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Vision-based range estimation via Immersion and Invariance for robot formation control

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Abstract— **The paper introduces a new vision-based range estimator based upon the Immersion and Invariance (I&I) methodology, for leader-follower formation control. The proposed reduced-order nonlinear observer achieves global exponential convergence of the observation error to zero and it is extremely simple to implement and to tune. A Lyapunov analysis is provided to show the stability of the closed-loop system arising from the combination of the range estimator and an input-state feedback controller. Simulation experiments illustrate the theory and show the effectiveness of the proposed design.**

I. Introduction

In the last few years we witnessed a growing interest in robotics, in motion coordination and cooperative control of multi-agent systems. In this respect, several new problems, such as, e.g., "consensus" [1], "rendezvous" [2], "coverage" [3], "formation control", etc., have been formulated and solved using tools coming from computer science and control theory. Among these, due to its wide range of applicability, the formation control problem received a special attention and stimulated a great deal of research [4]–[7]. By formation control we simply mean the problem of controlling the relative position and orientation of the robots in a group, while allowing the group to move as a whole.

In this paper we are interested to a *leader-follower* formation control approach, in which a leader robot moves along a predefined trajectory while the other robots, the followers, are to maintain a desired distance and orientation to it [6]. Leader-follower architectures are known to have poor disturbance rejection properties. In addition, the over-reliance on a single agent for achieving the goal may be undesirable, especially in adverse conditions. Nevertheless the leader-follower approach is particularly appreciated for its simplicity and scalability.

Recently, a special interest has been devoted to sensing devices for autonomous navigation of multi-robot systems. An inexpensive and challenging way to address the navigation problem is to use exclusively on-board passive vision sensors, which provide only the projection (or view-angle) to the other robots, but not the distance. In this respect, the formation control problem can be solved only if a *localization problem* has been addressed, i.e. only if a suitable observer providing an estimate of the relative distance and orientation of the robots w.r.t. a common reference frame, has been designed.

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A new observability condition for general nonlinear systems, based on the Extended Output Jacobian matrix, has been proposed in [8] and applied to the study of leader-follower formations localizability. The extended and unscented Kalman filters have been then used in [8] and [9], respectively, to estimate the robots relative distance (hereafter referred to as *range estimation*). Although widely used in the literature, these observers are known to have some serious drawbacks: they are difficult to tune and implement, the estimation error is not guaranteed to converge asymptotically to zero and an *a priori* knowledge about noise is usually required.

A new methodology, called *Immersion and Invariance* (hereafter, I&I), has been recently proposed to design reduced-order observers for general nonlinear systems [10]. Actually, the problem of constructing a reduced-order observer is formulated as a problem of rendering attractive an appropriately selected invariant manifold in the extended space of the plant and the observer. The effectiveness of the new observer design technique has been proved by Astolfi and coworkers through several academic and practical examples [10]–[12]. However, only a couple of other papers (see [13], [14]) exploited the I&I methodology to design nonlinear observers and no applications in formation control are reported in the robotics literature up to now.

The original contribution of this paper is twofold: first, we present an observer based upon the I&I technique for leader-follower range estimation using on-board camera information (bearing-only). The reduced-order observer provides a globally exponentially convergent estimate of the range. It can be easily tuned to achieve the desired convergence rate by acting on a single gain parameter and it is extremely simple to implement as well. As a second contribution, we present a input-state feedback control law and an accurate Lyapunov analysis to prove the stability of the closed-loop system arising from the combination of the range estimator and the formation controller.

The rest of the paper is organized as follows. Sect. II is devoted to the problem formulation. In Sect. III the basic theory related to the I&I observer design methodology is recalled. In Sect. IV the leader-to-follower range estimator is presented. In Sect. V an input-to-state feedback control scheme is designed and the stability of the closed-loop system is analytically proved via Lyapunov arguments. In Sect. VI simulation experiments confirm the effectiveness of the proposed designs. Finally, in Sect. VII the major contributions of the paper are summarized and future research lines are highlighted.

II. PROBLEM FORMULATION

The setup considered in the paper consists of two unicycle robots (see Fig. 1). One robot is the *leader*, whose control input is given by its translational and angular velocities, $u_L = [v_L \;\; \omega_L]^T$. The other robot is the *follower*, controlled by $u_F = [v_F \;\; \omega_F]^T$.

Each robot is equipped with an omnidirectional camera, which constitutes its only sensing device. Using well-known color detection and tracking algorithms [15], the leader is able to measure from the image, both the angle ζ (w.r.t. the camera of the follower) and the angle ψ (w.r.t. the colored marker P placed at a distance d along the follower translational axis) (see Fig. 1). Analogously, the follower can compute the angle ν using its panoramic sensor.
Note that the measurement of both the angles ζ and ψ by the leader, could not be a trivial task in practice, especially when the robots are distant. This problem has been addressed and solved in [9], where only the angle ζ needs to be computed.

As first shown in [6], the leader-follower kinematics can be easily expressed using polar coordinates $[\rho \;\; \psi \;\; \varphi]^T$, where ρ is the distance between the leader and the marker P on the follower and φ is the relative orientation between the two robots, i.e the bearing. It is easy to verify that

$$\varphi = -\zeta + \nu + \pi. \tag{1}$$

***Proposition** 1 ([8]):* Consider the setup in Fig. 1. The leader-follower kinematics can be expressed by the driftless system,

$$\begin{bmatrix} \dot{\rho} \\ \dot{\psi} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} \cos\gamma & d\sin\gamma & -\cos\psi & 0 \\ \frac{-\sin\gamma}{\rho} & \frac{d\cos\gamma}{\rho} & \frac{\sin\psi}{\rho} & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_F \\ \omega_F \\ v_L \\ \omega_L \end{bmatrix} \tag{2}$$

where $\gamma \triangleq \varphi + \psi$. ■

In order to simplify the subsequent derivations and without losing in generality, we will only consider formations with a single follower (nevertheless, the results of this paper can be immediately extended to the general case of n followers [8]).

The information flow between the robots is now briefly described. The follower transmits the angle ν to the leader and this robot computes the bearing φ using equation (1) (to simplify the discussion, we will henceforth refer *only* to the bearing φ implicitly assuming the transmission of ν). The leader can then measure a two dimensional vector,

$$y \triangleq [y_1 \;\; y_2]^T = [\psi \;\; \varphi]^T. \tag{3}$$

The estimation of the range is carried out by the leader which then uses it, together with (3), to compute the control input u_F. The leader subsequently transmits the vector $[v_F \;\; \omega_F]^T$ to the follower.

In Sect. IV we will design a nonlinear observer based upon the I&I methodology for the estimation of the range ρ given the angular measurements $[\psi \;\; \varphi]^T$, the control inputs of the robots and their derivatives.

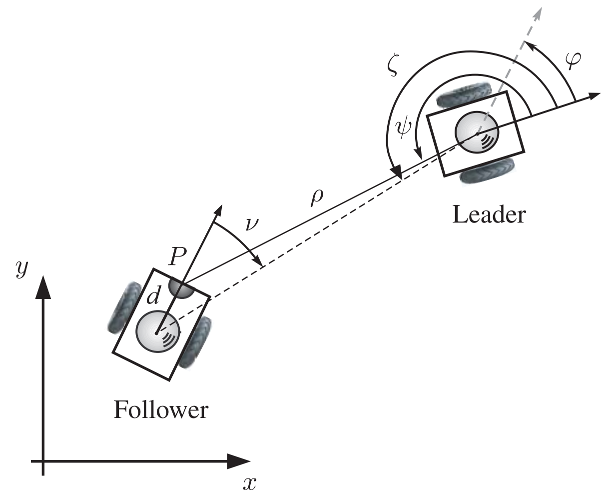

Fig. 1. Leader-follower setup

III. OBSERVER DESIGN VIA I&I

For the reader's convenience we provide here a brief overview of the basic theory related to the observer design via I&I [10], [12]. Consider generic nonlinear, time-varying systems described by

$$\dot{y} = f_1(y, \eta, t) \tag{4}$$
$$\dot{\eta} = f_2(y, \eta, t) \tag{5}$$

where $y \in I\!R^m$ is the measurable output and $\eta \in I\!R^n$ is the unmeasured state. The vector fields $f_1(\cdot)$, $f_2(\cdot)$ are assumed to be forward complete, i.e., trajectories starting at time t_0 are defined for all times $t \geq t_0$.

***Definition** 1:* The dynamical system

$$\dot{\xi} = \alpha(y, \xi, t) \tag{6}$$

with $\xi \in I\!R^p$, $p \geq n$, is called an *observer* for the system (4)-(5), if there exist mappings,

$$\beta(y, \xi, t) \colon I\!R^m \times I\!R^p \times I\!R^+ \to I\!R^p \;\text{ and }\; \phi_{y,t}(\eta) \colon I\!R^n \to I\!R^p$$

with $\phi_{y,t}(\eta)$ parameterized by y and t and left-invertible[1], such that the manifold

$$\mathcal{M}_t = \{(y, \eta, \xi) \in I\!R^m \times I\!R^n \times I\!R^p : \beta(y, \xi, t) = \phi_{y,t}(\eta)\}$$

has the following properties:

1) All the trajectories of the extended system (4)-(6) that start on the manifold $\mathcal{M}_t$ at time t remain on it for all times $\tau > t$, i.e., $\mathcal{M}_t$ is *positively invariant*.
2) All the trajectories of (4)-(6) that start in a neighborhood of $\mathcal{M}_t$ asymptotically converge to the manifold, i.e., $\mathcal{M}_t$ is *attractive*.

The above definition states that an asymptotic estimate $\hat{\eta}$ of the state η is given by $\phi^L_{y,t}(\beta(y, \xi, t))$, where $\phi^L_{y,t}$ denotes a left-inverse of $\phi_{y,t}$. The following proposition provides a

[1]A mapping $\phi_{y,t}(\eta) : I\!R^n \to I\!R^p$ (parameterized by y and t) is *left-invertible* if there exists a mapping $\phi^L_{y,t}(\eta) : I\!R^p \to I\!R^n$ such that $\phi^L_{y,t}(\phi_{y,t}(\eta)) = \eta$, for all $\eta \in I\!R^n$ (and for all y and t).

general tool for constructing a nonlinear observer of the form given in Definition 1.

***Proposition** 2 (I&I observer dynamics):* Consider the system (4)-(6) and suppose that there exist two mappings $\beta(\cdot) : I\!R^m \times I\!R^p \times I\!R^+ \to I\!R^p$ and $\phi_{y,t}(\cdot) : I\!R^n \to I\!R^p$ with left-inverse $\phi^L_{y,t}(\cdot) : I\!R^p \to I\!R^n$, such that the following conditions hold:

(A1) For all y, ξ and t, we have $\det\left(\frac{\partial\beta}{\partial\xi}\right) \neq 0$.

(A2) The system,

$$\dot{z} = -\frac{\partial\beta}{\partial y}\left(f_1(y,\hat\eta,t) - f_1(y,\eta,t)\right) + \left.\frac{\partial\phi_{y,t}}{\partial y}\right|_{\eta=\hat\eta} f_1(y,\hat\eta,t)$$
$$-\frac{\partial\phi_{y,t}}{\partial y} f_1(y,\eta,t) + \left.\frac{\partial\phi_{y,t}}{\partial\eta}\right|_{\eta=\hat\eta} f_2(y,\hat\eta,t) - \frac{\partial\phi_{y,t}}{\partial\eta} f_2(y,\eta,t)$$
$$+\left.\frac{\partial\phi_{y,t}}{\partial t}\right|_{\eta=\hat\eta} - \frac{\partial\phi_{y,t}}{\partial t} \qquad (7)$$

with $\hat\eta = \phi^L_{y,t}(\phi_{y,t}(\eta) + z)$, has an asymptotically stable equilibrium at $z = 0$, uniformly in η, y and t.

Under the assumptions (A1) and (A2), system (6) is a reduced-order observer for (4)-(5) with,

$$\alpha(y,\xi,t) = -\left(\frac{\partial\beta}{\partial\xi}\right)^{-1}\Bigg(\frac{\partial\beta}{\partial y} f_1(y,\hat\eta,t) + \frac{\partial\beta}{\partial t}$$
$$-\left.\frac{\partial\phi_{y,t}}{\partial y}\right|_{\eta=\hat\eta} f_1(y,\hat\eta,t) - \left.\frac{\partial\phi_{y,t}}{\partial\eta}\right|_{\eta=\hat\eta} f_2(y,\hat\eta,t) - \left.\frac{\partial\phi_{y,t}}{\partial t}\right|_{\eta=\hat\eta}\Bigg)$$

where $\hat\eta = \phi^L_{y,t}(\beta(y,\xi,t))$. The observer error dynamics are given by (7). ■

Remark 1: Prop. 2 provides an implicit description of the observer dynamics (6) in terms of the mappings $\beta(\cdot)$, $\phi_{y,t}(\cdot)$, $\phi^L_{y,t}(\cdot)$ which must then be selected to satisfy (A2). Hence, the problem of constructing a reduced-order observer for the system (4)-(5) reduces to the problem of rendering the system (7) asymptotically stable by assigning the functions $\beta(\cdot)$, $\phi_{y,t}(\cdot)$ and $\phi^L_{y,t}(\cdot)$. This peculiar stabilization problem can be extremely hard to solve, since, in general, it relies on the solution of a set of partial differential equations (or inequalities). However, as we will see in the next section, these equations are easily solvable in the problem under investigation.

IV. Range estimator

In order to apply the methodology described in the previous section, to design a nonlinear observer of the range ρ, system (2) should be recast in the form (4)-(5). In this respect, it is convenient to introduce the new variable $\eta \triangleq 1/\rho$, that is well-defined assuming $\rho \neq 0$. Using this transformation, system (2) becomes,

$$\begin{bmatrix}\dot\eta\\ \dot\psi\\ \dot\varphi\end{bmatrix} = \begin{bmatrix}-\eta^2\cos\gamma & -\eta^2 d\sin\gamma & \eta^2\cos\psi & 0\\ -\eta\sin\gamma & \eta\, d\cos\gamma & \eta\sin\psi & -1\\ 0 & -1 & 0 & 1\end{bmatrix}\begin{bmatrix}v_F\\ \omega_F\\ v_L\\ \omega_L\end{bmatrix}. \quad (8)$$

Recalling that $y \triangleq [\psi\ \ \beta]^T$, system (8) can be rewritten as,

$$\dot y = \underbrace{\begin{bmatrix}-\omega_L\\ \omega_L - \omega_F\end{bmatrix}}_{h(t)} + \underbrace{\begin{bmatrix}-v_F\sin\gamma + \omega_F\, d\cos\gamma + v_L\sin y_1\\ 0\end{bmatrix}}_{g(y,t)}\eta$$
$$\dot\eta = -\underbrace{\left(v_F\cos\gamma + \omega_F\, d\sin\gamma - v_L\cos y_1\right)}_{\ell(y,t)}\eta^2 \qquad (9)$$

where $\gamma \triangleq y_1 + y_2$.

The next proposition introduces a globally uniformly exponentially convergent observer of η.

***Proposition** 3 (Range estimator):* Suppose that the control inputs of the robots and the output y are bounded functions of time, i.e., v_L, ω_L, v_F, ω_F, $y \in \mathcal{L}^\infty$ and that v_L, v_F, ω_F are first order differentiable. Assume that the following condition is satisfied,

$$g_1(y,t) \neq 0,\ \ \forall\, t > 0, \qquad (10)$$

where $g_1(y,t)$ is the first component of the vector $g(y,t)$ defined in (9). Then:

$$\dot\xi = M[\,g_1(y,t)\,,\ -v_F\sin\gamma + \omega_F\, d\cos\gamma\,]\,(h(t) + g(y,t)\,\hat\eta)$$
$$+ M(\dot v_F\cos\gamma + \dot\omega_F\, d\sin\gamma - \dot v_L\cos y_1)$$
$$-\frac{\mathrm{sign}(g_1(y,t))}{g_1(y,t)^2}\Big([\ell(y,t)\,,\ v_F\cos\gamma + \omega_F\, d\sin\gamma]$$
$$\times(h(t) + g(y,t)\,\hat\eta)\,\hat\eta + (\dot v_F\sin\gamma - \dot\omega_F\, d\cos\gamma - \dot v_L\sin y_1)\,\hat\eta\Big)$$
$$+\frac{\mathrm{sign}(g_1(y,t))}{g_1(y,t)}\,\ell(y,t)\,\hat\eta^2 \qquad (11)$$

where M is a positive gain to be suitably tuned and

$$\hat\eta = \big(M\ell(y,t) - \xi\big)\big|g_1(y,t)\big|, \qquad (12)$$

is a globally uniformly exponentially convergent observer for system (9).

Proof: See the Appendix. ■

Some remarks are in order at this point:

- Equation (11) is a *reduced-order* observer for system (9): in fact it has lower dimension than the system.
- The observer can be easily tuned to achieve the desired convergence rate by acting on the single gain parameter M.
- There is an exact correspondence between (10), which is necessary to avoid singularities in (11), and the observability condition derived in [8] using the Extended Output Jacobian matrix. As pointed out in [8], this condition is extremely attractive since it allows one to define the set of all the trajectories of the leader preserving the observability of the system.

V. Formation control and closed-loop stability

Note that if the state $s = [\eta\ \ \psi\ \ \varphi]^T$ is perfectly known, then system (8) can be exactly input-state feedback linearized and the asymptotic convergence of s towards any desired state s^{des} is guaranteed. The stability analysis becomes more involved when an observer is present inside the control loop.

In Prop. 4, we design a formation control law and we study the effect of the I&I observer on the closed-loop stability through an accurate Lyapunov analysis.

For the next derivations, it is convenient to rewrite system (8) in the form:

$$\dot{s}_r = F(s)\, u_L + H(s)\, u_F \quad (13)$$

$$\dot{\varphi} = \omega_L - \omega_F \quad (14)$$

where $s_r \triangleq [\eta \;\; \psi]^T$ is the *reduced* state space vector and

$$H(s) = \begin{bmatrix} -\eta^2 \cos\gamma & -\eta^2 d \sin\gamma \\ -\eta \sin\gamma & \eta\, d \cos\gamma \end{bmatrix}, \; F(s) = \begin{bmatrix} \eta^2 \cos\psi & 0 \\ \eta \sin\psi & -1 \end{bmatrix}.$$

***Proposition** 4 (Control law and closed-loop stability):* Consider the system (13)-(14) with $v_L > 0$ and $\omega_L \le \omega_{Lmax}$, $\omega_{Lmax} \in I\!R^+$. For a given state estimate $\hat{s} = [\hat{\eta} \;\; \psi \;\; \varphi]^T$ (with $\hat{\eta} > 0$) provided by the I&I observer in Prop. 3 with gain M sufficiently large, the feedback control law

$$u_F = H^{-1}(\hat{s})\,(p - F(\hat{s})\, u_L) \quad (15)$$

with $p \triangleq -K(\hat{s}_r - s_r^{des})$, $K = \mathrm{diag}\{k_1, k_2\}$, $k_1, k_2 > 0$, $\hat{s}_r = [\hat{\eta} \;\; \psi]^T$, guarantees the asymptotic convergence of the control error $s_r - s_r^{des}$ to zero and the locally uniformly ultimate boundedness (UUB) of the internal dynamics φ.

Proof: Substituting (15) in (13) we obtain the dynamics of the controlled system:

$$\dot{s}_r = F(s)\, u_L + H(s)\, H^{-1}(\hat{s})(p\,(\hat{s}_r) - F(\hat{s})\, u_L)\,.$$

Since s_r^{des} is constant, the dynamics of the control error $e_r = s_r - s_r^{des}$ is simply given by,

$$\dot{e}_r = \underbrace{\begin{bmatrix} -k_1(\eta/\hat{\eta})^2 & 0 \\ 0 & -k_2(\eta/\hat{\eta}) \end{bmatrix}}_{A(t)} e_r + \underbrace{\begin{bmatrix} -k_1(\hat{\eta} - \eta)(\eta/\hat{\eta})^2 \\ \omega_L(\eta/\hat{\eta} - 1) \end{bmatrix}}_{b(t)} \quad (16)$$

where $\hat{s}_r = s_r + [\hat{\eta} - \eta, \; 0]^T$. To prove that the control error asymptotically converges to zero, we have then to study the stability of a linear time-varying system with perturbation $b(t)$.

First of all, let study the stability of the equilibrium point $e_r = 0$ of the *non-perturbed* system. Given the candidate Lyapunov function $V = e_r^T e_r$, we have:

$$\dot{V} = e_r^T \dot{e}_r + \dot{e}_r^T e_r = 2\, e_r^T A(t)\, e_r \le 2\,\lambda_M \|e_r\|^2 = 2\,\lambda_M\, V$$

where $\lambda_M = \max\{-k_1(\eta/\hat{\eta})^2, -k_2(\eta/\hat{\eta})\}$. Since $\hat{\eta} > 0$, then $\lambda_M < 0$, which implies that $e_r = 0$ is a globally exponentially stable equilibrium point for the non-perturbed system.

To study the stability of the *perturbed* system, let consider again the Lyapunov function $V = e_r^T e_r$ for which it results:

$$\begin{aligned} \dot{V} &= 2\, e_r^T A(t)\, e_r + 2\, e_r^T b(t) \\ &\le 2\,\lambda_M \|e_r\|^2 + 2\,\|e_r\|\,\|b\| \\ &\le 2\,(1-\theta)\,\lambda_M \|e_r\|^2 + 2\,\theta\lambda_M \|e_r\|^2 + 2\,\|e_r\|\,\delta \quad (17) \end{aligned}$$

where $0 < \theta < 1$ and $\|b\| \le \delta$. From the last inequality in (17) we have

$$\begin{aligned} \dot{V} &\le 2\,(1-\theta)\,\lambda_M\, \|e_r\|^2 < 0 \\ &\quad \text{if} \quad \delta \le -\,\theta\,\lambda_M\, \|e_r\| \;\text{ for all }\; e_r\,. \end{aligned} \quad (18)$$

Since by hypothesis $\omega_L \le \omega_{Lmax}$, we can choose $\delta = \left|\frac{\eta}{\hat{\eta}} - 1\right| \sqrt{\omega_{L\,max}^2 + \frac{k_1^2\,\eta^4}{\hat{\eta}^2}}$ and rewrite the second inequality in (18) as:

$$\|e_r\| \ge -\frac{1}{\theta\lambda_M} \left|\frac{\eta}{\hat{\eta}} - 1\right| \sqrt{\omega_{L\,max}^2 + \frac{k_1^2\,\eta^4}{\hat{\eta}^2}}\,. \quad (19)$$

We now study under which conditions (19) is verified, that is, $e_r = 0$ is an asymptotically stable equilibrium point for the perturbed system. If $\hat{\eta}$ rapidly converges to η, we can note that inequality (19) reduces to $\|e_r\| \ge 0$, that is always true. This implies that $e_r = 0$ is an asymptotically stable equilibrium point for system (16). At this point it is interesting to note that, due to the exponential convergence of the I&I observer estimation error to zero, there will exist two positive constants D and C such that

$$\hat{\eta} \ge D e^{-Ct} + \eta$$

or equivalently,

$$\left|1 - \frac{\hat{\eta}}{\eta}\right| \ge \frac{D}{\eta}\, e^{-Ct}. \quad (20)$$

Using inequality (20) in (19) and observing that parameter C is proportional to the observer gain M, we then note that the asymptotic convergence of the control error to zero can always be guaranteed by choosing a sufficiently large M.

It now remains to show that the internal dynamics φ is locally UUB. Taking ω_F from (15), equation (14) can be rewritten as,

$$\begin{aligned} \dot{\varphi} &= -\frac{v_L}{d} \sin\varphi - \frac{\sin\gamma}{\hat{\eta}^2 d}\, k_1\, e_r(1) \\ &\quad + \frac{\cos\gamma}{\hat{\eta}\, d}\, k_2\, e_r(2) - \omega_L\Big(\frac{\cos\gamma}{\hat{\eta}\, d} - 1\Big). \end{aligned} \quad (21)$$

It is convenient to write (21) synthetically in the form,

$$\dot{\varphi} = -\frac{v_L}{d} \sin\varphi + B(t, \varphi) \quad (22)$$

where $B(t, \varphi)$ is a nonvanishing perturbation acting on the nominal system $\dot{\varphi} = -\frac{v_L}{d} \sin\varphi$. The nominal system has a locally uniformly asymptotically stable equilibrium point in $\varphi = 0$ and its Lyapunov function $V = \frac{1}{2}\varphi^2$ satisfies the inequalities [16],

$$\alpha_1(|\varphi|) \le V \le \alpha_2(|\varphi|)\,, \quad -\frac{\partial V}{\partial \varphi} \frac{v_L}{d} \sin\varphi \le -\alpha_3(|\varphi|)$$

$$\left|\frac{\partial V}{\partial \varphi}\right| \le \alpha_4(|\varphi|)$$

in $[0, \infty) \times D$, where $D = \{\varphi \in I\!R : |\varphi| < \epsilon\}$, being ϵ a sufficiently small positive constant. $\alpha_i(\cdot)$, $i = 1, \ldots, 4$, are class $\mathcal{K}$ functions[2] defined as follows: $\alpha_1 = \frac{1}{4}\varphi^2$, $\alpha_2 = \varphi^2$, $\alpha_3 = \frac{v_L}{d}\varphi^2$ and $\alpha_4 = 2|\varphi|$. Since e_r is asymptotically convergent to zero and, by hypothesis, ω_L is bounded, there exist suitable velocities for the leader such that $B(t,\varphi)$ satisfies the uniform bound,

$$|B(t,\varphi)| \le \delta < \frac{\theta\,\alpha_3(\alpha_2^{-1}(\alpha_1(\epsilon)))}{\alpha_4(\epsilon)} \triangleq \frac{v_L\,\theta\,\epsilon}{8\,d}$$

for all $t \ge 0$, all $\varphi \in D$ and $0 < \theta < 1$. Then, for all $|\varphi(0)| < \alpha_2^{-1}(\alpha_1(\epsilon)) = \epsilon/2$, the solution $\varphi(t)$ of the perturbed system (22), satisfies

$$|\varphi(t)| \le \chi(|\varphi(0)|,\,t), \text{ for all } 0 \le t < t_1$$

and

$$|\varphi(t)| \le \sigma(\delta),\ \forall\, t \ge t_1$$

for some class $\mathcal{KL}$ function $\chi(\cdot,\cdot)$ and some finite time t_1, where $\sigma(\delta)$ is a class $\mathcal{K}$ function of δ defined by

$$\sigma(\delta) = \alpha_1^{-1}\big(\alpha_2\big(\alpha_3^{-1}\Big(\frac{\delta\,\alpha_4(\epsilon)}{\theta}\Big)\big)\big) \triangleq\ 2\sqrt{\frac{2\,d\,\delta\,\epsilon}{\theta\,v_L}}\,.$$

This proves that $\varphi(t)$ is locally UUB. ∎

VI. Simulation results

Simulation experiments have been conducted to illustrate the theory and show the effectiveness of the proposed designs. The leader is supposed to move along a circular path with velocities $v_L = 1\,\mathrm{m/s}$ and $\omega_L = \pi/10\,\mathrm{rad/s}$. The initial condition of system (8) is $[\eta(0)\ \psi(0)\ \varphi(0)]^T = [0.7186\ 1.5013\ 0.2618]^T$, the desired state $s_r^{des} = [1\ 2\pi/3]^T$ and $d = 0.1\,\mathrm{m}$. The gains of the observer and the controller are respectively $M = 13$ and $k_1 = k_2 = 0.1$. These values, as requested in Prop. 4, guarantee that the convergence rate of the observer is faster than that of the controller. We experimentally noticed that good closed-loop performances are assured by M from 1 to 2 orders of magnitude greater than k_1 and k_2 and that the size of M is not affected by the sampling time chosen to numerically integrate equation (11).

Fig. 2(a) shows the trajectory of the leader and the follower (in order to have a time reference, the robots are drawn every two seconds). In Fig. 2(b) the time history of the observation error $\rho - \hat{\rho}$ is provided. The error exponentially converges to zero as expected. In Fig. 2(c) the control errors $\rho - \rho^{des}$ and $\psi - \psi^{des}$ asymptotically converge to zero (recall that $\rho = 1/\eta$). Finally, Fig. 2(d) depicts the time history of the bearing angle φ. According to Prop. 4, the internal dynamics φ remains bounded while the robots move to reach the desired formation.

[2]For the definition of class $\mathcal{K}$ and class $\mathcal{KL}$ functions the reader is referred to [16], pag. 144.

VII. Conclusions and future work

The paper introduces an original vision-based leader-to-follower range estimator based upon the Immersion and Invariance methodology. The proposed reduced-order nonlinear observer achieves global exponential convergence of the observation error to zero and it is extremely simple to implement and to tune. An input-state feedback control law is designed and the closed-loop stability is proved through Lyapunov arguments. Simulation experiments illustrate the theory and show the effectiveness of the proposed designs.

Future research lines include the experimental validation of the proposed scheme and the extension of our results to vehicles with more involved kinematics (e.g., car-like robots).

Appendix
Proof of Proposition 3

With reference to the general design procedure presented in Sect. III, let suppose for simplicity, that $\phi_{y,t}(\eta) = \varepsilon(y,t)\,\eta$, where $\varepsilon(\cdot) \ne 0$ is a function to be determined [12]. Consider an observer of the form given in Prop. 2,

$$\begin{aligned}\dot{\xi} &= -\Big(\frac{\partial\beta}{\partial\xi}\Big)^{-1}\Big(\frac{\partial\beta}{\partial y}\,(h(t) + g(y,t)\,\hat{\eta}) + \frac{\partial\beta}{\partial t} - \\ &\frac{\partial\varepsilon}{\partial y}\,(h(t) + g(y,t)\,\hat{\eta})\,\hat{\eta} - \frac{\partial\varepsilon}{\partial t}\hat{\eta} + \varepsilon(y,t)\,\ell(y,t)\,\hat{\eta}^2\Big), \\ \hat{\eta} &= \varepsilon(y,t)^{-1}\,\beta(y,\xi,t).\end{aligned} \tag{23}$$

From (7) the dynamics of the error $z = \beta(y,\xi,t) - \varepsilon(y,t)\,\eta = \varepsilon(y,t)(\hat{\eta} - \eta)$ is given by,

$$\begin{aligned}\dot{z} = &-\Big(\frac{\partial\beta}{\partial y}\,g(y,t) - \frac{\partial\varepsilon}{\partial y}\,h(t) - \frac{\partial\varepsilon}{\partial t}\Big)\varepsilon(y,t)^{-1}z \\ &+\Big(\frac{\partial\varepsilon}{\partial y}\,g(y,t) - \varepsilon(y,t)\,\ell(y,t)\Big)(\hat{\eta}^2 - \eta^2).\end{aligned} \tag{24}$$

The observer design problem is now reduced to finding functions $\beta(\cdot)$ and $\varepsilon(\cdot) \ne 0$ that satisfy assumptions (A1)-(A2) in Prop. 2. In view of (24) this can be achieved by solving the partial differential equations

$$\frac{\partial\beta}{\partial y}\,g(y,t) - \frac{\partial\varepsilon}{\partial y}\,h(t) - \frac{\partial\varepsilon}{\partial t} = \kappa(y,t)\,\varepsilon(y,t) \tag{25}$$

$$\frac{\partial\varepsilon}{\partial y}\,g(y,t) - \varepsilon(y,t)\,\ell(y,t) = 0 \tag{26}$$

for some $\kappa(\cdot) > 0$. From (26) we obtain the solution $\varepsilon(y,t) = -\big(\,|g_1(y,t)|\,\big)^{-1}$ which by (10) is well-defined and nonzero for all y and t. Let,

$$\kappa(y,t) = M|g_1(y,t)|^3 + \Big(\frac{\partial\varepsilon}{\partial y}\,h(t) + \frac{\partial\varepsilon}{\partial t}\Big)\,|g_1(y,t)|.$$

By boundedness of the control inputs and $y(t)$, it is always possible to find $M > 0$ (sufficiently large) such that $\kappa(\cdot) > 0$. Equation (25) is now reduced to

$$\frac{\partial\beta}{\partial y}\,g(y,t) = -M g_1(y,t)^2$$

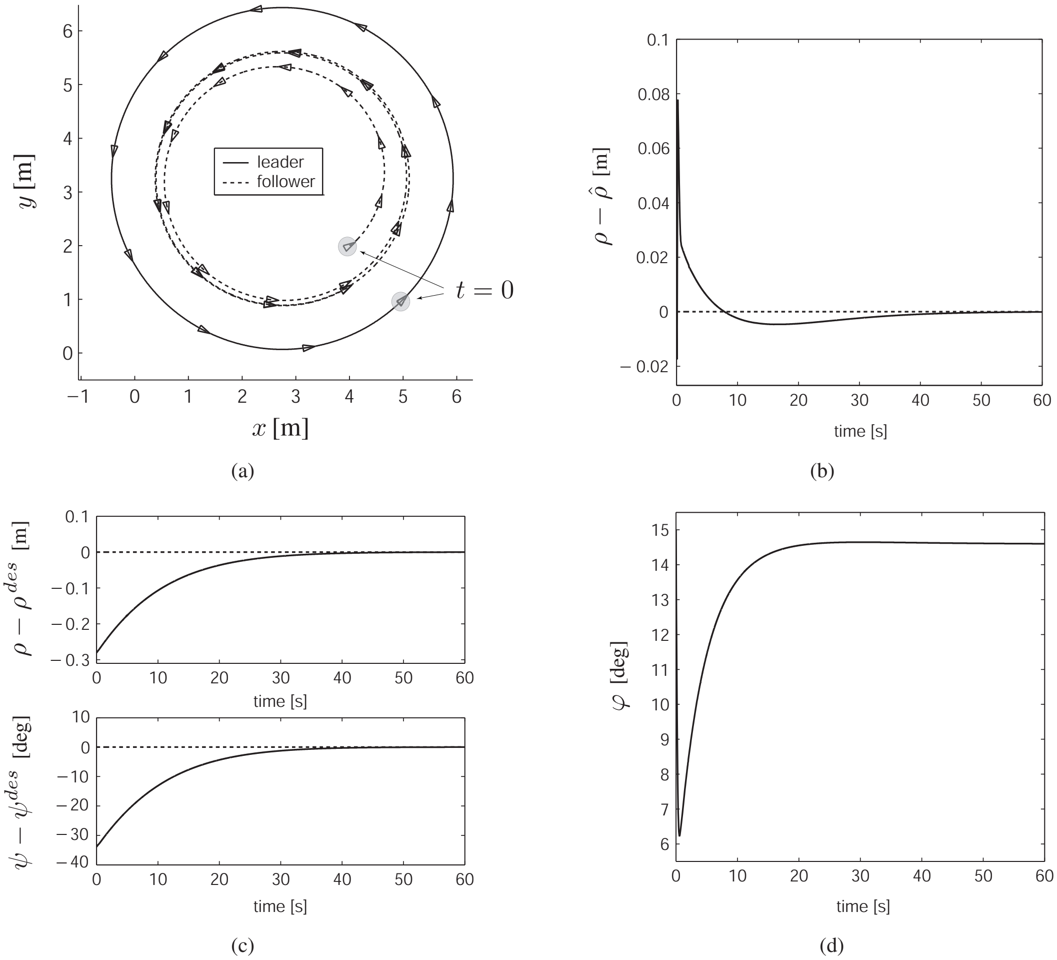

Fig. 2. (a) Trajectory of the leader and the follower; (b) Observation error $\rho - \hat{\rho}$; (c) Control errors $\rho - \rho^{des}$ and $\psi - \psi^{des}$; (d) Bearing angle φ.

which can be solved for $\beta(\cdot)$ yielding

$$\beta(y, \xi, t) = -M\ell(y, t) + \tau(\xi, t),$$

where $\tau(\cdot)$ is a free function. Selecting $\tau(\xi, t) = \xi$ ensures that assumption (A1) is satisfied. Substituting the above expression into (24) yields the equation $\dot{z} = -\kappa(y, t)\, z$ which is globally uniformly exponentially stable, hence assumption (A2) holds. By substituting the expressions of $\varepsilon(\cdot)$ and $\beta(\cdot)$ (with $\tau(\xi, t) = \xi$) in (23), equations (11)-(12) are obtained.

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