2008 IEEE International Conference on
Robotics and Automation
Pasadena, CA, USA, May 19-23, 2008

On the Hybrid Dynamics of Planar Mechanisms Supported by Frictional Contacts. I: Necessary Conditions for Stability

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Abstract— **This paper is concerned with the stability of planar mechanisms supported by multiple frictional contacts against gravity. Stability of equilibrium postures is investigated under initial perturbations which may involve sliding or separation at the contacts. The frictional dynamics is formulated using the notion of *contact modes*, and the related problems of solution ambiguity and inconsistency are reviewed. The paper then uses the condition of *strong equilibrium* to eliminate ambiguities, and defines a new condition of *kinematic-strong equilibrium* which additionally eliminates frictional inconsistencies. It is then proven that strong equilibrium is necessary for stability of frictional equilibrium postures, and that kinematic-strong equilibrium guarantees finite-time recovery of an initially perturbed contact. The results are demonstrated on a reduced model of a rigid body having a variable center-of-mass and supported by two frictional contacts. A companion paper completes the analysis of this reduced problem by investigating the overall hybrid dynamics and deriving sufficient conditions for its stability.**

I. Introduction

Performing quasistatic manipulation and locomotion robotic tasks requires transition through a sequence of equilibrium postures with multiple contacts. In order to execute these tasks reliably and successfully, the selected equilibrium postures must possess *dynamic stability* with respect to small position-and-velocity perturbations. In many cases, the perturbations may involve separation, sliding, or rolling of the contacts. Such perturbations may originate from external disturbances, localized surface irregularities, or inaccuracies in the coordinated (possibly over-constrained) motion of internal links of the robotic mechanism. The goal of this work is to analyze the dynamic response of a planar mechanism supported by multiple frictional contacts under *any* small perturbation about an equilibrium posture, and to derive conditions for posture's stability. The underlying dynamical system is essentially a *hybrid system*, consisting of multiple modes of contacts, each associated with its own constrained dynamics. The transitions between contact modes may either be smooth, as in switching from rolling to sliding contact, or nonsmooth, as in the case of contact recovery via a collision. This paper, which is the first of two parts, analyzes the constrained dynamics associated with each contact mode, derives necessary conditions for stability, and then establishes a sufficient condition for finite-time contact recovery. The second part, which appears in the companion paper [15], focuses on the reduced problem of a single rigid body supported by two frictional contacts, analyzes the hybrid dynamical system governed by collisions at the contacts, and derives sufficient conditions for its stability.

Previous works on quasistatic legged locomotion such as [2] only consider the static equilibrium constraints, but do not account for dynamic stability. The classical notion of dynamic stability requires convergence or boundedness of the dynamical system response to any small perturbation about an equilibrium point (e.g. [10]). In the robotics literature discussing the stability of multi-contact grasps, some works analyze the stability of force closure grasps under the assumption that the contact forces are actively controlled [4], [13]. Other works investigate the stability of force-closure grasps with passive fingers by modeling the natural compliance at the contacts and accounting for joints' stiffness [7], [9]. However, all of theses works analyze stability only with respect to perturbations under which all contacts are maintained fixed or rolling, but do not consider fully general perturbations under which the contacts can also slide or break. In the legged robots literature, several papers use the ZMP criterion as a stability condition for tasks involving quasistatic or dynamic motion [23]. This condition amounts to checking if the contact reaction forces required to constrain the dynamics of fixed contacts are feasible, but do not account for the possibility of initially sliding or breaking contacts and their recovery. Related stability measures for robotic vehicles focus on the energy required to generate tipover motion on rough terrains [21], again without explicitly considering the vehicle's full dynamics.

The dynamics of mechanical systems with multiple frictional contacts has been investigated in the context of dynamic simulations [22] and assembly planning [1]. The instantaneous dynamics can be formulated as a *linear complementarity problem* [20], which accounts for all possible contact modes in a unified framework. This formulation highlights some problems regarding solution existence and uniqueness. The first to identify paradoxical existence results related to frictional rigid-body dynamics was Painlevé [18], with his sliding rod example. For certain choices of physical parameters and initial conditions, the instantaneous solution of the rod's constrained dynamics is inconsistent. This paradox has been subsequently resolved in terms of impulsive contact forces causing a *tangential impact* event of discontinuous velocity jump followed by contact separation [5], [11]. This phenomenon has been recently demonstrated experimentally in [12], where a mechanism that mimics Painlevé's rod experiences a *dynamic jamming* event [3]. Another fundamental problem of frictional dynamics is *dynamic ambiguity*, at which the instantaneous dynamics has multiple solutions associated with different contact modes. Some attempts to address this problem use contact compliance in order to predict the dynamic response and thus resolve the ambiguity [8]. In order to retain the simplicity of the rigid-body paradigm and yet avoid the frictional dynamic ambiguity, Pang and Trinkle [19] introduced the condition of *strong equilibrium*, requiring that static equilibrium would be the only possible dynamic solution. This principle was subsequently used in applications such as sensorless manipulation [1] and climbing [6].

The main contributions of this paper are as follows. First, it proves that strong equilibrium is necessary for dynamic stability of frictional equilibrium postures. Second, it defines a new criterion, called *kinematic-strong equilibrium*, which additionally eliminates frictional inconsistencies. The kinematic-strong equilibrium criterion is proven to serve as a key component in stability, by guaranteeing finite-time recovery of an initially perturbed contact.

The structure of the paper is as follows. Section II formulates the dynamics of planar frictional systems, and defines the notions of frictional equilibrium postures and frictional stability. Section III defines the criteria of strong equilibrium and kinematic-strong equilibrium, and establishes their relation to frictional stability and finite-time contact recovery. Section IV demonstrates the results on a reduced model of a planar rigid body having a variable center-of-mass and supported by two frictional contacts. Finally, the concluding section discusses possible extensions of this work.

II. Planar Frictional Dynamics and Stability

This section formulates the dynamics of a planar mechanism supported by frictional contacts against gravity, then defines the notion of frictional stability of an equilibrium posture.

A. Basic Terminology

Consider a planar mechanism $\mathcal{M}$ supported via k frictional contacts by a static piecewise-linear terrain. Let $q \in I\!R^n$ denote the mechanism's configuration,[1] and let $\dot{q}$ and $\ddot{q}$ denote its generalized velocity and acceleration. The mechanism is subject to k unilateral constraints of the form $h_i(q) \geq 0$, representing the contacts. We assume that all contacts occur between linear segments of the terrain and vertex points of the mechanism's links. Let t_i and n_i denote the unit tangent and unit normal at the i-th contact, such that n_i points away from the terrain segment. Then n_i is locally constant, and the contact constraints are:

$$h_i(q) = (x_i(q) - x_i^o) \cdot n_i \geq 0 \quad \text{for } i = 1 \ldots k, \qquad (1)$$

where $x_i(q)$ is the position of the link's vertex associated with the i-th contact and x_i^o is a fixed point on the i-th linear terrain segment. Let $J_i(q) = \frac{\partial x_i(q)}{\partial q}$ be the Jacobian matrix associated with the i-th contact, and let $v_i(q, \dot{q}) = J_i(q)\dot{q}$ be the velocity of the i-th contact. The state of $\mathcal{M}$, $(q, \dot{q})$, is constrained to lie within the *collision-free region* in state space, defined by

$$\mathcal{F} = \left\{ (q, \dot{q}) : \begin{array}{l} h_i(q) \geq 0 \quad \text{for } i = 1 \ldots k \quad \text{s. t.} \\ \text{if } h_i(q) = 0 \text{ then } n_i \cdot v_i(q, \dot{q}) \geq 0 \end{array} \right\}. \qquad (2)$$

The dynamics of $\mathcal{M}$ is governed by the equation of motion,

$$M(q)\ddot{q} + B(q, \dot{q})\dot{q} + G(q) = \sum_{i=1}^{k} J_i^T(q) f_i \qquad (3)$$

where $M(q)$ is the mechanism's inertia matrix, $B(q, \dot{q})$ is the matrix of velocity-dependent generalized forces, and $G(q)$ is the vector of generalized gravitational forces. On the right hand side of (3), $f_i \in I\!R^2$ is the i-th contact reaction force acting at x_i. Assuming Coulomb's friction law, each contact force f_i must lie in a *friction cone*, denoted $\mathcal{C}_i$, which is given by

$$\mathcal{C}_i = \{ f_i : |t_i \cdot f_i| \leq \mu (n_i \cdot f_i) \}, \qquad (4)$$

where μ is the coefficient of friction. This paper focuses on analyzing frictional equilibrium postures, defined as follows:

Definition 1. *A k-contact configuration q_0 of $\mathcal{M}$ is a* **frictional equilibrium posture** *if there exist contact forces $f_i \in \mathcal{C}_i$ for $i = 1 \ldots k$ such that (3) is satisfied with $q = q_0$, $\dot{q} = 0$, and $\ddot{q} = 0$.*

B. Planar Frictional Dynamics

In order to investigate the stability of a frictional equilibrium posture, one needs to analyze the solution of (3) in response to small perturbations about the equilibrium. Since $(q, \dot{q})$ are known quantities, $h_i(q)$ and $v_i(q, \dot{q})$ are known at every time instant. In order to obtain an instantaneous solution for the unknowns $\ddot{q}$ and $f_1, \ldots, f_k$ in (3), one needs to invoke additional relations between the contact forces and contact velocities. Each *active* contact which satisfies $h_i(q(t)) = 0$ is governed by one of four distinct modes. The four modes, denoted S, F, R, and L, correspond to contact separation, fixed (or rolling) contact, right-sliding and left-sliding, respectively. For a k-contact arrangement, a contact mode is encoded by a k-letter word from the alphabet $\{S, F, R, L\}$. For example, the contact mode SR of a 2-contact arrangement means that the contact x_1 is instantaneously separating, while the contact x_2 is sliding to the right. Each contact mode is associated with linear equality and inequality constraints in f_i and v_i as summarized in Table I.

When a particular contact mode is active, its equality constraint on velocities are differentiated with respect to

TABLE I

The possible contact modes at a planar frictional contact.

contact mode	physical meaning	kinematic constraints	force constraints
S	Separation	$v_i \cdot n_i > 0$	$f_i = 0$
F	Fixed/rolling	$v_i = 0$	$\|f_i \cdot t_i\| \leq \mu (f_i \cdot n_i)$
R	Right sliding	$v_i \cdot n_i = 0$ $v_i \cdot t_i > 0$	$f_i \cdot t_i = -\mu (f_i \cdot n_i)$ $f_i \cdot n_i \geq 0$
L	Left sliding	$v_i \cdot n_i = 0$ $v_i \cdot t_i < 0$	$f_i \cdot t_i = \mu (f_i \cdot n_i)$ $f_i \cdot n_i \geq 0$

[1] Since this paper focuses on small neighborhoods about equilibrium points, we may use a local coordinate chart on $I\!R^n$

time, giving similar constraints on the contact accelerations, denoted a_i for $i = 1 \dots k$. Using the relation $a_i = J_i(q)\ddot{q} + \dot{J}_i(q)\dot{q}$ yields equality constraints in $\ddot{q}$. When the $\ddot{q}$ constraints are augmented with the force constraints and the equation of motion (3), one obtains a square linear system in $\ddot{q}$ and $f_1, \dots, f_k$. This system is generically full rank, and has a unique solution for the instantaneous dynamics under the given contact mode. Thus for a given initial state of $\mathcal{M}$, one identifies the active contact mode, computes its instantaneous dynamic solution, then integrates eq. (3) for some finite time interval in order to obtain the *dynamic solution* $q(t)$. Note, however, that each contact mode is also associated with inequality constraints which must be satisfied in order for the dynamic solution to be consistent. In some cases, a consistent solution may not exist, or might be non-unique. These well-known problems are addressed in the next section. Nevertheless, this definition of dynamic solution is sufficient for introducing the notion of frictional stability.

C. Definition of Frictional Stability

We now define the notion of frictional stability of an equilibrium posture. Let $N_\epsilon(q_0)$ denote an ϵ-neighborhood about an equilibrium point $(q_0, 0)$: $N_\epsilon(q_0) = \{(q, \dot{q}) : \ \|q - q_0\| < \epsilon \text{ and } \|\dot{q}\| < \epsilon\}$.

Definition 2. *Let q_0 be a k-contact frictional equilibrium posture of $\mathcal{M}$. Then q_0 is* **frictionally stable** *if for any small $\epsilon > 0$ there exists a sufficiently small $\delta > 0$ such that for all perturbations $(q(0), \dot{q}(0)) \in N_\delta(q_0) \cap \mathcal{F}$ the dynamic solution $q(t)$ is consistent and converges to some equilibrium posture while staying within $N_\epsilon(q_0)$.*

The definition is an adaptation of the classical Lyapunov stability [10] to hybrid systems. It combines *stability of invariant sets* with *Lagrange Stability* which requires boundedness within an arbitrarily small neighborhood [24]. The definition captures the inherent neutrality of frictional equilibrium postures which commonly lie within a continuum of equilibria. Note that the definition considers only perturbations within $\mathcal{F}$ i.e., no interpenetrations are allowed.

III. Necessary Conditions for Frictional Stability

This section derives necessary conditions for the frictional stability of an equilibrium posture. Focusing on the dynamics under zero-velocity initial conditions, we first review the strong equilibrium condition which eliminates frictional ambiguity. This criterion is proven to be necessary for frictional stability. Then we define a more restrictive kinematic-strong equilibrium condition, which additionally eliminates frictional inconsistency. Finally, kinematic-strong equilibrium is proven to guarantee finite-time recovery of an initially perturbed contact.

A. The Strong Equilibrium Criterion

Consider the frictional dynamics (3)-(4) under zero-velocity initial conditions, $(q(0), \dot{q}(0)) = (q_0, 0)$, where q_0 is a k-contact equilibrium posture. Since the contact velocities are initially zero, they do not determine a unique contact mode. Hence one must consider *each* contact mode, compute its associated instantaneous dynamic solution, then check its consistency. The consistency of each contact mode is checked according to its contact force inequalities, as well as its kinematic inequalities in which *contact velocities are replaced by contact accelerations*. For instance, in a single-contact posture consistency of contact separation is checked by the inequality $n_1 \cdot a_1 > 0$, while consistency of right-sliding is checked by the inequalities $n_1 \cdot f_1 > 0$ and $t_1 \cdot a_1 > 0$. Since q_0 is an equilibrium configuration, the contact mode F^k (all k contacts are fixed) is consistent. Hence if some non-static contact mode is also consistent at $(q_0, 0)$, the dynamic solution is *ambiguous* and the Coulomb's friction model cannot determine which contact mode is actually evolving. In order to avoid this intricate phenomenon, Trinkle and Pang [19] introduced the following notion of strong equilibrium.

Definition 3. *A k-contact frictional equilibrium posture q_0 of $\mathcal{M}$ is a* **strong equilibrium** *if the dynamic solution of all non-static modes under the initial condition $(q_0, 0)$ is inconsistent.*

The strong equilibrium condition was originally called *strong stability* [19]. However, its relation to the classical notion of dynamic stability justifies the strong equilibrium terminology. The following theorem establishes the relation between strong equilibrium and frictional stability.

Theorem 1 ([14]). *Let q_0 be a k-contact frictional equilibrium posture of a planar mechanism $\mathcal{M}$. Then strong equilibrium is* **necessary** *for the posture's frictional stability.*

Proof sketch: Assume that q_0 is *not* a strong equilibrium. In this case there exists a non-static contact mode, MODE, which is consistent under initial conditions $(q_0, 0)$. In particular, it is associated with some consistent kinematic inequalities involving contact *accelerations*. For instance, if MODE is associated with contact separation at x_i, its dynamic solution satisfies $n_i \cdot a_i > 0$ according to Table I. Since the dynamic solution under MODE is continuous with respect to $(q, \dot{q})$, there exists an open set of initial conditions about $(q_0, 0)$ under which $n_i \cdot a_i$ is still positive, and the contact x_i initially accelerates *away* from its nominal position. Similarly, if MODE is associated with a right-sliding at x_i, its dynamic solution at $(q_0, 0)$ satisfies $t_i \cdot a_i > 0$, and for sufficiently small perturbations the initial sliding accelerates x_i away from its nominal position. The trajectory $(q(t), \dot{q}(t))$ therefore cannot be bounded in an *arbitrarily small* neighborhood of $(q_0, 0)$, contradicting the definition of frictional stability. □

The intuition behind the theorem is as follows. Generic position-and-velocity perturbations incur nonzero initial contact velocities. Such perturbations uniquely determine some initial non-static contact mode. If the initial non-static mode is consistent, then strong equilibrium guarantees that the mechanism will accelerate in such a way as to *recover* the perturbed contacts. This behavior is necessary for frictional stability of the unperturbed equilibrium posture.

B. The Kinematic-Strong Equilibrium Criterion

While strong equilibrium eliminates dynamic ambiguities, a contact mode dictated by an initial perturbation about $(q_0, 0)$ may be *initially inconsistent*. This scenario, known as Painlevé's paradox [11], [18], is characterized by a sliding motion under which the instantaneous dynamic solution violates one or more inequality constraints on the contact forces. Another problematic scenario occurs when an initially consistent motion carries the trajectory $(q(t), \dot{q}(t))$ into a point of inconsistency. This event, known as *dynamic jamming* [3], [12], is characterized by a sliding motion under which the contact forces and accelerations at one or more sliding contact diverge to infinity in finite time. In both cases the solution inconsistency can be resolved in terms of impulsive forces which cause a non-smooth transition to a contact separation mode [5], [11]. In order to guarantee contact recovery as a first step toward frictional stability, one needs to avoid such cases of inconsistency. This requirement is captured by the condition of kinematic-strong equilibrium, whose definition is based on the distinction between kinematic constraints and force constraints (see Table I).

Definition 4. *A k-contact frictional equilibrium posture q_0 of a mechanism $\mathcal{M}$ is a* **kinematic-strong equilibrium** *if the dynamic solution of each non-static mode under the initial condition $(q_0, 0)$ satisfies all force constraints but violates at least one of the kinematic constraints.*

Note that kinematic-strong equilibrium is more restrictive than strong equilibrium, which only requires that the dynamic solution of each non-static contact mode would violate one constraint, without specifying which type. The following theorem establishes the relation between kinematic-strong equilibrium and finite-time recovery of an initially perturbed contact.

Theorem 2 ([14]). *Let q_0 be a k-contact frictional equilibrium posture of a planar mechanism $\mathcal{M}$. Then for any arbitrarily small $t_0 > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that for all $(q(0), \dot{q}(0)) \in N_\delta(q_0) \cap \mathcal{F}$, there exists a* **finite time** *$t' < t_0$ such that the initial contact mode is consistent during the time interval $[0, t')$, and stays within $N_\epsilon(q_0)$. Moreover, at $t = t'$ either an initially sliding contact becomes stationary (or rolling), or an initially separated contact recovers contact.*

Proof sketch: Consider a perturbation which imposes a non-static contact mode MODE. The kinematic-strong equilibrium condition implies that at the state $(q_0, 0)$, all force constraint associated with MODE are satisfied, and at least one kinematic constraints is violated *in the acceleration level*. For instance, if MODE is associated with contact separation at x_i, then its dynamic solution at $(q_0, 0)$ may satisfy $n_i \cdot a_i < 0$. Since the dynamic solution under MODE is continuous with respect to $(q, \dot{q})$, there exist an open set of initial conditions about $(q_0, 0)$ under which the force constraints are still satisfied, and MODE is initially consistent. Moreover, for sufficiently small initial velocity, $n_i \cdot a_i$ is still negative, and x_i accelerates towards contact recovery via a collision. Similarly, if MODE is associated with a right-sliding at x_i, its dynamic solution may satisfy $t_i \cdot a_i < 0$, and for sufficiently small perturbations, MODE is initially consistent, and the sliding of x_i decelerates to a full stop in finite time. □

The kinematic-strong equilibrium is a key component of frictional stability, since it guarantees that the initial response under small perturbations is bounbded and contacts are recoverred in finite time. However, note that it only guarantees recovery of a *single contact*, and not convergence to an equilibrium point. Moreover, recovery an initially seperated contact reults in an *impact* event. Analysis of the hybrid dynamics associated with a sequence of impacts is relegated to the companion paper [15].

IV. Two-Contact Rigid Body Postures

This section demonstrates the computation of kinematic-strong equilibrium postures for a rigid body $\mathcal{B}$ supported by two frictional contacts in a planar gravitational field. The rigid body model serves as a simplification of a two-legged mechanism whose complex kinematic structure is lumped into $\mathcal{B}$'s center-of-mass, denoted $\boldsymbol{x}$, which varies as a free parameter. The mass of $\mathcal{B}$ is denoted m and its radius of gyration is denoted ρ. The contact points are x_1 and x_2 and the contact reaction forces are f_1 and f_2, where $f_i \in \mathcal{C}_i$ for $i = 1, 2$. The equation of motion of $\mathcal{B}$ is given by

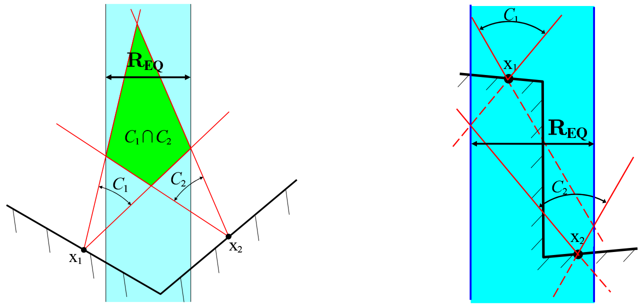
$$\begin{aligned} & f_1 + f_2 + f_g = ma \\ & (x_1 - \boldsymbol{x})^T J f_1 + (x_2 - \boldsymbol{x})^T J f_2 = m\rho^2 \alpha, \end{aligned} \tag{5}$$

where $a \in I\!R^2$ is $\mathcal{B}$'s center-of-mass linear acceleration, $\alpha \in I\!R$ is $\mathcal{B}$'s angular acceleration, f_g is the gravitational force acting at $\boldsymbol{x}$, and $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. For given contacts and friction cones, a posture is fully characterized by the center-of-mass location $\boldsymbol{x}$ and the radius of gyration ρ. We first review the computation of the region of center-of-mass positions achieving frictional equilibrium postures. Then we demonstrate the phenomena of dynamic ambiguity and inconsistency in terms of center-of-mass position. Finally, for a given ρ we compute the center-of-mass region achieving kinematic-strong equilibrium postures.

A. Center-of-Mass Feasible Equilibrium Region

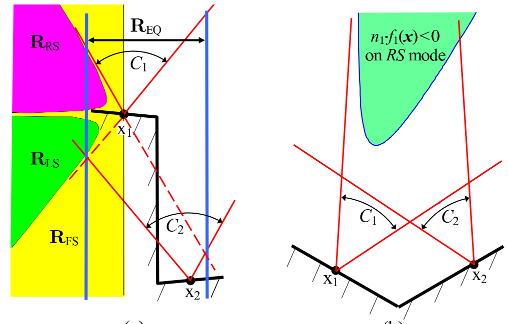
We first review the computation of the center-of-mass region which results in frictional equilibrium postures [17]. Given two contacts, the *feasible equilibrium region*, denoted $\mathcal{R}_{EQ}$, is defined as the region of center-of-mass positions $\boldsymbol{x}$ for which there exist contact forces $f_i \in \mathcal{C}_i$ $(i = 1, 2)$ satisfying (5) with zero accelerations $a = \alpha = 0$. Note that the equilibrium condition (5) and the frictional constraints (4) are linear in f_1, f_2 and $\boldsymbol{x}$. Hence the boundaries of $\mathcal{R}_{EQ}$ are computable as linear programming problems. It can be easily shown that $\mathcal{R}_{EQ}$, if nonempty, is a single connected vertical strip [17].

Graphical Examples: We briefly show computation results of $\mathcal{R}_{EQ}$ for several 2-contact postures. First consider the special case where the two contacts lie on a flat horizontal



(a) (b)

Fig. 1. Graphical characterization of $\mathcal{R}_{EQ}$ in two cases



(a) (b)

Fig. 2. Examples of (a) frictional ambiguity and (b) frictional inconsistency for 2-contact rigid body stances

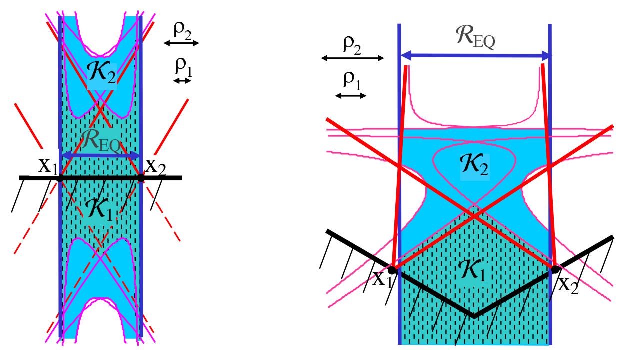
floor. In this case $\mathcal{R}_{EQ}$ is precisely the vertical strip spanned by the two contacts. Examples of $\mathcal{R}_{EQ}$ in 2-contact arrangements of more complex geometry are shown in Fig. 1. Fig. 1(a) shows a V-shaped terrain with $\mu = 0.3$. In this case $\mathcal{R}_{EQ}$ is the vertical strip spanned by the intersection of the two friction cones emanating from the contacts. Fig. 1(b) shows a terrain consisting of a high step with $\mu = 0.5$, and the resulting region $\mathcal{R}_{EQ}$. Note that in this case $\mathcal{R}_{EQ}$ exceeds horizontally beyond the contacts.

B. Frictional Dynamics of Variable Center-of-Mass Postures

We now consider the contact modes governing the dynamics of $\mathcal{B}$ under initial conditions $(q_0, 0)$, where q_0 is a frictional equilibrium posture. Enumerating all $4^2 = 16$ possible 2-contact modes, then disregarding the order of contacts and direction of sliding, gives five representative modes: FF, SS, FS, RS, and RR. For each particular contact mode, the instantaneous dynamic solution for f_1, f_2 and a, α in (5) depends on the center-of-mass position $\boldsymbol{x}$ and the radius of gyration ρ. Assuming that ρ is a known constant, the inequality constraints associated with a particular contact mode define a *feasible region* of center-of-mass locations for which this contact mode is consistent. These regions are denoted $\mathcal{R}_{SS}, \mathcal{R}_{FS}, \mathcal{R}_{RS}$, and so on. Note that the feasible region of the static contact mode FF is precisely the feasible equilibrium region $\mathcal{R}_{EQ}$. The feasible regions of all non-static contact modes are bounded by curves which are linear and quadratic in $\boldsymbol{x}$, and their computation is detailed in [14].

In the following examples we use the feasible regions in order to demonstrate the phenomena of frictional ambiguity and frictional inconsistency. Fig. 2(a) depicts the feasible equilibrium region $\mathcal{R}_{EQ}$ for the 2-contact posture of Fig. 1(b). The figure also plots, in shaded regions, the feasible region of the contact modes FS, RS, and LS, associated with pure rolling, right sliding, and left sliding at x_1, and separation at x_2. Note that the portion of $\mathcal{R}_{EQ}$ lying to the left of x_1 *overlaps* with the regions $\mathcal{R}_{FS}, \mathcal{R}_{RS}$, or $\mathcal{R}_{LS}$. This overlap corresponds to *frictional ambiguity* of the static equilibrium with these non-static contact modes. When $\boldsymbol{x}$ is positioned in the overlap region, $\mathcal{B}$ can be in static equilibrium, but small perturbations of rolling or sliding will result in non-recoverable motion. Therefore, such postures violate the strong equilibrium condition, and are essentially *unstable*, as stated in Theorem 1. The phenomenon of *frictional inconsistency* is demonstrated in Fig. 2(b) for a 2-contact posture with $\mu = 0.5$. Consider a small velocity perturbation imposing the contact mode RS (right sliding at x_1 and separation at x_2). The associated dynamic solution for f_1 depends of the center-of-mass $\boldsymbol{x}$, and the region of $\boldsymbol{x}$ satisfying $n_1 \cdot f_1(\boldsymbol{x}) < 0$ is depicted in Fig. 2(b) (shaded region). When $\boldsymbol{x}$ lies in this region, no finite-force consistent solution exists under such perturbations, and the only consistent solution is tangential impact followed by immediate contact separation. Thus, such postures violate the kinematic-strong equilibrium condition.

Finally, we demonstrate the computation of the kinematic-strong equilibrium postures for a rigid body $\mathcal{B}$ supported by two contacts having a coefficient of friction μ. For a known radius of gyration ρ, the region of center-of-mass locations $\boldsymbol{x}$ of kinematic-strong equilibrium postures is denoted $\mathcal{K}$. Fig. 3 shows two postures with $\mu = 0.5$, along with the kinematic-strong equilibrium regions $\mathcal{K}_1$ (dashed) and $\mathcal{K}_2$ (shaded), associated with radii of gyration ρ_1 and ρ_2, such that $\rho_2 = 2\rho_1$. The true lengths of ρ_1 and ρ_2 relative to the contacts' geometry are shown on the figures. While the feasible equilibrium region $\mathcal{R}_{EQ}$ (also shown on the figures) does not depend on ρ, the kinematic-strong equilibrium region is significantly affected by ρ as follows. The hyperbolic curves bounding the region $\mathcal{K}$ have fixed asymptotes, and become "sharper" as ρ decreases, removing a larger portion of $\mathcal{R}_{EQ}$. Hence when ρ increases i.e., when $\mathcal{B}$'s mass is distributed farther from its center, $\mathcal{K}$ becomes larger. For postures satisfying $\boldsymbol{x} \in \mathcal{K}$, finite-time recovery of a perturbed contact is guaranteed by Theorem 2.



(a) (b)

Fig. 3. Kinematic-strong equilibrium regions $\mathcal{K}_1$ (dashed) and $\mathcal{K}_2$ (shaded) for radii of gyration ρ_1 and ρ_2 on (a) a flat terrain (b) a V-shaped terrain.

V. Concluding Discussion

This paper discussed the stability of planar mechanisms supported by multiple frictional contacts against gravity. We reviewed the notion of strong equilibrium which eliminates dynamic ambiguity, and defined the notion of kinematic-strong equilibrium which additionally eliminates dynamic inconsistency. We established that strong equilibrium is necessary for frictional stability, and that kinematic-strong equilibrium guarantees finite-time recovery of an initially perturbed contact. The results were demonstrated on a simplified mechanism model of a rigid body having a variable center of mass and supported by two frictional contacts. The results indicate that the criterion of kinematic-strong equilibrium is a key frictional stability component. It provides a significant step toward incorporating dynamic stability considerations into automated planning of robotic locomotion and manipulation supported by multiple frictional contacts against gravity.

We now discuss two generalizations of the results. First consider a planar legged robot moving quasistatically on a frictional terrain. While the kinematic structure of the robot can be lumped into a single rigid body $\mathcal{B}$ having a variable center-of-mass, the inertial forces generated by motion of internal limbs can be lumped as an additional wrench (i.e. force and torque) acting on $\mathcal{B}$'s center-of-mass. Thus, accounting for this internal motion can be reduced to computation of equilibrium postures which are *robust* with respect to a given neighborhood of external wrenches surrounding the nominal gravitational wrench. A posture q_0 is robust with respect to a wrench neighborhood $\mathcal{W}$ if it forms a kinematic-strong equilibrium posture for *any* external wrench in $\mathcal{W}$. While robustness of frictional equilibrium postures is analyzed in [17] and robustness of strong equilibrium postures is analyzed in [16], computation of robust kinematic-strong equilibrium postures is an open problem. Next consider the generalization of the results to three dimensions. Under Coulomb's friction law, each contact force $f_i \in I\!R^3$ must satisfy the *quadratic* constraint $\|f_i^t\| \leq \mu(n_i \cdot f_i)$, where $f_i^t \in I\!R^2$ is the tangential projection of f_i. Thus, under a given contact mode, the dynamics yields a system of quadratic equations in the unknowns f_i and $\ddot{q}$, whose solution cannot be obtained in closed form. A possible approach for computing approximate solutions is to replace the quadratic friction cones with polyhedral cones, and formulate the instantaneous dynamics as a linear complementarity problem (e.g. [20]). The extension of this approach for computation of strong and kinematic-strong equilibrium postures is a future challenge.

Finally, this paper provided only *necessary* conditions for frictional stability. This is because it focused on the solution boundedness of the constrained dynamics associated with the initial contact mode until it switches, either by sliding halt or by collision. The companion paper [15] analyzes the hybrid dynamics associated with a sequence of collisions, and concatenates the constrained dynamics phases with the hybrid dynamics phases. The companion paper derives *sufficient* conditions for frictional stability of equilibrium postures for the simplified problem of a planar rigid body supported by two frictional contacts.

References

[1] D. J. Balkcom, E. J. Gottlieb, and J. C. Trinkle. A sensorless insertion strategy for rigid planar parts. In *IEEE Int. Conf. on Robotics and Automation*, pages 882–887, 2002.

[2] T. Bretl. Motion planning of multi-limbed robots subject to equilibrium constraints: The free-climbing robot problem. *Int. J. of Robotics Research*, 25(4):317–342, Apr 2006.

[3] P. E. Dupont and S. P. Yamajako. Jamming and wedging in constrained rigid-body dynamics. In *IEEE Int. Conf. on Robotics and Automation*, pages 2349–2354, 1994.

[4] F. Fen, M. Shoham, and R. Longman. Lyapunov stability of force-controlled grasps with a multifingered hand. *Int. J. of Robotics Research*, 15(2):137–154, 1996.

[5] F. Génot and B. Brogliato. New results on Painlevé paradoxes. *European J. of Mechanics– A/Solids*, 18(4):653–677, 1999.

[6] A. Greenfield, A. Rizzi, and H. Choset. Dynamic ambiguities in frictional rigid-body systems with application to climbing via bracing. In *IEEE Int. Conf. on Robotics and Automation*, pages 1959–1964, 2005.

[7] W. S. Howard and V. Kumar. On the stability of grasped objects. *IEEE Trans. on Robotics and Automation*, 12(6):904–917, 1996.

[8] P. R. Kraus, V. Kumar, and P. Dupont. Analysis of frictional contact models for dynamic simulation. In *IEEE Int. Conf. on Robotics and Automation*, pages 2822–2827, 1998.

[9] Q. Lin, J. W. Burdick, and E.Rimon Computation and analysis of natural compliance in grasping and fixturing arrangements. *IEEE Trans. on Robotics and Automation*, 20(4):651–667, 2004.

[10] A. M. Lyapunov. *Stability of motion*. Academic Press, New-York and London, 1966.

[11] M. T. Mason and Y. Wang. On the inconsistency of rigid-body frictional planar mechanics. In *IEEE Int. Conf. on Robotics and Automation*, pages 524–528, 1988.

[12] D. Meltz, Y. Or, and E. Rimon. Experimental verification and graphical characterization of dynamic jamming in frictional rigid-body mechanics. In *IEEE Int. Conf. on Robotics and Automation*, pages 580 – 585, 2007.

[13] V.-D. Nguyen. Constructing stable grasps. *The Int. Journal of Robotics Research*, 8(1):27–37, 1988.

[14] Y. Or. *Frictional equilibrium postures for robotic locomotion - computation, geometric characterization, and stability analysis*. PhD thesis, in *robots.technion.ac.il*, Technion, Israel, 2007.

[15] Y. Or and E. Rimon. On the hybrid dynamics of planar mechanisms supported by frictional contacts. II: Stability of two-contact rigid body postures. To appear in *IEEE Int. Conf. on Robotics and Automation*, 2008.

[16] Y. Or and E. Rimon. Robust multiple contact postures in a two-dimensional gravitational field. In *IEEE Int. Conf. on Robotics and Automation*, pages 4783–4788, 2004.

[17] Y. Or and E. Rimon. Computation and graphical characterization of robust multiple- contact postures in two-dimensional gravitational environments. *Int. J. of Robotics Research*, 25(11):1071–1086, 2006.

[18] P. Painlevé. Sur les lois du frottement de glissement. *Comptes Rendus De L'Acadamie Des Sciences*, (121):112–115, 1895.

[19] J. S. Pang and J. C. Trinkle. Stability characterizations of rigid body contact problems with Coulomb friction. *Zeitschrift für Angewandte Mathematik und Mechanik*, 80(10):643–663, 2000.

[20] J.S. Pang and J.C. Trinkle. Complementarity formulations and existence of solutions of dynamic multi-rigid-body contact problems with Coulomb friction. *Mathematical Programming*, 74:199–226, 1996.

[21] E. Papadopoulos and D. Rey. A new measure of tipover stability margin for mobile manipulators. In *IEEE Int. Conf. on Robotics and Automation*, pages 3111–3116, 1996.

[22] D. E. Stewart and J. C. Trinkle. An implicit time-stepping scheme for rigid body dynamics with inelastic collisions and Coulomb friction. *Int. J. of Numerical Methods in Engineering*, 39:2673–2691, 1996.

[23] M. Vukobratovic and B. Borovac. Zero moment point — thirty five years of its life. *Int. J. of Humanoid Robotics*, 1(1):157–173, 2004.

[24] H. Ye, A. N. Michel, and L. Hou. Stability theory for hybrid dynamical systems. *IEEE Trans. on Automatic Control*, 43(4):461–474, 1998.