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On the Hybrid Dynamics of Planar Mechanisms Supported by Frictional Contacts. II: Stability of Two-Contact Rigid Body Postures

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Abstract— **This paper is concerned with the hybrid dynamics and stability of a planar rigid body supported by two frictional contacts in a gravitational field. The stability of equilibrium postures is investigated under initial perturbations which may involve sliding or separation of the contacts. The paper formulates the hybrid dynamics induced by collisions and impacts at the two contacts. The Zeno behavior of bouncing and clattering motion converging to limit points of one- or two-contact re-establishment are analyzed, and a condition guaranteeing convergence to a Zeno point is derived. Finally, this condition is combined with the results of the companion paper to derive sufficient conditions for stability of frictional two-contact equilibrium postures of a planar rigid body.**

I. INTRODUCTION

Performing robotic tasks of quasistatic manipulation and locomotion with frictional contacts is based on transition through equilibrium postures. In order to enhance reliability, the selected postures must be *dynamically stable* with respect to small perturbations, which, in many practical cases, involve separation, sliding, or rolling of the contacts. The goal of this paper is to focus on the simplest possible model of a planar rigid body supported by two frictional contacts, analyze its hybrid dynamics under small perturbations about an equilibrium posture, and derive conditions for posture's stability. The rigid body, having a variable center-of-mass, serves as a simplification for a two-legged mechanism moving quasistatically on a rough terrain. A companion paper [12] analyzed the constrained frictional dynamics of such multiple-contact mechanisms under perturbations about an equilibrium posture, and derived a necessary condition for stability and a sufficient condition for finite-time recovery of initially perturbed contact. Focusing on the reduced problem of a rigid-body with two contacts, this paper analyzes the hybrid dynamical system which is dominated by collisions at the contacts. The two main contributions of this paper are as follows. First, it derives a condition guaranteeing that the dynamic response of the perturbed rigid body converges in finite time to a limit point at which either one or two contacts are re-established. Second, it combines this condition with the results of [12] and establishes sufficient conditions for stability of two-contact equilibrium postures.

The classical notion of stability focuses on the solution convergence or boundedness, for a dynamical system under small perturbations about equilibrium points (e.g. Lyapunov stability for continuous dynamical systems [8]). A mechanical system with intermittent contacts is a special subclass of *hybrid dynamical systems*, in which phases of continuous dynamics are interleaved by discrete events of non-smooth changes due to collisions. [7], [14]. A classical technique for analyzing the dynamics and stability of such systems is the *Poincaré map* [6], [10], which is based on sampling the dynamic solution of the hybrid system at the collision times, and thus reducing to a discrete-time dynamical system. Wang [15] used linearization of the Poincaré map to analyze the stability of a rigid body under sequential impacts at a single contact. Goyal et al. [4], [5] analyzed the linearized discrete dynamics of a symmetric rod undergoing alternating collisions at its two endpoints under the simplifying assumption of zero gravity. Another central approach for stability analysis of hybrid systems is based on using generalized Lyapunov functions for hybrid systems [10], [16]. However, frictional equilibrium postures typically form a continuum of configurations which do not lie at minima of the potential energy. Thus they are not associated with any obvious candidate for a Lyapunov function.

A key feature of hybrid dynamical systems, which is strongly related to stability, is the *Zeno solution* [13], [17]. This special type of solution involves an infinite number of switches (i.e. collisions) and converges in *finite time* to some limit point. While the hybrid system cannot predict the dynamic solution past the limit point, Ames et al. proposed the concept of *completed hybrid system* [1]. This concept postulates that after convergence of the Zeno solution, the system switches to a holonomically constrained dynamical system, thus enabling composition of the hybrid phases and the constrained phases of the dynamics. This principle is key to the stability analysis presented in this paper.

The structure of the paper is as follows. The next section reviews the notions and main results presented in [12]. Section III formulates the impact-induced hybrid dynamics and demonstrates the Zeno phenomenon. Section IV presents sufficient conditions for finite-time convergence to contact re-establishment. Section V combines the results with the results of [12], and derives sufficient conditions for frictional stability. Finally, the concluding section discusses limitations and possible extensions of the results.

II. PROBLEM STATEMENT

This section defines the basic terminology and reviews the notions and main results presented in [12]. Consider a planar rigid body $\mathcal{B}$ having mass m and moment of inertia $I=m\rho^2$, where ρ is the radius of gyration. In its nominal posture, $\mathcal{B}$ is supported by two point contacts on a frictional terrain

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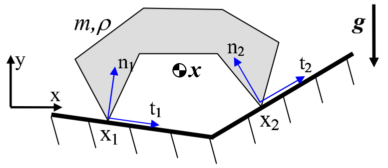

Fig. 1. A planar rigid body supported by two contacts.

under gravity. The nominal position of $\mathcal{B}$'s center-of-mass is denoted $\boldsymbol{x}$, and the nominal position of the contacts are x_1 and x_2. For convenience, it is assumed that the contacts are made between two vertex points of $\mathcal{B}$ and two segments of a piecewise-linear terrain. Thus, the tangent and normal unit vectors at the vicinity of the i-th contact, denoted t_i and n_i, are constant. The terrain is assumed as *upward facing*, in the sense that $n_i \cdot \boldsymbol{e}_y > 0$ for $i = 1, 2$, where $\boldsymbol{e}_y$ is the upward vertical direction. The configuration of $\mathcal{B}$ is parametrized by the coordinates $q = (\boldsymbol{r}, \theta)$, where $\boldsymbol{r} = (r_x, r_y)$ is the displacement of $\mathcal{B}$'s center-of-mass from its nominal position and θ is the orientation of $\mathcal{B}$ relative to its orientation at the nominal posture. The positions of the two vertex points of $\mathcal{B}$ are given by $\boldsymbol{r}_i(q) = \boldsymbol{r} + R(\theta)(x_i - \boldsymbol{x})$, for $i = 1, 2$, where $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Since $\mathcal{B}$ is not allowed to penetrate the terrain, its configuration must satisfy the constraints given by

$$h_i(q) = n_i \cdot (\boldsymbol{r}_i(q) - x_i) \geq 0 \ , \ i = 1, 2. \qquad (1)$$

Let $\dot{q}, \ddot{q}$ denote the generalized velocity and acceleration of $\mathcal{B}$. The *state* of $\mathcal{B}$ is thus constrained to lie within the *collision-free region* in state space, defined by

$$\mathcal{F} = \left\{ (q, \dot{q}) : \begin{array}{c} h_i(q) \geq 0 \text{ for } i = 1,2 \text{ such that} \\ \text{if } h_j(q) = 0 \text{ then } \nabla h_j(q) \cdot \dot{q} \geq 0 \end{array} \right\}. \qquad (2)$$

A contact force $f_i \in I\!R^2$ acts at each contact point $\boldsymbol{r}_i$ that satisfies $h_i(q) = 0$. According to Coulomb's friction law, each contact force must lie within a *friction cone*, denoted $\mathcal{C}_i$, which is given by $\mathcal{C}_i = \{f_i : |t_i \cdot f_i| \leq \mu(n_i \cdot f_i)\}$, where μ is the *coefficient of friction*. The dynamics of $\mathcal{B}$ is governed by its *equation of motion*, given by

$$M\ddot{q} + G = J_1^T(q) f_1 + J_2^T(q) f_2 \ , \quad \text{where}$$
$$M = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m\rho^2 \end{pmatrix}, \ G = \begin{pmatrix} 0 \\ mg \\ 0 \end{pmatrix}, \ J_i(q) = \frac{\partial \boldsymbol{r}_i(q)}{\partial q} \qquad (3)$$

and g is the gravitational acceleration. The nominal configuration $q_0 = 0$ of $\mathcal{B}$ is characterized by the nominal positions of the center-of-mass $\boldsymbol{x}$ and the contacts x_1 and x_2. q_0 is called a *frictional equilibrium posture* if there exist contact forces $f_i \in \mathcal{C}_i$, $i = 1, 2$ that satisfy (3) with $q = q_0$, $\dot{q} = 0$, and $\ddot{q} = 0$. For given nominal contacts x_1 and x_2, the *feasible equilibrium region*, denoted $\mathcal{R}_{EQ}$, is the region of center-of-mass locations $\boldsymbol{x}$ that form a frictional equilibrium posture. This region is an infinite vertical strip, whose computation is analyzed in [11] and reviewed in [12]. In the following, we assume that q stays within a small neighborhood U of $q = 0$. Hence, we simplify the notation by using a local chart of U and treat both q and $\dot{q}$ as elements of $I\!R^3$. An ϵ-neighborhood of the equilibrium state $(q, \dot{q}) = (0, 0)$ is then defined as $N_\epsilon = \{(q, \dot{q}) : \|q\| < \epsilon \text{ and } \|\dot{q}\| < \epsilon\}$. Using this notion, the definition of frictional stability is as follows:

Definition 1 (Frictional stability). *Consider the planar rigid body supported by two frictional contacts, whose dynamics is given in (3). A frictional equilibrium posture $q = 0$ possesses* frictional stability *if for any arbitrarily small $\epsilon > 0$, there exists sufficiently small $\delta > 0$, such that for any initial conditions $(q(0), \dot{q}(0)) \in N_\delta \cap \mathcal{F}$, the dynamic solution $q(t)$ exists, and converges to an equilibrium posture while staying within the neighborhood N_ϵ of the original equilibrium.*

Note that the notion of dynamic solution is not completely defined at this stage. It is composed of two different phases, namely the constrained frictional dynamics and the impact-induced hybrid dynamics. The results of [12] regarding the constrained frictional dynamics are briefly reviewed below, while the impact-induced hybrid dynamics will be analyzed in sections III-IV. Assume that $\mathcal{B}$ is given an initial position-and-velocity perturbation which imposes contact constraints on its motion. In order to compute the resulting trajectory $q(t)$ of $\mathcal{B}$, one needs to solve (3) under the constraints, as follows. Let $v_i = J_i(q)\dot{q}$ denote the velocity at the i-th contact. Each contact is governed by one of four distinct modes, denoted S, F, R, and L, which correspond to contact separation, fixed (or rolling) contact, right-sliding and left-sliding, respectively. For two contacts, a contact mode is thus encoded by a two-letter word. For example, the contact mode SR means that the contact x_1 is instantaneously separating, while the contact x_2 is sliding to the right. Each contact mode is associated with equality and inequality constraints on the contact force f_i and on the contact velocity v_i as summarized in Table I. Choosing a particular contact mode, its equality constraints on velocities v_i are differentiated with respect to time, yielding affine constraints on the acceleration $\ddot{q}$. Augmenting the constraints with the equation of motion (3), one obtains a square linear system in $\ddot{q}, f_1, f_2$, generically having a unique solution for the instantaneous dynamics under the chosen contact mode. However, each contact mode must also satisfy its associated inequality constraints in order to be consistent. A well-known observation is that in some cases, no consistent solution exists, and the only way to resolve the inconsistency is to incorporate impulsive forces [9]. In order to avoid such scenarios, the companion paper [12] introduced the notion of *kinematic-strong equilibrium*, which is briefly reviewed here.

TABLE I

THE POSSIBLE CONTACT MODES AT A PLANAR FRICTIONAL CONTACT.

contact mode	**physical meaning**	**kinematic constraints**	**force constraints**
S	Separation	$v_i \cdot n_i > 0$	$f_i = 0$
F	Fixed/rolling	$v_i = 0$	$\|f_i \cdot t_i\| \leq \mu(f_i \cdot n_i)$
R	Right sliding	$v_i \cdot n_i = 0$ $v_i \cdot t_i > 0$	$f_i \cdot t_i = -\mu(f_i \cdot n_i)$ $f_i \cdot n_i \geq 0$
L	Left sliding	$v_i \cdot n_i = 0$ $v_i \cdot t_i < 0$	$f_i \cdot t_i = \mu(f_i \cdot n_i)$ $f_i \cdot n_i \geq 0$

Consider now a frictional equilibrium posture with *zero initial velocity*. Since such initial conditions do not determine a unique contact mode, one needs to consider each non-static contact mode, compute its associated instantaneous dynamic solution, and then check its consistency, where under zero velocity, the kinematic constraints are evaluated *with contact velocities* v_i *replaced by contact accelerations*. Using these notions, the definition of kinematic-strong equilibrium is as follows. An equilibrium posture $q_0 = 0$ is a kinematic-strong equilibrium if for each non-static contact mode, the instantaneous dynamic solution under zero velocity satisfies *all* force constraints, and violates *at least one* kinematic constraint. The following theorem establishes the relation of kinematic-strong equilibrium with the boundedness of dynamic solution and recovery of initially-perturbed contacts.

Theorem 1 ([11]). *Let* q_0 *be an equilibrium posture of* $\mathcal{B}$. *Then for any arbitrarily small* $t_0, \epsilon > 0$, *there exists a sufficiently small* δ, *such that under* any *initial conditions* $(q(0), \dot{q}(0)) \in N_\delta(q_0) \cap \mathcal{F}$, *there exists a time* $t' < t_0$ *such that the initial contact mode is consistent during the time interval* $[0, t')$, *and the solution stays within the neighborhood* $N_\epsilon(q_0)$. *Moreover, at the time* $t = t'$, *either an initially sliding contact becomes stationary (or rolling), or an initially separated contact recovers via a collision.*

III. Impact-Induced Hybrid Dynamics

This section formulates the hybrid dynamics of $\mathcal{B}$ and reviews the phenomenon of Zeno behavior, demonstrated on the classical bouncing ball example. When the state $(q, \dot{q})$ of $\mathcal{B}$ lies in the collision-free region $\mathcal{F}$ defined in (2), its free-flight is governed by the *unconstrained dynamics* $M\ddot{q} + G = 0$, where M and G are given in (3). When $\mathcal{B}$ reaches a contact with a non-zero approach velocity, a collision occurs. This event is characterized in terms of $(q, \dot{q})$ as $h_i(q) = 0$ and $\nabla h_i(q) \cdot \dot{q} < 0$, for some $i \in \{1, 2\}$. The collision is modeled as an instantaneous event of discontinuous velocity change, while the configuration q remains unchanged. The velocity change due to a collision at the i-th contact in a configuration q is given by $M\Delta\dot{q} = J_i^T(q)P_i$, where P_i is the *contact impulse*. In order to compute the post-collision velocity, one needs to establish a *collision law*, which is a relationship between the pre-collision velocity $\dot{q}(t^-)$ and the impulse P_i, which, in turn, determines the post-collision velocity $\dot{q}(t^+)$. In this work we adopt the *frictionless restitution law* [1], which assumes that P_i acts in the direction of the contact normal and its magnitude is determined such that the change in the normal velocity at the contact, denoted v_i^n, satisfies $v_i^n(t^+) = -e v_i^n(t^-)$, where $e \in (0, 1)$ is termed the *coefficient of restitution*. Under these assumptions, the velocity change due to collision in a configuration q satisfies the linear relationship given by ([1]):

$$\dot{q}(t^+) = A_i(q)\dot{q}(t^-), \text{ where}$$
$$A_i(q) = I - \frac{1+e}{\nabla h_i(q)^T M^{-1} \nabla h_i(q)} M^{-1} \nabla h_i(q) \nabla h_i(q)^T \quad (4)$$

and I is the identity matrix. Examples of more complicated collision laws that account for friction at the contacts can be found in [2], [3]. Using a simplified version of the notation in [1], a *solution* of the hybrid dynamics under a given initial state $(q(0), \dot{q}(0))$ is defined by a piecewise-smooth trajectory $q(t)$ and a countable (possibly infinite) set of collision times $\mathcal{T} = \{t_1, t_2, \ldots\}$. For $t \in (t_k, t_{k+1})$, the trajectory $q(t)$ satisfies the unconstrained dynamics. At each collision time $t_k \in \mathcal{T}$, the system's state satisfies $h_i(q(t_k)) = 0$ and $v_i^n(t_k) < 0$ for some $i \in \{1, 2\}$, corresponding to a collision at the i-th contact, and the velocity is changed according to the collision law $\dot{q}(t_k^+) = A_i(q(t_k))\dot{q}(t_k^-)$.

A fundamental phenomenon in hybrid dynamical systems is the *Zeno behavior*, at which the solution $q(t)$ converges to a limit point through an infinite sequence of collisions occuring in *finite time*. A classical example of Zeno behavior is a ball modeled as a point mass m, which bounces on a horizontal floor under gravity. The unconstrained dynamics of the ball's height $y(t)$ is $m\ddot{y} = -mg$. The contact is represented by the unilateral constraint $y \geq 0$, and the collision law (4) reduces to $\dot{y}(t^+) = -e\dot{y}(t^-)$. It is then easily shown that the discrete-time dynamics of the post-collision velocity (i.e. the Poincaré map) is $\dot{y}(t_{k+1}^+) = e\dot{y}(t_k^+)$, and that y and $\dot{y}$ converge asymptotically to zero for any $e \in (0, 1)$. Moreover, the infinite sequence of collision times t_k converges to a finite limit t_∞. The Zeno behavior is of crucial importance in the ensuing stability analysis, since it guarantees contact re-establishment in finite time.

In order to complete the formulation of the hybrid dynamics of $\mathcal{B}$ under small initial perturbations, we define a new set of coordinates as $q' = (h_1, h_2, \theta)$ where $h_1(q), h_2(q)$ are given in (1), and *linearize* the hybrid dynamics about the state $(0, 0)$. The linearized unconstrained dynamics is given in terms of $(q', \dot{q}')$ as ([11]):

$$\begin{pmatrix} \ddot{h}_1 \\ \ddot{h}_2 \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} -g_1 \\ -g_2 \\ 0 \end{pmatrix}, \quad (5)$$

where $g_i = -n_i \cdot \boldsymbol{g}$ for $i = 1, 2$, and $\boldsymbol{g}$ is the vector of gravity acceleration. The linearized collision laws expressed in terms of $(q', \dot{q}')$ are given by ([11]):

$$\dot{q}'(t^+) = A_i \dot{q}'(t^-), \ i = 1, 2, \text{ where}$$
$$A_1 = \begin{pmatrix} -e & 0 & 0 \\ (1+e)\psi_1 & 1 & 0 \\ (1+e)\varphi_1 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & (1+e)\psi_2 & 0 \\ 0 & -e & 0 \\ 0 & (1+e)\varphi_2 & 1 \end{pmatrix},$$
$$\varphi_i = -\frac{b_i}{b_i^2+\rho^2}, \ \psi_i = -\frac{b_1 b_2 + (n_1 \cdot n_2)\rho^2}{b_i^2+\rho^2}, \text{ and } b_i = t_i \cdot (x_i - \boldsymbol{x}). \quad (6)$$

IV. Conditions for Convergence to Zeno Points

This section analyzes the linearized hybrid dynamics of $\mathcal{B}$ and derives conditions guaranteeing that its solution converges to a Zeno point involving re-establishment of one or two contacts in finite time.

A. One- and Two- Contact Zeno Solutions

We now describe two types of Zeno solutions of the hybrid dynamics of $\mathcal{B}$, named the *bouncing* and *clattering* motions.

The bouncing motion is a Zeno solution involving an infinite sequence of collisions at a *single* contact r_i, which converges in finite time to a limit point named *1-Zeno point*, that satisfies $h_i = 0$, i.e. the contact at r_i is re-established. Note that the linearized hybrid dynamics of a single coordinate h_i under collisions at r_i only is identical to the hybrid dynamics of a bouncing ball. Thus, it is easily shown that there exists an open set of initial conditions leading to a Zeno solution of single-contact bouncing without any collision at the other contact.

When the solution of the hybrid dynamics involves collisions at *both* contacts, proving existence of general Zeno solutions becomes highly complicated. In this work we focus on the simplest Zeno solution involving collisions at two contacts, named the *clattering motion* [4], and derive conditions for its convergence. Clattering motion involves an infinite sequence of *alternating* collisions at both contact, which converge in finite time to a limit point named *2-Zeno point*, that satisfies $h_1 = h_2 = 0$, i.e. both contacts are re-established. We now formulate the Poincaré map for the linearized hybrid dynamics of $h_1(t)$ and $h_2(t)$ under clattering motion. First, we define the *alternating* indices $\{a, b\}$, where a is the index of the colliding contact, and b is the index of the free-flying contact at time t_k. Without loss of generality, we assume that $a = 1, b = 2$ for odd k, and $a = 2, b = 1$ for even k. Let us denote $h_{ik} = h_i(t_k^+)$ and $u_{ik} = \dot{h}_i(t_k^+)$ for $i \in \{a, b\}$, where by convention, $k = 0$ corresponds to the time t_1^-. Note that, by definition, $h_{ak} = 0$ for all k. The Poincaré map of clattering motion is then given by ([11]):

$$\begin{aligned} &u_{a,k+1} = e\sqrt{v_{bk}^2 + 2g_b h_{bk}} \\ &u_{b,k+1} = u_{ak} - u_{bk} - (\psi_b(1+e)+1)\sqrt{v_{bk}^2 + 2g_b h_{bk}} \\ &h_{b,k+1} = u_{ak}\tau_{bk} - g\tau_k^2/2, \\ &\text{where } \tau_k = (u_{bk} + \sqrt{v_{bk}^2 + 2g_b h_{bk}})/g_b. \end{aligned} \tag{7}$$

The intermediate variable τ_k is the time to next collision, i.e. $\tau_k = t_{k+1} - t_k$. The dynamics of clattering motion (7) must also satisfy an auxiliary *persistence condition* requiring that the alternating order of collisions is maintained, which is formulated as

$$\eta_k < 1 \text{ for } k \geq 1, \text{ where } \eta_k = \frac{\tau_k g_a}{2u_{ak}}. \tag{8}$$

B. Conditions for Convergence to Zeno Points

We now derive conditions for convergence of clattering motion, and for overall convergence to 1- or 2- Zeno points. First, note that (7) is a nonlinear discrete-time dynamical system whose right-hand side is *not differentiable* at $u_{ak} = u_{bk} = h_{bk} = 0$. Thus, its stability properties cannot be determined by conventional linearization. Motivated by the work of Goyal et al. [4], [5] who analyzed the dynamics of clattering under the simplifying assumption of *zero gravity*, we define the ratio $\epsilon_k = 2g_b h_{bk}/u_{bk}^2$, and require that $\epsilon_k \to 0$. The physical interpretation of this requirement is that at times t_k^+, the separation distance of the free-flying contact h_{bk} is sufficiently small compared to its approach velocity u_{bk}, thus the change in the velocities $\dot{h}_a, \dot{h}_b$ due to gravity during the next flight phase is negligible. Let $\bar{u}_{ak}, \bar{u}_{bk}, \bar{h}_{bk}$ denote the dynamic solution of u_{ak}, u_{bk}, h_{bk} in (7) in the limit of $\epsilon_k \to 0$ (i.e. zero gravity). The discrete-time dynamics of $\bar{u}_{ak}, \bar{u}_{bk}, \bar{h}_{bk}$ is then given by

$$\begin{aligned} &\bar{u}_{a,k+1} = -e\bar{u}_{bk} \\ &\bar{u}_{b,k+1} = \bar{u}_{ak} + \psi_b(1+e)\bar{u}_{bk} \\ &\bar{h}_{b,k+1} = \bar{u}_{ak}\bar{\tau}_k \text{ , where } \bar{\tau}_k = \bar{h}_{bk}/|\bar{u}_{bk}|. \end{aligned} \tag{9}$$

Note that the dynamics of $\bar{u}_{ak}$ and $\bar{u}_{bk}$ in (9) is linear and decoupled from $\bar{h}_{bk}$. It is shown in [11] that if the parameters e, ψ_1, ψ_2 satisfy the condition given by

$$\psi_1 > 0, \ \psi_2 > 0, \text{ and } \sqrt{\psi_1\psi_2} > \frac{e^{1/3} + e^{2/3}}{1+e}, \tag{10}$$

then the solutions of (9) for $\bar{u}_{1k}, \bar{u}_{2k}, \bar{h}_{bk}$ and $\bar{\epsilon}_k$ converge asymptotically to zero in an infinite number of discrete-time steps completed in finite time (i.e. the sum of the infinite sequence $\bar{\tau}_k$ is finite), where $\bar{\epsilon}_k$, is defined as $\bar{\epsilon}_k = 2g\bar{h}_{bk}/\bar{u}_{bk}^2$. The following lemma states that (10) also guarantees convergence of clattering motion under *nonzero* gravity, and is based on comparing the solutions of the dynamical systems (9) and (7) under the same initial conditions.

Lemma IV.1 ([11]). *Consider the solutions of (7) and (9) under initial conditions satisfying $\bar{h}_{b1} = h_{b1} > 0$, $\bar{u}_{a1} = u_{a1} \geq 0$, and $\bar{u}_{b1} = u_{b1} < 0$. Then these solutions satisfy the following inequalities:*

1) $h_{bk} \leq \bar{h}_{bk}$ *for all* $k > 1$
2) $\tau_{bk} \leq \bar{\tau}_{bk}$ *for all* $k > 1$
3) $\epsilon_k \leq \bar{\epsilon}_k$ *for all* $k > 1$

Moreover, if the condition (10) is satisfied, then both ϵ_k and η_k decrease monotonously to zero with k.

The results implied by this lemma are as follows. First, note that once a collision at a contact r_1 is followed by a collision at r_2, the initial value of η satisfies $\eta_1 < 1$, and condition (10) then implies that η_k is monotonously decreasing. As a result, the solution is guaranteed to satisfy the persistence condition (8), along with finite-time convergence to a 2-Zeno point. Thus, condition (10) is named the *clattering convergence condition*. Another implication is that condition (10) guarantees that the solution of the linearized hybrid system converges in finite time to either 1- or 2- Zeno point, under *any* given initial conditions. Finally, the results regarding the linearized hybrid system can be extended to the *nonlinear* hybrid dynamics of $\mathcal{B}$ under sufficiently small initial perturbation about equilibrium. All these implications are summarized in the following theorem.

Theorem 2. *Let $q = 0$ be an equilibrium posture of $\mathcal{B}$ with two contacts on a piecewise-linear terrain, and assume that the clattering convergence condition (10) is satisfied. Then for any arbitrarily small $t_0, \epsilon > 0$, there exists a sufficiently small δ, such that under any initial conditions $(q(0), \dot{q}(0)) \in N_\delta \cap \mathsf{int}(\mathcal{F})$, the solution of the hybrid dynamics converges in finite time $t' < t_0$ to a Zeno point at which either $h_1 = 0$ or $h_2 = 0$ or both, while staying within the neighborhood N_ϵ during the entire time interval $[0, t']$.*

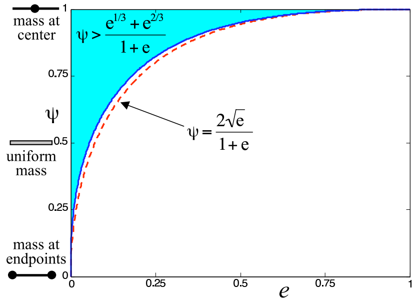

Fig. 2. Clattering convergence region in (e, ψ) plane.

C. Example - the Symmetric Rod on Flat Floor

We now demonstrate the application of Theorem 2 to the example of a slender horizontal rod on a flat horizontal floor, which was previously analyzed in [4], [5] under the simplifying assumption of zero gravity. The length of the rod is $2L$, and its radius of gyration is ρ. It is assumed that the mass distribution of the rod is symmetric about its center, and that it makes contact with the terrain only at its endpoints. Due to symmetry, the two parameters ψ_1 and ψ_2 in (6) are equal and given by $\psi_1 = \psi_2 \triangleq \psi = (L^2 - \rho^2)/(L^2 + \rho^2)$. The parameter $\psi \in [0, 1]$ (denoted q in [5]) characterizes the mass distribution of the rod as follows. $\psi = 1$ corresponds to a concentrated mass at the center of the rod. $\psi = 0.5$ corresponds to uniformly distributed mass. $\psi = 0$ corresponds to concentrated masses at the rod's endpoints. The conditions of clattering convergence (10) reduce to a single inequality in the parameters (e, ψ). Graphically, this inequality defines a region in the plane of (e, ψ), which is the shaded region shown in Fig. 2. Note that in order to ensure clattering convergence, the coefficient of restitution e must be sufficiently small, and the parameter ψ must sufficiently large, i.e. the rod's mass is concentrated close to its center. The dashed curve in Fig 2a corresponds to the condition derived in [5] for clattering convergence. Note that this condition is less conservative than our condition (10), since we account for the effect of gravity and impose the additional requirement $\epsilon_k \to 0$. Another key difference is that [5] did not account for bouncing motion, which is governed by gravity effects, while in our setting, solutions of clattering motion and of bouncing motion are both possible, depending on initial conditions. The reader is referred to [11] for the closed-from dependence on initial conditions and for numerical simulation results

V. Frictional Stability of Two-Contact Postures

This section combines the results above with the results of the companion paper [12], and presents sufficient conditions for frictional stability of two-contact frictional equilibrium postures of a planar rigid body.

A. The Completed Hybrid System

We now consider the composition of the two different phases of the dynamics of $\mathcal{B}$, namely, the impact-induced hybrid dynamics and the constrained frictional dynamics analyzed in [12]. Theorem 2 gives sufficient conditions for finite-time convergence of the hybrid dynamics solution to a Zeno point. Note that *Zeno points are not physical equilibrium points*, as they involve nonzero velocity of $\mathcal{B}$. Thus, one needs to determine the dynamic solution *past* a Zeno point. A recent work by Ames et al. [1] postulates that at the Zeno time t_∞, the hybrid systems *switches to a holonomically constrained dynamical system*, with the initial conditions $q(t_\infty), \dot{q}(t_\infty)$. This composition of dynamical systems, termed *a completed hybrid system* in [1], applies naturally to the two-contact hybrid dynamics of $\mathcal{B}$, as follows. When the hybrid dynamics converges to a 1-Zeno point of via bouncing motion, it switches to the constrained frictional dynamics of a single-contact sliding or rolling, i.e. one of the contact modes RS, FS, or LS. When the hybrid dynamics converges to a 2-Zeno point of a via clattering motion, it switches to the constrained frictional dynamics of a two-contact sliding, i.e. contact modes RR or LL. Finally, when the solution of the constrained dynamics of single-contact sliding or rolling reaches a collision at the free-flying contact, it undergoes an infinite sequence of alternating collisions while both contacts are maintained. Under the clattering convergence condition (10), this sequence converges to a 2-Zeno point, and the dynamics eventually switches to a constrained two-contact sliding. This special Zeno solution, occurring *in zero time* , is termed *chattering Zeno* in [1].

B. Sufficient Conditions for Frictional Stability

We now finally address the problem of frictional stability. The following theorem gives sufficient conditions for frictional stability of two-contact equilibrium postures of $\mathcal{B}$.

Theorem 3 ([11]). *Let q_0 be a frictional equilibrium posture of a planar rigid body $\mathcal{B}$ on an upward-facing piecewise-linear terrain under gravity. Then if both kinematic-strong equilibrium condition and clattering convergence condition (10) are satisfied at q_0, then it possesses frictional stability.*

The main idea of the proof is based on the observation that these two conditions impose a chronological order on the composed phases of solution under small initial perturbations, as follows. First, consider a small perturbation at which both contacts are initially separated. Since q_0 is a kinematic-strong equilibrium posture, the solution of the free-flight dynamics must reach a collision in finite time. Then, the clattering convergence condition implies that the hybrid dynamics solution converges to a 1- or 2-Zeno point. In the case of a 2-Zeno point, the dynamics then switches to constrained motion of two-contact sliding. In the case of a 1-Zeno point, the dynamics switches to constrained motion of single-contact sliding or rolling. Then again, the solution must reach a collision event at the free-flying contact, which results in convergence to a 2-Zeno point via a chattering Zeno sequence and then switching to constrained two-contact sliding. Finally, the kinematic-strong equilibrium condition implies that the two sliding contacts are decelerating, and $\mathcal{B}$ stops at a nearby equilibrium posture in finite time. In case where the initial perturbation imposes a constrained motion rather than free-flying, the solution simply starts from an advanced stage in the process described above, and proceeds similarly. Note that the solution undergoes a finite number of stages, each occurring in finite time, and stays within a

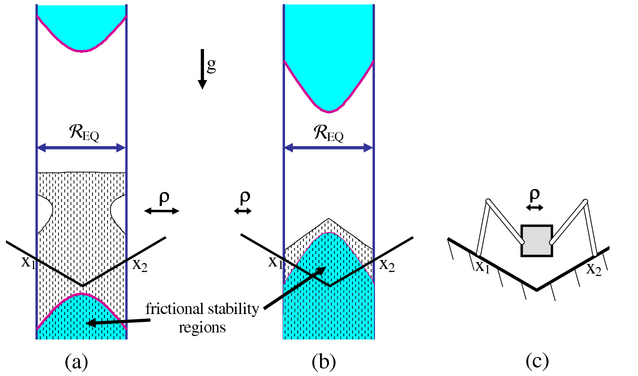

Fig. 3. (a)-(b) Center-of-mass stable equilibrium regions for two stances with different radii of gyration, (c) Illustration of a stable equilibrium stance.

bounded neighborhood of q_0. This neighborhood, as well as the total time of motion, can be made arbitrarily small by setting the initial perturbation sufficiently small.

C. The Center-of-Mass Region of Frictional Stability

We now demonstrate computation results of the region of center-of-mass locations x achieving equilibrium and frictional stability of $\mathcal{B}$ with two given contacts. Figs. 3a-b shows the center-of-mass stable equilibrium region for two stances on a symmetric V-shaped terrain with coefficient of friction $\mu = 0.5$ and coefficient of restitution $e = 0.1$. These two stances differ in the value of $\mathcal{B}$'s radius of gyration ρ, whose lengths are shown on both figures. The feasible equilibrium region $\mathcal{R}_{EQ}$ is a vertical strip that does not depend on ρ. The shaded regions in both figures are center-of-mass regions satisfying the kinematic-strong equilibrium condition. The dashed regions are center-of-mass regions satisfying the clattering convergence condition (10). The intersection of these two regions gives center-of-mass locations achieving equilibrium and frictional stability. Note that there is a tradeoff regarding the radius of gyration of $\mathcal{B}$, as follows. Increasing ρ results in larger region of kinematic-strong equilibrium, but smaller region of clattering convergence, as in the example of Fig. 3a, where the intersection region lies entirely under the terrain. In the example of Fig. 3b, with ρ twice smaller, the kinematic-strong equilibrium region is smaller, but the clattering convergence region is enlarged, and the intersection region has a portion above the terrain, making stable equilibrium postures practically achievable. Finally, Fig. 3c shows an illustration of a two-legged mechanism (treated as a single rigid body) positioned in a stable equilibrium posture by keeping the center-of-mass location sufficiently low. The small radius of gyration is achieved by a massive central body and relatively thin limbs.

VI. Concluding Discussion

This paper analyzed the impact-induced hybrid dynamics of a planar rigid body with two contacts, and derived a condition guaranteeing that its solution converges in finite time to a Zeno point of either one- or two-contact re-establishment. Using the concept of completed hybrid system, it then proved that this condition, augmented with the kinematic-strong equilibrium condition (derived in the companion paper [12]), are sufficient for stability of a two-contact frictional equilibrium posture.

We now briefly list the main limitations of the results and discuss possible extensions. First, note that this paper only provided *sufficient conditions* for frictional stability. Though the component of kinematic-strong equilibrium seems also necessary, the clattering convergence condition might be too conservative. When clattering convergence is not satisfied, the hybrid dynamics results in complex sequences of collisions at the two contacts, which may be periodic, quasi-periodic, or even chaotic. Analyzing these sequences and deriving conditions for their convergence to Zeno points is a challenging open problem. Second, the analysis of the hybrid dynamics in this paper is limited to two contacts, and the obvious extension to analysis of three or more contacts remains as a future challenge. Finally, the hybrid dynamics was formulated for the simple case of a single rigid-body. Future extension of the analysis to robotic mechanisms supported by frictional contacts must account for the *control laws* of the actuated joints and their influence on stability.

References

[1] A. D. Ames, H. Zheng, R. Gregg, and S. Sastry. Is there life after Zeno? Taking executions past the breaking (Zeno) point. In *American Control Conference (ACC'06),* Minneapolis, MN, 2006.

[2] B. Brogliato. *Nonsmooth Mechanics: Models, Dynamics and Control*. Springer-Verlag, London, 1999.

[3] A. Chatterjee and A. Ruina. A new algebraic rigid body collision law based on impulse space considerations. *Journal of Applied Mechanics*, 65(4):939–951, 1998.

[4] S. Goyal, J. M. Papadopoulos, and P. A. Sullivan. The dynamics of clattering I: Equation of motion and examples. *ASME J. of Dynamic Systems, Measurement, and Control*, 120:83–93, 1998.

[5] S. Goyal, J. M. Papadopoulos, and P. A. Sullivan. The dynamics of clattering II: Global results and shock protection. *ASME J. of Dynamic Systems, Measurement, and Control*, 120:94–102, 1998.

[6] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag, NY, 1983.

[7] W. M. Haddad, V. Chellaboina, and S. G. Nersesov. *Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity, and Control*. Princeton Univ. Press, New Jersey, 2006.

[8] A.M. Lyapunov. *Stability of motion*. Academic Press, New-York and London, 1966.

[9] M. T. Mason and Y. Wang. On the inconsistency of rigid-body frictional planar mechanics. In *IEEE Int. Conf. on Robotics and Automation*, pages 524–528, 1988.

[10] S. Nersesov, V. Chellaboina, and W. Haddad. A generalization of Poincaré's theorem to hybrid and impulsive dynamical systems. *Int. J. Hybrid Systems*, 2:35–51, 2002.

[11] Y. Or. *Frictional equilibrium postures for robotic locomotion - computation, geometric characterization, and stability analysis*. PhD thesis, in *robots.technion.ac.il*, Technion, Israel, 2007.

[12] Y. Or and E. Rimon. On the hybrid dynamics of planar mechanisms supported by frictional contacts. I: Necessary conditions for stability. To appear in *IEEE Int. Conf. on Robotics and Automation*, 2008.

[13] J. Shen and J.S. Pang. Linear complementarity systems: Zeno states. *SIAM J. on Control and Optimization*, 44(3):1040–1066, 2005.

[14] A. van der Schaft and H. Schumacher. *An Introduction to Hybrid Dynamical Systems*. Lecture Notes in Control and Information Sciences 251, Springer-Verlag, 2000.

[15] Y. Wang. Dynamic modeling and stability analysis of mechanical systems with time-varying topologies. *ASME J. of mechanical design*, 155(4):808–816, 1993.

[16] H. Ye, A. N. Michel, and L. Hou. Stability theory for hybrid dynamical systems. *IEEE Trans. on Automatic Control*, 43(4):461–474, 1998.

[17] J. Zhang, K. H. Johansson, J. Lygeros, and S. Sastry. Zeno hybrid systems. *Int. J. Robust and Nonlinear Control*, 1(2):435–451, 2001.