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Real-time Clothoid Approximation by Rational Bezier Curves

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Abstract- **This paper presents a novel technique for implementing Clothoidal real-time paths for mobile robots. As first step, Rational Bezier curves are obtained as approximation of the Fresnel integrals. By rescaling, rotating and translating the previously computed RBC, an on-line Clothoidal path is obtained. In this process, coefficients, weights and control points are kept invariant. This on-line approach guarantees that an RBC has the same behavior as the original Clothoid using a low curve order. The resulting Clothoidal path allows any two arbitrary poses to be joined in a plane. RBCs working as Clothoids are also used to search for the shortest bounded-curvature path with a significant computational cost reduction. In addition to this, the proposed technique is tested on a real mobile robot for trajectory generation and kinematic control. To the authors' knowledge, the present approach is the first technique which allows real-time Clothoidal path computation.**

I. Introduction

Trajectory generation for autonomous vehicles has been subject to extensive research in recent decades. The simplest solution is to generate a trajectory concatenating line and arc segments [2], [3]. The main disadvantage of this technique is the curvature discontinuity between segments. This problem can be overcome by using smooth transition curves between straight lines and arc segments. In addition, constantly varying centrifugal acceleration, *jerk* is desirable in order to minimize wheel slip problems. This type of curves is commonly known as Clothoid or Cornu spiral. In this sense, Clothoids have been used in mobile robot trajectory generation [5], [6]. Another possibility is to combine only piecewise clothoids to join two poses (x,y,θ) in a plane [7], [8]. In these papers, authors introduced the concept of Elementary paths, that is, two equal concatenate piecewise clothoids to join symmetrical poses. They also introduced the concept of Bi-Elementary paths to join two arbitrary poses in a plane just only combining two different elementary paths.

It is interesting to note that Clothoids have also been used, for many years, as transition curves in road design [4] given that constant *jerk* guarantees passenger comfort.

Unfortunately, Clothoids are transcendental curves defined in terms of Fresnel integrals which cannot be solved analytically. For this reason, in recent years, research efforts have been focused on finding continuous function approximation techniques [9]-[12]. However, these techniques, successful in CAD platforms, cannot be used in real-time path planning generation due to the great computational cost of high order approximation curves. In this sense, [10] uses a 26^{th} order continuous function, which is unacceptable in real-time systems to the authors' opinion. In [12], a Fixed Point iteration technique is used to find an approximated solution of Fresnel integrals which requires a high computational cost. The results obtained in [11] do not guarantee inherent properties of Clothoids. In addition to mentioned problems to approximate a Clothoid, joining piecewise Clothoids requires additional iterative methods [7], [8].

The goal of our research is to present a technique that computes a general continuous curve approximation of Clothoids at the lowest possible degree, guaranteeing, at the same time, Clothoidal behavior.

The process generates first an off-line general approximation of Fresnel integrals, and then particularizes it by rescaling, rotating and translating an on-line curve is obtained. Therefore, Clothoidal path construction is carried out without iteration, suitable for real-time applications.

Our previous work [1] presented a general off-line approximation of the Fresnel integrals into Rational Bezier Curves (RBC). This paper represents a continuous progression of [1], in the sense that, elementary paths are constructed simply by rescaling, rotating and translating the general off-line formulation, keeping coefficients, weights and control points of RBCs invariant.

This paper is organized as follows. In section II, a brief review of Clothoid curve properties is presented. Section III shows a brief review of the methodology to approximate Fresnel integrals by Rational Bezier Curves [1]. Section IV establishes the methodology to compute Bi-elementary paths using the general off-line Clothoidal formulation. In Section V, the methodology is tested to search for the shortest bounded-curvature paths. In Section VI, Clothoidal paths as control references are tested on a real mobile robot. Conclusions and future works are presented in Section VII.

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II. PROPERTIES OF CLOTHOID CURVES

The Cornu spiral or Clothoid curve is defined parametrically in terms of Fresnel integrals as follows:

$$Q(\gamma)=\begin{pmatrix}x(\gamma)\\y(\gamma)\end{pmatrix}=K\cdot\begin{pmatrix}C(\gamma)\\S(\gamma)\end{pmatrix}=K\cdot\begin{pmatrix}\int_0^\gamma \cos\frac{\pi\cdot\xi^2}{2}d\xi\\ \int_0^\gamma \sin\frac{\pi\cdot\xi^2}{2}d\xi\end{pmatrix} \quad (1)$$

where K is a positive real number, γ is a non-negative real number. Clothoid curves have the following properties:

1. Angle of tangent: $\tau=\pi\cdot\gamma^2/2$
2. Curvature: $k=\pi\cdot\gamma/K$, Radius $R=1/k$
3. Arc length L: $L=K\cdot\gamma=\sqrt{\pi}\cdot A\cdot\gamma$, where A is the well-known clothoid constant parameter.
4. Homotecial factor $K=\sqrt{\pi}\cdot A$

The most attractive property of the clothoid curve is that:

$$\frac{1}{k}=R=\frac{A^2}{L} \quad (2)$$

where R is the radius of the curvature. This property guarantees smooth transitions establishing, at the same time, a linear relation between the curvature and the arc length. In addition, the variation of the centrifugal acceleration, J, is defined by the topographers as [4]:

$$A^2=\frac{V^3}{J}$$

where V is vehicle velocity. As mentioned in the introduction, Fresnel integrals must be solved numerically. Approximation methods use polynomial and non-polynomial functions. In particular, all existing techniques involving non-polynomial functions [9] are only useful when approximating Fresnel integrals in a single point. However, CAD/CAM systems or mobile robot trajectory generation modules require a continuous function. For this purpose, polynomial functions are the ideal solution.

The standard polynomial functions commonly used in CAD/CAM are Bezier, Rational Bezier, B-spline and NURBS. Some of these curves have been used for Clothoid approximation [10]-[12].

III. PREVIOUS WORKS

Our previous work [1] presented an off-line methodology to approximate the Clothoid by Rational Bezier Curve for a selected working interval. The RBC has the following formulation:

$$P_u=\frac{\sum_{k=0}^{N} w_k\cdot C_k\cdot\frac{N!}{k!\cdot(N-k)!}\cdot(u)^k\cdot(1-u)^{N-k}}{\sum_{k=0}^{N} w_k\cdot\frac{N!}{k!\cdot(N-k)!}\cdot(u)^k\cdot(1-u)^{N-k}}$$

where:

C_k: Control points
N: Order of the RBC
w_k: Weights of the control points
u: Intrinsic parameter [0…1]

In order to construct a Clothoid-like Rational Bezier, it is necessary to change the variable,

$$u=1-\frac{\gamma_e-\gamma}{\gamma_e-\gamma_i}$$

where γ_i and γ_e are the limits of the selected working interval, $\gamma_i\leq\gamma\leq\gamma_e$, for the Fresnel integrals which can be calculated from Clothoid properties explained in Section II. This is based on the tangent angle, curvature and the arc length of the clothoid as seen in Section II and Fig.3.
The RBC has two degrees of freedom corresponding to control points and weights. In [1], first the control points are computed forcing the weights to 1. This translates the RBC to a Bezier curve that can be expressed as a linear equation of these control points, as follows,

$$P_\gamma=C_0\cdot B_\gamma^0+C_1\cdot B_\gamma^1+\ldots+C_N\cdot B_\gamma^N$$

where B_γ^k is the kth Bernstein basis function:

$$B_\gamma^k=\frac{N!}{k!\cdot(N-k)!}\cdot\left(1-\frac{\gamma_e-\gamma}{\gamma_e-\gamma_i}\right)^k\cdot\left(\frac{\gamma_e-\gamma}{\gamma_e-\gamma_i}\right)^{N-k}$$

In that case, Fresnel integrals $P_\gamma=(C(\gamma), S(\gamma))$ are obtained using a non-polynomial approximation explained in [9], that computes the Fresnel integrals with an accuracy of $2\cdot10^{-10}$. The resulting linear equations can be solved by least squares techniques obtaining the control points of the RBC that approximate the Fresnel points. This approximation does not touch the start and end points, nor does it guarantee the required C^2 continuity at the start point. To overcome this problem, the control points have to be forced at these locations at the cost of decreasing the approximation accuracy unless the order of the curve is increased.

An alternative to increasing the order of the Bezier curve is to compute the weights using the previously computed control points and force the start and end points to the correct locations. In that case, the weights are also computed using least squares techniques.

As shown in [1], the RBC has the same behaviour as the Clothoid with homothetial factor equal to 1 and error in the approximation less than $1\cdot10^{-20}$. Fig.1 shows a block diagram of the methodology.

In order to obtain a general off-line formulation of the clothoid, one of the main properties of the parametric curves, that is, *transformation invariance* [14] is used. This property allows the RBC to be rotated, translated and rescaled through the control points. Note that the homothetial factor, K, of the Clothoid is included in the RBC as a scaling factor, as shown in (3).

$$P(\gamma)=\frac{\sum_{k=0}^{N} w_k\cdot(K\cdot C_k)\cdot B_\gamma^k}{\sum_{k=0}^{N} w_k\cdot B_\gamma^k} \quad (3)$$

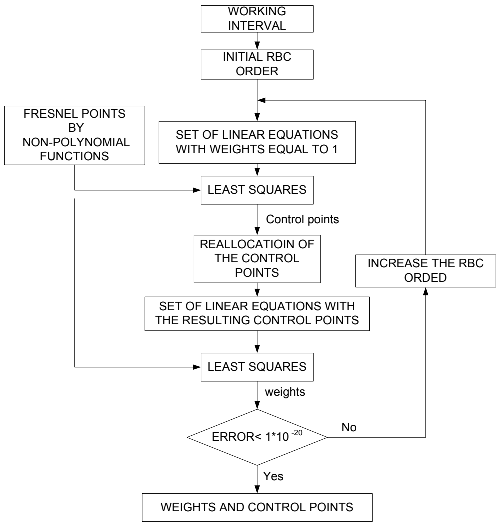


Fig. 1 Block diagram of the approximation technique

The selected working interval depends on the application. For a 2π rotation, four Clothoids with a working interval of $[0,\pi/2]$ have to be combined. In this case, the approximation technique determines that an 11th order RBC guarantees Clothoidal behavior. The corresponding weights and control points are shown in Table I.

TABLE I
COEFFICIENTS OF THE RBC FOR A PATH PLANNING

i	C		S	
	C_i	w_i	C_i	w_i
0	0	1	0	1
1	0.0909	$1\text{-}77\cdot 10^{-7}$	0	1
2	0.1818	$1\text{+}77\cdot 10^{-8}$	0	1
3	0.2727	$1\text{-}17\cdot 10^{-7}$	0.003	$1\text{-}27\cdot 10^{-6}$
4	0.3636	$1\text{+}15\cdot 10^{-7}$	0.012	$1\text{+}92\cdot 10^{-7}$
5	0.4540	$1\text{-}18\cdot 10^{-7}$	0.031	$1\text{+}14\cdot 10^{-7}$
6	0.5422	$1\text{-}82\cdot 10^{-8}$	0.0634	$1\text{-}47\cdot 10^{-7}$
7	0.6251	$1\text{-}68\cdot 10^{-8}$	0.1107	$1\text{+}36\cdot 10^{-7}$
8	0.6973	$1\text{-}22\cdot 10^{-8}$	0.1755	$1\text{-}19\cdot 10^{-7}$
9	0.7513	$1\text{-}40\cdot 10^{-9}$	0.2564	$1\text{+}66\cdot 10^{-8}$
10	0.7797	$1\text{-}47\cdot 10^{-8}$	0.3473	$1\text{-}27\cdot 10^{-9}$
11	0.7798	1	0.4382	1

The most interesting aspect to remark is that the homothetial factor K acts as a scaling factor. Therefore, the coefficients are computed just once for any group of Clothoids.

IV. BI-ELEMENTARY PATH

In this paper, an elementary path is considered as a concatenation of two equal piecewise Clothoids. Furthermore, two poses in the plane can be obtained by concatenating two different elementary paths to form a *Bi-elementary path* [7], [8]. In particular, to link the start pose, $\mathbf{p}_i = (x_i, y_i, \theta_i)$, and end pose, $\mathbf{p}_e = (x_e, y_e, \theta_e)$, it is necessary to compute the split pose, $\mathbf{p}_s = (x_s, y_s, \theta_s)$ which is a symmetric pose with respect to the start and end poses [13].

From the split pose, the tangent angles of the first Clothoid,(τ_1, τ_2) are obtained as:

$$\tau_1 = \left(\left\|\theta_s\right| - \left|\theta_i\right\|\right)/2 \quad (4)$$

$$\tau_2 = \left(\left\|\theta_e\right| - \left|\theta_s\right\|\right)/2 \quad (5)$$

The second Clothoid is equal and symmetric to the first one. This means that the control points of the second clothoid can be obtained by simply taking the symmetric control points of the first Clothoid, see Fig.2.

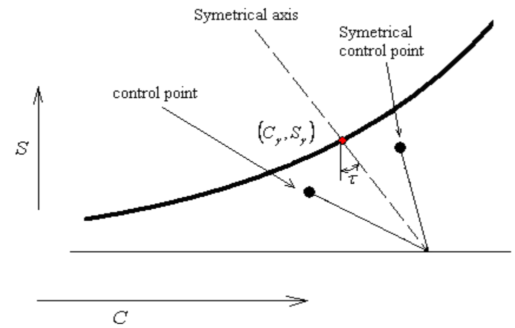


Fig. 2. Control points symmetry: Construction of an elementary path.

In addition, a pivot intermediate point ($C\gamma$, $S\gamma$), defining a symmetrical axis between the two elementary paths is obtained. This pivot point is calculated introducing the tangent angles, (τ_1, τ_2) in the RBC, as described in Fig. 3.

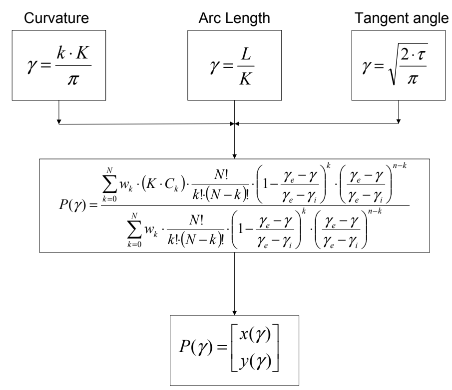
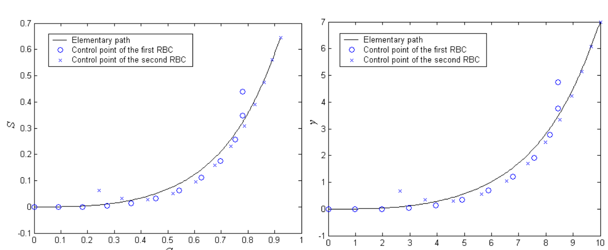


Fig 3 Block diagram of RBC reconfiguration

As the tangent angle does not depend on the homothetial factor, it is possible to construct normal elementary paths with *K=1*. Afterwards, the specific elementary paths will be recomputed using the homothetial factor as a scaling factor, as seen in Fig. 4.



Without scaling factor With scaling factor

Fig 4. Elementary path construction with end pose (10,7, 0.7854)

Therefore, the next step will be to scale the control points considering that the last control point of the second clothoid, $\left(C_{C_N}^2, C_{S_N}^2\right)$, of the first elementary path and the last control point of second clothoid, $\left(C_{C_N}^4, C_{S_N}^4\right)$, belong to the second elementary path has to coincide with the split pose and the end pose respectively, that is:

$$K_1 = \frac{\left\| |x_s| - |x_i| \right\|}{C_{C_N}^2} = \frac{\left\| |y_s| - |y_i| \right\|}{C_{S_N}^2} \quad (6)$$

$$K_2 = \frac{\left\| |x_e| - |x_s| \right\|}{C_{C_N}^4} = \frac{\left\| |y_e| - |y_s| \right\|}{C_{S_N}^4} \quad (7)$$

Without losing generality, both elementary paths are constructed with respect to the coordinate origin. Fig.5 is a diagram of elementary path construction.

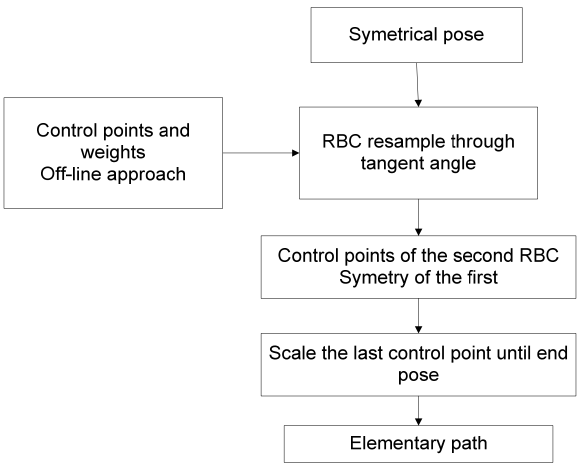


Fig 5. Diagram of the elementary path construction

The last step is to translate and rotate the control points of the RBCs until the correct pose, defining a Bi-Elementary path. Figure 6 shows an example of Bi-elementary path construction.

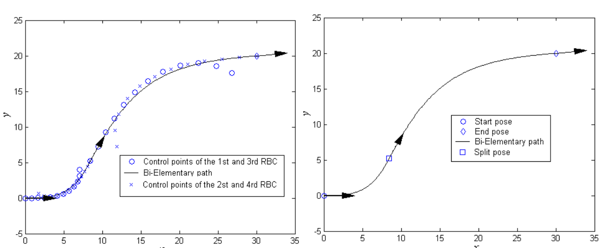


Fig 6 Bi-elementary path: Start pose:(0,0,0), End pose: (30,20,0.1)

The working interval of the resulting RBCs for the four Clothoids are $[0\ldots\gamma(\tau_1)]$, $]\gamma(\tau_1)\ldots 0]$, $]0\ldots\gamma(\tau_2)]$, $]\gamma(\tau_2)\ldots 0]$ respectively. In contrast to the method described in [7] and [8], this method avoids iterative procedures when computing elementary paths.

V. THE SHORTEST BOUNDED-CURVATURE PATH

As demonstrated in [13], the *loci* of split poses (intermediate poses) joining the start and end poses with a Bi-elementary path is a circle. Therefore, it exists an infinite set of solutions (Bi-elementary paths with different lengths and curvatures) joining start and end poses. In particular, we are interested in obtaining the shortest bounded-curvature path that satisfies kinematic curvature constrains of vehicle-like mobile robots $k \le k_{\max}$. Unfortunately, this solution can not be obtained with analytical methods; requiring heuristic algorithms.

Based on the ideas from [15], we have developed a heuristic method to find the shortest bounded-curvature path. In this paper, the shortest bounded-curvature path is obtained by concatenating circular arcs with the maximum allowed curvature and straight lines. However, results from [15] do not satisfy curvature continuity condition. On the contrary, our approach solves this problem using a Bi-elementary path that inherently satisfies this condition. Obviously, our solution obtain one elementary path with the maximum curvature and minimum scaling factor (similar to the maximum curvature circle) and another elementary path with minimum curvature and maximum scaling factor (similar to a straight line).

Based on [13], the center (x_c, y_c) and radius r of the circle (*loci* of split poses) depends on the start pose, $\mathbf{p}_i$, and end pose $\mathbf{p}_e$, generally expressed as:

$$(x_c, y_c, r_c) \equiv \mathbf{f}(\mathbf{p}_i, \mathbf{p}_e)$$

Additionally, the split pose depends on the independent variable θ (angle the circle) as shown in Fig. 7:

$$\mathbf{p}_s \equiv \mathbf{h}(\mathbf{p}_i, \mathbf{p}_e, \theta)$$

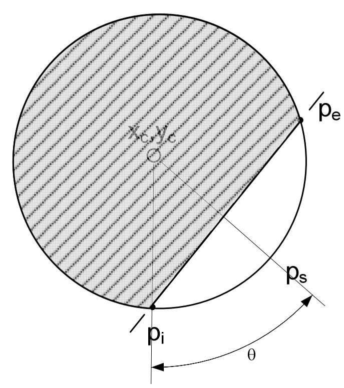


Fig 7. Diagram of the elementary path construction

Therefore, the problem is stated as follows:

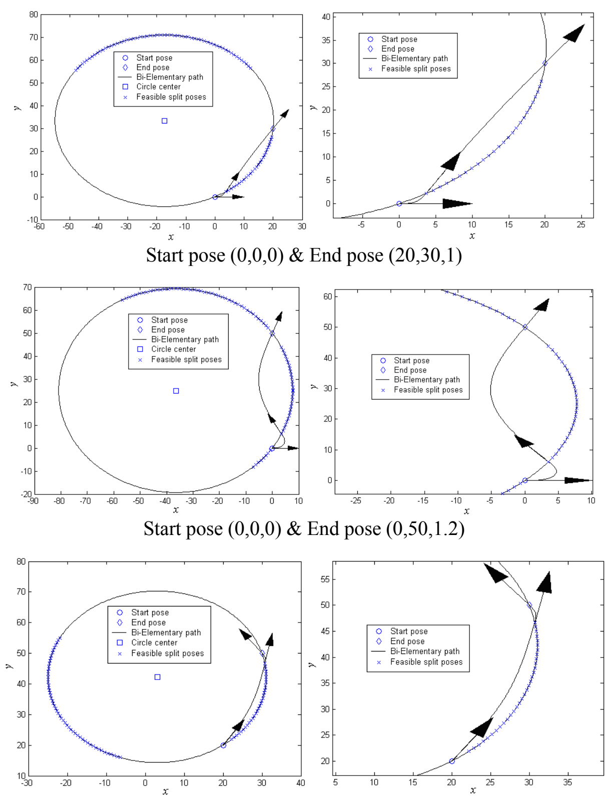
$$L_{\min} = \min_{\mathbf{p}_i}\left(Length\left(BiElementary\left(\mathbf{p}_s, \mathbf{p}_e, \mathbf{p}_i\right)\right)\right) \text{ with } k \le k_{\max}.$$

It can be shown that the split pose for the optimal solution is always located on the non-shadowed area H_1 of the circle depicted in Fig. 7 if only if the curvature condition is satisfied in the area. Otherwise, the solution will be necessary located on the shadowed area H_2. Therefore, our search procedure tries first to find a solution in H_1 and if doesn't exists, then looks for solution in H_2.

Based on [15], our search procedure looks for the pose that satisfies the curvature condition and it is closest to either start or end pose.

It is important to remark that in our method it is not necessary to develop the whole Bi-elementary path to compute the length of the path. This can be done firstly, computing the tangent angle of the Elementary paths using Equations (4) and (5). Later, terms K_1 and K_2 of the Fresnel integrals (scaling factors of the RBCs) are calculated through Equations (6) and (7). In this case, only the last control point of the second Clothoid of the Elementary path, $\left(C^2_{C_N}, C^2_{S_N}\right)$, $\left(C^4_{C_N}, C^4_{S_N}\right)$ it is required to compute the arc length, which can be done simply by projecting the first control point of the first Clothoid.

Fig. 8 depicts three geometric loci (possible split poses within bounded-curvature) where the optimal solution has been found in H_1.



Start pose (0,0,0) & End pose (20,30,1)

Start pose (0,0,0) & End pose (0,50,1.2)

Start pose (20,20,1.2) & End pose (30,50,2.2)

Fig. 8. Bounded-curvature shortest path with RBCs as clothoids

Fig. 9, shows the case where the solution could not be found in H_1 and therefore the search procedure was focused on H_2.

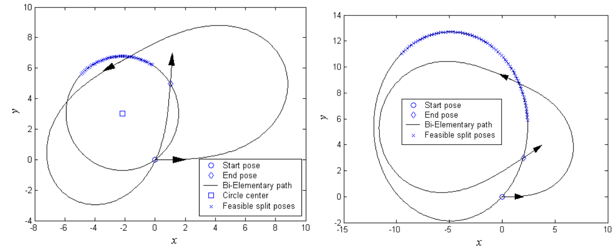


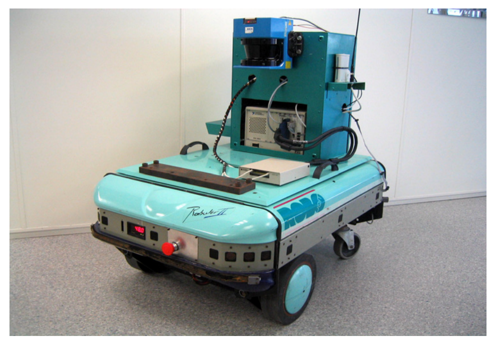
Fig. 9. Bounded-curvature shortest path out of the shortest path

With this heuristic method, the computational cost of the search procedure is significantly reduced compared to brute-force search on the whole circle.

In our simulations using a 2.4 GHz Pentium IV, If the solution exists in H_1, the mean time required was 13 *ms*. Otherwise, several iterations will be required at the cost of *0.26 ms/iteration*.

VI. Experimental Results

In order to test the new formulation, a trajectory generation module has been implemented on the differential mobile robot shown in Picture 1.



Picture 1 Differential mobile robot

This robot is equipped with an industrial PC NI PXI-8186 with a Pentium IV at 2.2 GHz. With this processor, Clothoidal paths are computed in *15 ms.*

In real applications, it is particularly important to resample the RBC, especially when using computed paths as control references. In any case, as the RBC is a continuous function, continuous resampling is always possible.

Curvature, arc length and tangent angles depend on each vehicle and path requirement, as explained in Section IV. In particular, for a tricycle-like mobile robot, the curvature is related to the steering wheel turning radius while arc length is considered for differential mobile robots.

The main property of the clothoid is that it guarantees a constant *Jerk* which minimizes wheel slip. This produces errors that cannot be measured by the encoders and therefore the mobile robot is equipped with inertial sensors capable of measuring variation in the tangential acceleration. The open-loop control structure used in our experiments is shown in Fig. 10.

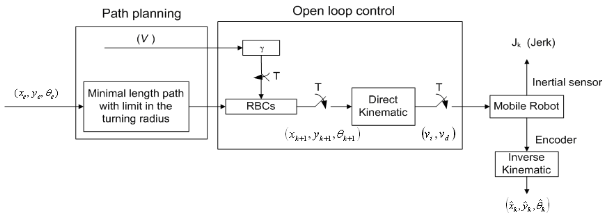


Fig. 10. Open control loop

where γ parameter can be easily calculated as

$$\gamma = \frac{V \cdot n \cdot T}{K}$$

and

n: number of samples in the RBC.
T: sampling period.
V: mobile robot velocity.
K: scaling factor.

An example of the bi-elementary path followed by the robot is show in Fig. 11. (left). Start pose is selected to (0,0,0) and end pose to (5,12,1). The robot velocity is selected to 0.25 m/s. the centrifugal acceleration suffers by the robot is show in Fig.11 (right).

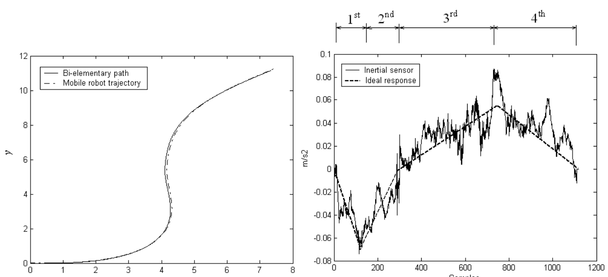


Fig. 11. Clothoidal path (right). Inertial sensor measurement (left)

VII. CONCLUSIONS AND FUTURE WORKS

This paper presents a method for obtaining real-time Clothoidal paths in mobile robotics. The method involves two steps: 1) to define off-line approximations of Clothoids as Rational Bezier Curves (RBC); 2) To generate on-line paths by simply rescaling, rotating and translating the previous off-line formulation. In whole process, coefficients, weights and control points are kept invariant.

One of the main advantages of this method is that a low order curve is obtained guaranteeing Clothoidal behavior. In addition to this, the curved is computed off-line, which is one of the most time-consuming tasks. Another advantage of this method is that construction of Elementary paths and Bi-elementary path is computed without requiring iterative methods. As a consequence, the method can be implemented in systems under real-time requirements.

As it has been shown, there are infinite solutions to join two arbitrary poses in a plane using Bi-Elementary paths. In particular, we are interested in the shortest-bounded curvature path, which can be obtained based on some heuristic rules. It has been shown that the Bi-elementary parameter computation takes about *0.26 ms of* computational time. As a result, robot path planning only requires about *15 ms* to find the shortest bounded-curvature path using a heuristic iterative method.

In real applications, it is particularly important to resample the RBC, especially when using the computed paths as control references. Given that RBCs are continuous functions, continuous resampling is always possible. In the paper, it has been shown, different cases to resample a RBC based on: curvature, arc length and tangent angle. In particular, for tricycle-like mobile robots, the curvature case, since it is related to the steering wheel turning radius while arc length case is considered for differential mobile robots.

As a conclusion, it is also important to remark that the technique presented in this paper is a novel approach and, to the authors' knowledge, it is the first proposal to allow Clothoidal paths implementing real-time.

As further research, it will be interesting to extend 2D Fresnel integrals associated to Clothoids to 3D. This will imply the application of the proposed path planning method to robots with 6 d.o.f. or even more. This method can also be used to generate trajectories for high speed CNC machines.

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