma v = u - \Lambda^T\lambda, \tag{18}$$

where

$$\Lambda \triangleq AW^{-T}. \tag{19}$$

It can be verified that the quasi–velocities satisfy the following Paraffin constraint equation:

$$\Lambda v = 0. \tag{20}$$

Also, (20) may suggest that Λ be taken as the Jacobian of the constraint with respect to the quasi–coordinates. However, this is true only if the quasi–coordinates ever exist. This means that, in general, system (18) together with (20) most likely constitutes a non-holonomic system even though the configuration–variables q satisfies a holonomic constraint equation.

Since W is a full-rank matrix, we can say rank(Λ) = rank$(A) = r$, where $r \leq m$ is the number of independent constraints. Then, according to the *singular value decomposition* (SVD) there exist unitary (orthogonal) matrices $U = [U_1 \;\; U_2] \in \mathbb{R}^{m\times m}$ and $V = [V_1 \;\; V_2] \in \mathbb{R}^{n\times n}$ (i.e., $U^TU = I_m$ and $V^TV = I_n$) such that

$$\Lambda = U\Sigma V^T \quad \text{where} \quad \Sigma = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \tag{21}$$

and $S = \mathrm{diag}(\sigma_1, \cdots, \sigma_r)$ with $\sigma_1 \geq \cdots \geq \sigma_r > 0$ being the singular values [21]. The unitary matrices are partitioned so that the dimensions of the submatrices U_1 and V_1 are consistent with those of S. That is the columns of U_1 and V_2 are the corresponding sets of orthonormal eigenvalues which span the range space and the null space of Λ, respectively.

Now, we take advantage of the arbitration in choosing the square–root factorization to find a particular one that leads to decoupling of the equations of motion and those of constrained force. Consider the unitary transformation (8) where the orthogonal matrix V corresponds to decomposition (21). Then, the equations of motion expressed in terms of the new quasi–variables become

$$\dot{\bar{v}} + \bar{\Gamma}\bar{v} = \bar{u} - \bar{\Lambda}^T\lambda, \qquad (22)$$

where $\bar{\Lambda} \triangleq \Lambda V$ and $\bar{\Gamma}$ has been already defined in (11). Again, it can be easily verified that the new quasi–velocities satisfy the following Pfaffian constraints:

$$\bar{\Lambda}\bar{v} = 0. \qquad (23)$$

At the first glance, the transformed system (22)–(23) reassembles (18)–(20) without gaining any simplification. However, it is the structure of $\bar{\Lambda}$ that will result in further simplification. Using (21) in the definition of $\bar{\Lambda}$ gives

$$\bar{\Lambda} = \begin{bmatrix}\Lambda_r & 0_{m\times(n-r)}\end{bmatrix} \quad \text{where} \quad \Lambda_r \triangleq U_1 S. \qquad (24)$$

Since $\Lambda_r \in \mathbb{R}^{m\times r}$ is a full-rank matrix, it can be inferred from (23) that the first rth elements of the transformed quasi–velocity $\bar{v}$ must be zero. That is,

$$\bar{v} = \begin{bmatrix}0_{r\times 1}\\ v_r\end{bmatrix}, \qquad (25)$$

where $v_r \in \mathbb{R}^{n-r}$ represents a set of *reduced quasi–velocities*– in the following, the subscript $_r$ denotes variables associated with the reduced-order variables. Clearly, the zero components of the transformed quasi–velocities are due to the r–independent constraints. It can be verified that (25) is equivalent to

$$V_2^T v = v_r. \qquad (26)$$

Now, by using (26) in the reciprocal of relation (6), we can show that there is a one-to-one correspondence between v and $\dot{q}$ as

$$\dot{q} = W^{-T}V_2 v_r, \quad \text{and} \quad v_r = V_2^T W^T \dot{q}. \qquad (27)$$

Moreover, by virtue of (25), we partition the quasi–force accordingly as

$$\bar{u} = \begin{bmatrix}u_o\\ u_r\end{bmatrix}, \quad \text{where} \quad \begin{matrix}u_o \triangleq V_1^T W^{-1} f\\ u_r \triangleq V_2^T W^{-1} f\end{matrix}. \qquad (28)$$

In addition, we assume that matrix $\bar{\Gamma}$ is divided into four block matrices

$$\bar{\Gamma}_{ij} = V_i^T\big(\Gamma - \Omega\big)V_j, \qquad i,j = 1,2, \qquad (29)$$

and then define

$$\Gamma_r \triangleq \bar{\Gamma}_{22}, \quad \text{and} \quad \Gamma_o \triangleq \bar{\Gamma}_{12}. \qquad (30)$$

Now, substituting (25) into (22) and then using definitions (28) and (29), we arrive at

$$\dot{v}_r + \Gamma_r v_r = u_r, \qquad (31a)$$

and

$$\Lambda_r^T\lambda + \Gamma_o v_r = u_o \qquad (31b)$$

Apparently, (31a) and (31b) represent the equations of motion and those of constraint force which are completely decoupled from each other. Note that the partitioned components of the quasi–force, i.e., $\bar{u}_r$ and $\bar{u}_o$, contribute exclusively to the motion system and the constraint force system, respectively. Now, we are ready to combine (31a) and (27) into the state–space form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}q\\ v_r\end{bmatrix} = \begin{bmatrix}W^{-T}V_2\\ -\Gamma_r\end{bmatrix} v_r + \begin{bmatrix}0\\ I\end{bmatrix} u_r. \qquad (32)$$

The Lagrangian multipliers can be uniquely obtained from (31b) through matrix inversion only if $r = m$, i.e., in the presence of no redundant constraints. Otherwise, there are fewer equations than unknowns, and hence there is no unique solution to (31b). Nevertheless, the minimum norm solution can be found by

$$\min\|\lambda\| \quad \leftarrow \quad \lambda = U_1 S^{-1}\big(u_o - \Gamma_o v_r\big). \qquad (33)$$

B. Calculating the Coriolis Term

The Coriolis force term Γ_r itself characterized completely the motion dynamics of a constrained mechanical system expressed by reduced quasi–velocities. In this section, we describe Γ_r expressed in terms of v_r that appears to be simpler than (30). First, in view of (5) and the facts that $v = V_2 v_r$ and $\|v_r\| = \|v\|$, one can verify that

$$\frac{\partial v_r}{\partial q} = \frac{\partial v}{\partial q}V_2. \qquad (34)$$

Now, consider the relation between v_r and $\dot{q}$ as

$$v_r = W_r^T(q)\dot{q},$$

where $W_r = WV_2$. Then, from (7b), (10), (30), and (35) we obtain

$$\begin{aligned}\Gamma_r &= V_2^T W^{-1}\big(\dot{W} - \frac{\partial v}{\partial q}\big)V_2 + V_2\dot{V}_2\\ &= V_2^T W^{-1}\big(\dot{W}_r - W\dot{V}_2 - \frac{\partial v}{\partial q}V_2\big) + V_2^T\dot{V}_2\\ &= V_2^T W^{-1}\big(\dot{W}_r - \frac{\partial v_r}{\partial q}\big).\end{aligned} \qquad (35)$$

Finally, by noting that $V_2^T W^{-1} = W_r^+$ is a left inverse of W_r, that is, $W_r^+ W_r = I$, we can express (35) by

$$\Gamma_r = W_r^+\big(\dot{W}_r - \frac{\partial v_r}{\partial q}\big), \qquad (36)$$

which closely resembles the Coriolis term of unconstrained mechanical systems in (7b). It is interesting to note that $W_r \in \mathbb{R}^{n\times(n-r)}$ can be thought of as the factorization of the semi-positive "mass matrix" $M_r = W_r W_r^T = WPW^T$, where $P = V_2V_2^T$ is a projection matrix which projects vectors from $\mathbb{R}^n$ to the null space of system (20). A comparison between systems (31a)–(36) and (7) reveals that the formulation of constrained mechanical systems remains essentially similar to that of unconstrained mechanical systems if the quasi–velocity is simply replaced by a reduced quasi–velocity.

Finally, a development similar to (35) shows that

$$\Gamma_o = W_r^{-}\big(\dot{W}_r - \frac{\partial v_r}{\partial q}\big),$$

where $W_r^{-} = V_1^T W^{-1}$ is an annihilator for W_r, i.e., $W_r^{-} W_r = 0$.

IV. Control

In general, it should be always possible to choose a minimal set of independent velocity coordinate, equal in number of the degrees-of-freedom (DOF) exhibited by the mechanical system. However, a minimal set of independent generalized coordinates may not exist; a well-known example is the orientation configuration of a rigid-body that can not be expressed by a three-dimensional vector. However, the conventional control of constrained mechanical system relies on the existence of a minimal set of parameters defining the configuration of a constrained MBS. In this section, we provide velocity and position feedbacks from (reduced) quasi–velocities and (dependent) configuration variables, respectively, for tracking control and regulating a constrained MBS. Interestingly enough, the control challenge, then, becomes similar to that of non-holonomic systems, as the configuration of MBS can not be represented by any quasi–coordinates.

A. Properties

First, we explore some properties of system (31) that will be useful in control design purposes.

Remark 2: Using (13) and the fact that Ω is a skew-symmetric matrix in definition (29), we can say

$$v_r^T \Gamma_r v_r = 0.$$

Assume that c_1 denote the minimum eigenvalue of M for all configurations q, that is, $c_1 I \leq M(q)$. Then, using the norm properties leads to

$$\|W^{-1}\| \leq c_1^{-1/2}, \tag{37}$$

which in turn leads to the following.

Remark 3: Matrices Γ_r and Γ_o satisfy

$$\|\Gamma_r\| \leq \gamma \|v_r\| \quad \text{and} \quad \|\Gamma_o\| \leq \gamma \|v_r\| \tag{38}$$

for some bounded constant $\gamma > 0$ (see Appendix A for details).

Furthermore, we assume that $W_r(q)$ is a sufficiently smooth function so that it satisfies the Lipschitz condition, i.e., there exists a finite scalar $c_2 > 0$ such that

$$\|W_r^T(q) - W_r^T(q^*)\| \leq c_2 \|q - q^*\| \qquad \forall q, q^* \in \mathbb{R}^n. \tag{39}$$

B. Tracking Control

We adopt a Lyapunov-based control scheme [22, p. 74] for designing a feedback control in terms of quasi–velocities. Define the composite error

$$\epsilon \triangleq \tilde{v}_r + W_r^T(q) K_p \tilde{q}, \tag{40}$$

where $K_p > 0$ and $\tilde{v}_r = v_r - v_{r_d}$. Also, define the new variable as $s = v_{r_d} - W_r^T K_p \tilde{q}$, which is used in the following control law:

$$u_r = \dot{s} + \Gamma_r s - K_d \epsilon, \tag{41}$$

where $K_d > 0$. Applying control law (41) to system (31a) gives the dynamics of the error ϵ in terms of the first-order differential equation:

$$\dot{\epsilon} = -\big(\Gamma_r + K_d\big)\epsilon. \tag{42}$$

As shown in Appendix B, the solution of (42) is bounded by

$$\|\epsilon\| \leq \|\epsilon(0)\| e^{-\eta_1 t}, \tag{43}$$

where $\eta_1 = \lambda_{\min}(K_d)$, and hence the composite error ϵ is exponentially stable. On the other hand, pre-multiplying both sides of (40) by $W^{-T}(q)V_2(q)$, the resultant equation can be rearranged to the following differential equation

$$\begin{aligned}\dot{\tilde{q}} = &- W^{-T}(q)V_2(q)\big[W_r^T(q) - W_r^T(q_d)\big]W^{-T}(q_d)V_2(q_d)v_d \\ &- K_d\tilde{q} + W^{-T}(q)V_2(q)\epsilon. \end{aligned} \tag{44}$$

Assuming that $\|\dot{q}_d\| \leq c_4$, $c = c_2 c_4 / c_1$ and $\eta_2 = \lambda_{\min}(K_d) - c$ and using (37) and (39), we can show that the solution of the above differential equation satisfies

$$\|\tilde{q}\| \leq \left(\|\tilde{q}(0)\| - \kappa\|\epsilon(0)\|\right) e^{-\eta_2 t} + \kappa \|\epsilon(0)\| e^{-\eta_1 t}, \tag{45}$$

where $\kappa^{-1} = c_1^{1/2}(\eta_2 - \eta_1)$; see Appendix C for details. Equation (45) implies that $\tilde{q}$ exponentially converges to zero as t goes to infinity if $\eta_2 > 0$, i.e., T

$$\lambda_{\min}(K_p) > c. \tag{46}$$

The above development can be summarized in the following theorem.

Theorem 1: Assume that the mass matrix factorization is a smooth function satisfying the Lipschitz condition and that $\|v_d\|$ is bounded. Then, for a sufficiently large position gain, i.e., (46) is satisfied, the error trajectories of the configuration-variables and quasi–velocities of a constrained MBS under control law (40)–(41) exponentially converge to zero.

Tracking of the desired constraint force λ_d can be achieved simply by compensating for the velocity perturbation term in (31b), i.e.,

$$u_o = \Lambda_r^T \lambda_d + \Gamma_o v_r. \tag{47}$$

C. Regulation

The advantages of using the notion quasi–velocity for control of serial manipulators have been recognized by many researchers and various setpoint PD controllers based on the quasi–velocity feedbacks have been proposed [6], [11], [23], [24]. In this section, we extend such a feedback control for hybrid motion/force control of constrained mechanical systems.

Consider the following control law for system (31a)

$$u_r = -K_d v_r - V_2^T(q) W^{-1}(q) K_p \tilde{q}, \tag{48}$$

where K_d and K_p are *positive definite* gain matrices, and $\tilde{q} = q - q_d$ is the configuration error. Then, the dynamics of the closed–loop system becomes

$$\dot{v}_r = -\Gamma_r v_r - K_d v_r - V_2^T W^{-1} K_p \tilde{q}. \tag{49}$$

Choose the following standard Lyapunov function

$$V = \frac{1}{2}\|\tilde{v}_r\|^2 + \frac{1}{2}\tilde{q}^T K_p \tilde{q}. \tag{50}$$

Then, using Remark 2 in the time-derivative of (50) along (49) yields

$$\dot{V} = -v_r^T K_d v_r,$$

which is negative-semidefinite. Clearly, we have $\dot{V} = 0$ only if $v_r = 0$, and hence the largest invariant set with respect to system (49) can be found as follows: $\mathcal{S} = \{v_r, \tilde{q} : v_r = 0, \quad V_2^T W^{-1} K_p \tilde{q} = 0\}$. On the other hand, $V_2^T W^{-1} K_p$ is a full-rank matrix and thus the vector equation inside the set $\mathcal{S}$ can only hold if the configuration error $\tilde{q}$ vanishes. Therefore, according to LaSalle's Global Invariant Set Theorem [25], [26, p.115], the solution of system (49) asymptotically converges to the invariant set $\mathcal{S}$, that is, $q \to q_d$ as t goes to infinity.

In the case of full-rank Jacobian, the Lagrangian multiplier can be simply regulated to its desired value λ_d by

$$u_o = \Lambda_r^T \lambda_d. \tag{51}$$

Substituting (51) into (31b) and using Remark 3, we get

$$\|\lambda - \lambda_d\| \leq \frac{\gamma}{s_r}\|v_r\|^2.$$

Since $v_r \to 0$, then $\lambda \to \lambda_d$ as t goes to infinity.

D. Gauge Invariant

A problem that often arises in robotics, namely hybrid control or minimum solution to joint rate or joint force, is that generalized coordinate q may have a combination of rotational and translational components that can be even compounded by having combination of rotational and translational constraints [27]. This may lead to inconsistent results, i.e., results that are invariant with respect to changes in dimensional units unless adequate weighting matrixes are used [16], [27]–[29].

An important property of the reduced quasi–velocity and quasi–force is that they always have homogenous units. As a matter of fact, since

$$\|v_r\| = \|v\| = \sqrt{2T},$$

we can say that all elements of the vector of quasi–velocity v or v_r must have a homogenous unit $[\sqrt{\text{kg}}\text{m/s}]$. This is true even if the vector of the generalized coordinate or the constraints have combinations of rotational and translational components. Similarly, one can argue that the elements of the quasi–force have always identical unit $[\sqrt{\text{kg}}\text{m/s}^2]$, regardless of the units of the generalized force of the constraint wrench.

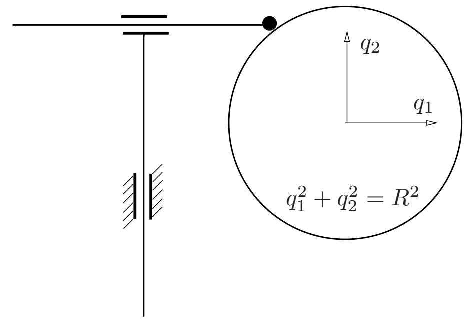

Fig. 1. Constrained motion of a two-DOF manipulator.

E. An Analytical Example

Consider a Cartesian robot with 2 DOF whose end-effector is constrained to move on a circle as illustrated in Fig.1. Let us assume $M = \text{diag}(m_1, m_2)$ denote the inertia matrix of the robot. Then, a simple choice for the square–root factorization of the inertia matrix is $W = \text{diag}(\sqrt{m_1}, \sqrt{m_2})$. In addition, the constraint equation is trivially

$$\Phi(q) = q_1^2 + q_2^2 - R^2 = 0.$$

Thus, $A = \begin{bmatrix} 2q_1 & 2q_2 \end{bmatrix}$ and the unitary matrix whose columns span the null space of $\Lambda = \begin{bmatrix} q_1/\sqrt{m_1} & q_2/\sqrt{m_2} \end{bmatrix}$ and its orthogonal complement can be derived as

$$V = \frac{1}{g(q)} \begin{bmatrix} \sqrt{m_2} q_1 & -\sqrt{m_1} q_2 \\ \sqrt{m_1} q_2 & \sqrt{m_2} q_1 \end{bmatrix}.$$

where $g(q) \triangleq (m_2 q_1^2 + m_1 q_2^2)^{1/2}$. Therefore, according to (27), the reduced-order quasi–velocity is

$$v_r = \frac{g \dot{q}_1}{q_2}, \tag{52}$$

and

$$\Lambda_r = \frac{2g}{\sqrt{m_1 m_2}}. \tag{53}$$

Having obtained variables (52) and (53), one can design an force/motion setpoint controller according to (48) and (51). Derivation of the full–dynamic model of the robotic system, say for a model-based control, requires the angular rate matrix Ω. In the two–dimensional Euclidean space, the angular rate matrix takes the following simple form:

$$\Omega = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix},$$

where scalar ω can be calculated by virtue of (10) as

$$\omega = \frac{\sqrt{m_1 m_2} R^2}{g^2} v_r.$$

Having obtained Ω, we can implement the reference tracking controller (40)–(41).

V. Various Square-Root Decompositions

An important result of Remark 1 is that the quasi–velocity of a particular system is not unique. Rather, different factorizations of the inertia matrix can be used to transform velocity coordinates to quasi–velocities. Selecting an adequate factorization can play an important role in model simplification and/or computational efficiency. Fortunately, there are a variety of algorithms for computing numerical and symbolical factorizations some of which will be reviewed in this section.

A. Cholesky Decomposition

According to the *Cholesky decomposition*, a symmetric and positive-definite matrix M can be decomposed efficiently into $M = LL^T$, where L is a lower–triangular matrix with strictly positive–diagonal elements; L is also called the *Cholesky triangle*. The following formula can be used to obtain the Cholesky triangle through some elementary operations

$$l_{ii} = \big(m_{ii} - \sum_{k=1}^{i-1} l_{ik}^2\big)^{1/2} \qquad \forall i = 1, \cdots, n \tag{54}$$
$$l_{ji} = \big(m_{ji} - \sum_{k=1}^{i-1} l_{jk} l_{ik}\big)/l_{ii} \qquad \forall j = i+1, \cdots, n$$

Since L is a lower-triangular matrix, its inverse can be simply computed by the back substitution technique.

B. Positive Square Root

According to the spectral theorem, there is an operator $W = M^{1/2}$ such that $M^{1/2}$ is itself positive and $(M^{1/2})^2 = M$. The operator $M^{1/2}$ is the unique *positive square root* factorization of M. Consider the diagonalized form of M as $M = ZDZ^T$, where the diagonal matrix D contains the eigenvalues of M and the columns of Z consist of the corresponding orthogonal eigenvectors. Then, the factorization can be found by

$$M^{1/2} = ZD^{1/2}Z^T.$$

Although this factorization is conceptually simple, it does not appear to be computationally efficient. This is because computing the symmetric square–root factorization requires the eigenvalues and eigenvectors, whereas the Cholesky decomposition only needs simple algebraic operations. Nevertheless, it is known that symmetric square–root of a matrix can be obtained numerically through a fast converging iteration loop [30]. A method to robustly handle the mass matrix eigenfactor derivatives was developed in [20] based on the square–root algorithm to solve the matrix Riccati differential equation [19].

C. Spatial Operator Factorization

An alternative to the above numerical factorization approach is provided by the results on operator factorization of the manipulator mass matrix [5], [6], [31], where closed–form expressions of the mass matrix factorization and its inverse and time–derivatives are provided. Rodriguez *et al* [5] developed a recursive factorization of the mass matrix of serial chain manipulators that can be directly obtained from the link geometric and inertial parameters. It was shown that the mass matrix has factorization $W = \big(I + H\Phi G\big)D^{1/2}$, where H and Φ are given by a known link geometric parameters while the diagonal matrix D and matrix G are related to the link masses [5], [31]. This factorization has been referred to as the ”Innovative Factorization” [5]. Since $I + H\Phi G$ is a lower triangular matrix, the inversion of the factorization can be readily obtained. Moreover, a closed–form expression of the time–derivative of the Innovative Factorization was presented by Jain *et al.* [6].

VI. Conclusions

Quasi-velocities had been proven to be a strong tool for efficient modeling of MBSs. In this paper, we have extended the concept of quasi–velocities for comprehensive and efficient modeling of the constrained MBSs. Taking advantage of the fact that the square–root factorization is invariant under unitary transformations, we showed that the quasi–velocities associated with a given system are not unique rather they are related by unitary transformation. Subsequently, it has been shown that the unitary matrix corresponding to the kernel of the Paffinian constraints of the quasi–velocities could lead to a significant simplification of the dynamics formulation of constrained MBS if the equations are expressed in terms of the reduced quasi–velocities. The expression of the Coriolis term was presented in a unified way applicable for unconstrained and constrained MBSs alike. In it, the number of required differentiation operations was reduced, which can lead to computational efficiency. Furthermore, the transformation naturally led to the decoupling of the equations of motion than those of constraint forces such that each system has its own independent force input. This allowed the possibility to develop a simple force control action that is totally independent from the motion control action. Another import aspect of this formulation is that the invariance problems rising in hybrid force/motion control of constrained MBS due to the use of selection matrices becomes a nonissue. This is because the the square root of the mass matrix nicely transforms disparate units into homogeneous units for all the quasi–velocities.

The structure of the formulation was proved to be also suit control application. To this end, some properties of the quasi–velocity dynamic formulation that could be useful for control purposes were presented. It has been followed by development of the tracking control of a constrained MBS based on the composite feedback on the vectors of quasi–velocity and configuration–variables, where the latter vector contains redundant variables.

Appendix A

In view of (27) and (37) and knowing that $\|V_2\| = 1$, we can say

$$\|\dot{q}\| \leq \|W^{-T}\|\|V_2\|\|v_r\| \leq c_1^{-1/2}\|v_r\|. \tag{55}$$

Assuming that the factorization $W(q)$ is a sufficiently smooth function, then all the partials in (14) are bounded and hence there exists a finite $c_3 > 0$ such that $\|R\| \leq c_3\|\dot{q}\|$. Using this result, (55) and (37) in (7b), we get

$$\|\Gamma\| \leq \gamma\|v_r\|, \tag{56}$$

where $\gamma = c_3/c_1$. Finally, knowing that $\|V_1\| = \|V_2\| = 1$, we can infer (38) from (29) and (56).

Appendix B

Consider the following positive–definite function:

$$V = \frac{1}{2}\|\epsilon\|^2$$

In view of Remark 2, the time-derivative of the above function along the error trajectory (42) is obtained as

$$\dot{V} = -\epsilon^T K_d \epsilon$$

which gives

$$\dot{V} \leq -2\lambda_{\min}(K_d)V.$$

Thus

$$V \leq V(0)e^{-2\lambda_{\min}(K_d)t},$$

which is equivalent to (43).

Appendix C

Consider the following positive–definite function

$$V = \frac{1}{2}\|\tilde{q}\|^2, \tag{57}$$

whose time–derivative along (44) gives

$$\dot{V} = -\tilde{q}^T K_d \tilde{q} - \tilde{q}^T W^{-T}(q) V_2(q)\epsilon \\ - \tilde{q}^T W^{-T}(q) V_2(q)\big[W_r^T(q) - W_r^T(q_d)\big] W^{-T}(q_d) V_2(q_d)\dot{v}_d.$$

From (37) and (39), we can find a bound on $\dot{V}$ as

$$\dot{V} \leq -\lambda_{\min}(K_d)\|\tilde{q}\|^2 + c_1^{-1/2}\|\tilde{q}\|\|\epsilon\| + c_1^{-1}c_2c_4\|\tilde{q}\|^2 \\ \leq -2\eta_2 V + \sqrt{2}c_1^{-1/2}V^{1/2}\|\epsilon\|, \tag{58}$$

which is in the form of a Bernoulli differential inequality. The above nonlinear inequality can be linearized by the following change of variable $U = \sqrt{V}$, i.e.,

$$\dot{U} \leq -\eta_2 U + (2c_1)^{-1/2}\|\epsilon\| \tag{59}$$

In view of the comparison lemma [26, p. 222] and (43), one can show that the solution of (59) must satisfy

$$U \leq U(0)e^{-\eta_1 t} + \frac{\|\epsilon(0)\|}{\sqrt{2c_1}}\int_0^t e^{-\eta_2(t-\tau)-\eta_1\tau}d\tau,$$

which is equivalent to (45).

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