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Nonlinear Observability of the Centralized Multi-vehicle SLAM Problem

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***Abstract*—This paper investigates the Centralized Multi-vehicle Simultaneous Localization and Mapping (CMSLAM) problem in the context of the nonlinear observability. Theory is first developed for the nonlinear observability of CMSLAM using the relatively simple unicycle vehicle model, which gives rise to a CMSLAM problem in control affine form. Conditions required for nonlinear observability of CMSLAM when estimating 1. One landmark and 2. More than one landmark are detailed. The theory developed is then extended for more practical car-like vehicle models. CMSLAM simulations and experiments are demonstrated showing the effects of nonlinear observability.**

***Index Terms*—SLAM, observability, multi-vehicle**

I. Introduction

Autonomous vehicle positioning applications [1] often rely on GPS or knowledge of *a priori* known maps. However, constructing maps from scratch for localization is a difficult task. GPS systems on the other hand are affected by signal blockages resulting from interference, multi-path reflections off the ground and/or surrounding structures and partial satellite occlusion. The Simultaneous Localization and Mapping (SLAM) solution [2] has been introduced with the intention of overcoming most of the above problems in autonomous vehicle positioning. Mapping from a large number of vehicles is useful in several applications such as autonomous surveying, surveillance, mining, cargo handling, underwater missions, space exploration and military. Gathering, processing and utilizing maps and localization information from a large number of vehicles is particularly useful where centralized fleet monitoring and control is used. This is typical of applications such as mining, exploration and surveying. In this context the CMSLAM problem ([9] and [10]) or centralized simultaneous localization and mapping using multiple vehicles is considered in this paper. However, CMSLAM is computationally very demanding in maintaining map vehicle correlations [5] and in data association [6] unless the size of the map is efficiently managed. Therefore, understanding the observability and solving computational and memory constraints of the highly nonlinear CMSLAM is beneficial in many aspects. Nonlinear observability theory [3] has been applied for the analysis of the observability of SLAM in [7], [4] and [12] and in multi-vehicle SLAM in [8]. [12] investigates the nonlinear observability of single vehicle localization problem using unicycle vehicle model and bearing only observations with known and unknown landmarks. [7] analyzes the nonlinear local weak observability of one landmark single vehicle SLAM problem using a car-like vehicle model, range bearing sensors and the nonlinear observability theory. [4] extends [7] into single vehicle SLAM with any number of landmarks. [8] in particular analyses the nonlinear observability of 2 vehicle SLAM using unicycle vehicle models. However, neither of these works generalizes to estimating any number of unknown landmarks and vehicles or discusses any other important properties and implications of the nonlinear observability of SLAM and multi-vehicle SLAM in detail.

In this work we develop theory on the nonlinear observability [3] properties of CMSLAM involving any number of vehicles and landmarks. It is interesting to know that nonlinear observability properties of CMSLAM are different depending on the number of estimated landmarks in the map is equal to one or grater than one. The CMSLAM problem, in contrast to many independent single vehicle SLAM problems, represents dependencies among all vehicles and landmarks consistently and allows inclusion of observations of vehicles by other vehicles also into the problem. It is also argued that there is no requirement for maintaining a very large number of landmarks and correlations among them if the CMSLAM problem satisfies full nonlinear observability conditions. Therefore, it is suggested that estimated landmarks which violate the conditions necessary for the full nonlinear observability of CMSLAM be removed from the estimation algorithm and stored for future use

The paper is organized as follows. Section II introduces the CMSLAM problem. Section III describes the theory of nonlinear observability. Section IV provides rigorous proofs on the nonlinear observability properties of the CMSLAM problem using unicycle vehicle models. Section V then extends the results of Section IV to the use of the bicycle model. Section VI provides simulations and experiments to substantiate the theoretical results claimed. Section VII discusses the results and concludes the work.

II. Centralized Multi-vehicle SLAM Problem

A. Centralized Multi-vehicle SLAM Problem

Let there be n vehicles moving on a 2D flat surface estimating their poses and the location states of m landmarks. Let

$$\mathbf{x}_{v,i}(t) = \begin{bmatrix} x_{v,i}(t) & y_{v,i}(t) & \theta_{v,i}(t) \end{bmatrix}^T \tag{1}$$

where $x_{v,i}(t)$, $y_{v,i}(t)$ and $\theta_{v,i}(t)$ are the longitudinal and lateral coordinates and the heading of the i^{th} vehicle. $\mathbf{x}_{v,i}(t)$ is the pose state of the i^{th} vehicle. The map state $\mathbf{m}(t)$ is;

$$\mathbf{m}(t) = \begin{bmatrix} x_1(t) & y_1(t) & \dots & x_m(t) & y_m(t) \end{bmatrix}^T \tag{2}$$

where $x_i(t)$ and $y_i(t)$ are the longitudinal and lateral coordinates of the i^{th} estimated landmark. The combined

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map vehicle state $\mathbf{x}_{n,m}(t)$ and the process model without noise terms of the CMSLAM are given by;

$$\mathbf{x}_{n,m}(t) = \left[\mathbf{x}_{v,1}^T(t) \quad \mathbf{x}_{v,2}^T(t) \quad .... \quad \mathbf{x}_{v,n}^T(t) \quad \mathbf{m}^T(t) \right]^T \quad (3)$$

$$\dot{\mathbf{x}}_{n,m}(t) = \mathbf{f}(\mathbf{x}_{n,m}(t),\ \mathbf{u}) \quad (4)$$

where $\mathbf{u}$ is the control inputs to all the vehicles and $\mathbf{f}$ is the nonlinear process model. The map model is;

$$\dot{\mathbf{m}}(t) = 0 \quad (5)$$

The structure and the complexity of $\mathbf{f}(.)$ therefore mostly depend on the vehicle model used in CMSLAM. The measurement model of the CMSLAM problem assuming range and bearing sensors in all vehicles and excluding additive noise for simplicity is given by $\mathbf{z}(t) = \mathbf{h}(\mathbf{x}_{n,m}(t))$;

$$\mathbf{h}(.) = [\mathbf{h}_1^T \quad \mathbf{h}_2^T \quad ... \quad \mathbf{h}_n^T]^T \quad (6)$$

Assuming all the vehicles observe all the estimated landmarks, $\mathbf{h}_i$ for any *i* is given by;

$$\mathbf{h}_i = [\mathbf{h}_{i,1}^T \quad \mathbf{h}_{i,2}^T \quad ... \quad \mathbf{h}_{i,m}^T]^T \quad (7)$$

$$\mathbf{h}_{i,j} = \begin{bmatrix} \sqrt{(x_j(t) - x_{v,i}(t))^2 + (y_j(t) - y_{v,i}(t))^2} \\ \tan^{-1}\{(y_j(t) - y_{v,i}(t)) / (x_j(t) - x_{v,i}(t))\} - \theta_{v,i}(t) \end{bmatrix} \quad (8)$$

where $\mathbf{h}_{i,j}$ is the part of the observation model $\mathbf{h}(t)$ corresponding to i^{th} vehicle observing j^{th} landmark. For simplicity of notation we hereinafter remove the symbol *t* denoting time from the variables and denote the measurement model of *n* vehicle *m* landmark CMSLAM by $\mathbf{h}(n,m)$ and the process model by $\mathbf{f}(n,m)$.

B. Vehicle Models

The vehicle kinematic models such as the unicycle model and the bicycle model [2] are widely used in autonomous vehicle navigation applications. Depending on the requirements to model complex vehicle dynamics even more complex vehicle models can also be utilized. Here, we consider the unicycle and bicycle models, which are more than adequate to describe the CMSLAM problem on a 2D plane. The unicycle model for the i^{th} vehicle is;

$$\dot{\mathbf{x}}_{v,i} = \left[v_i \cos(\theta_{v,i}) \quad v_i \sin(\theta_{v,i}) \quad \omega_i \right]^T \quad (9)$$

where v_i is the speed input and ω_i is the angular velocity input of the heading in the 2D plane of the i^{th} vehicle. In the unicycle model ω_i can be independently set from v_i. In the simple car-like (bicycle) model ω_i is dependent on v_i. i.e.

$$\omega_i = v_i \tan(\gamma_i) / W_i \quad (10)$$

where γ_i is the steering angle input and W_i is the wheel base of the i^{th} vehicle.

III. THEORY OF NONLINEAR OBSERVABILITY

A. Nonlinear Observability of Systems

The SLAM problem and in particular CMSLAM problem are highly nonlinear. As a result the linear techniques of observability analysis are not appropriate. In the case of highly nonlinear systems, linear observability theory can only be applied under certain assumptions and linearization of process and measurement models. The main disadvantage of the linear observability analysis is it does not take the effects of inputs into consideration. The linear observability analysis tests the observability of a system over a finite amount of time steps and therefore is a global phenomenon. The nonlinear observability theory according to [3] on the other hand is more of a local phenomenon which is much more suitable for the analysis of highly nonlinear systems. The basic theory of nonlinear observability according to [3] can be given in a nutshell as follows. Let $\mathbf{\Sigma}$ be a nonlinear state estimation problem defined by

$$\mathbf{\Sigma}\{\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\ \mathbf{u}) \text{ and } \mathbf{z} = \mathbf{h}(\mathbf{x}) \quad (11)$$

where $\mathbf{x}$ is the state vector estimated, $\mathbf{u}$ is the control input, $\mathbf{z}$ is the measurement vector and $\mathbf{f}(.)$ and $\mathbf{h}(.)$ are nonlinear functions designating the process and the measurement models respectively.

Theorem 1: Let $\mathbf{S}_O = \mathbf{d}L_{\mathbf{f}}(.....(L_{\mathbf{f}}(h_i)).....)$ denote a linear space comprising finite linear combinations of the gradients of the Lie derivatives of zero and higher orders of the components of the measurement model denoted by h_i. Here, $\mathbf{h}(.) = \left[h_1 \quad h_2 \quad .... \quad h_m \right]$, $\mathbf{d}$ is the gradient operator with respect to $\mathbf{x}$ and $L_{\mathbf{f}}^k(h_i)$ is the Lie derivative of order *k* of h_i with respect to $\mathbf{f}(.)$. $\mathbf{\Sigma}$ is locally weakly observable at $\mathbf{x} = \mathbf{x}^*$ if $\mathbf{S}_O(\mathbf{x}^*)$ satisfies the observability rank condition at $\mathbf{x} = \mathbf{x}^*$, where $\mathbf{S}_O(\mathbf{x}^*)$ denotes the $\mathbf{S}_O$ evaluated at $\mathbf{x} = \mathbf{x}^*$ and the observability rank condition is satisfied if the dimension of $\mathbf{S}_O(\mathbf{x}^*)$ is equal to *n* (the dimension of the state vector $\mathbf{x}$).

B. Nonlinear Observability of Control Affine Systems

For a special class of nonlinear problems [11] has shown the following interesting result (Theorem 2).

Theorem 2: If $\mathbf{\Sigma}$ is in control affine form $\mathbf{f}(\mathbf{x},\ \mathbf{u}) = \mathbf{g}^0(\mathbf{x}) + \sum \mathbf{g}^i(\mathbf{x}) u_i$ where $\mathbf{x}$ is a vector of *n* state variables occupying an open subset Ξ of $\mathbb{R}^n$, $\mathbf{g}^0(.)$ **,....,** $\mathbf{g}^i(.)$ are *n* dimensional vector analytic functions in Ξ, the measurement function $\mathbf{h}(.)$ is an analytic function of $\mathbb{R}^m$ and $\mathbf{u}$ is an analytic function of time comprising distinct scalar controls u_i, then $\mathbf{\Sigma}$ is locally weakly observable if the matrix $\mathbf{O}_\Sigma$ (hereinafter referred to as the nonlinear observability matrix) given below has rank *n* (i.e satisfies nonlinear observability rank condition).

$$\mathbf{O}_\Sigma = \left[(\mathbf{d}L_{\mathbf{f}}^0\mathbf{h})^T \quad (\mathbf{d}L_{\mathbf{f}}^1\mathbf{h})^T \quad ... \quad (\mathbf{d}L_{\mathbf{f}}^{n-1}\mathbf{h})^T \right]^T \quad (12)$$

Conversely for systems that are control affine, considering Lie derivatives of order zero to *n*-1 in $\mathbf{O}_\Sigma$ is adequate to determine the nonlinear observability.

IV. NONLINEAR OBSERVABILITY OF THE CMSLAM PROBLEM

A. Nonlinear Observability Matrix of the CMSLAM

When the motions of vehicles are modeled using a unicycle kinematic model, the CMSLAM problem can be written in the following control affine form.

$$\dot{\mathbf{x}}_{n,m} = \sum_i \left(\mathbf{g}_i^1(\mathbf{x}_{n,m}) v_i + \mathbf{g}_i^2(\mathbf{x}_{n,m}) \omega_i \right) \quad (13)$$

$$\mathbf{g}_i^1(\mathbf{x}_{n,m}) = \left[\mathbf{0}_{i,1}^T \quad \cos(\theta_{v,i}) \quad \sin(\theta_{v,i}) \quad \mathbf{0}_{i,2}^T \right]^T \quad (14)$$

$$\mathbf{g}_i^2(\mathbf{x}_{n,m}) = \left[\mathbf{0}_{i,3}^T \quad 1 \quad \mathbf{0}_{i,4}^T \right]^T \quad (15)$$

where $\mathbf{0}_{i,1}$, $\mathbf{0}_{i,2}$, $\mathbf{0}_{i,3}$, and $\mathbf{0}_{i,4}$ are null vectors with appropriate dimensions. $\mathbf{0}_{i,1}$ does not exist for *i*=1. Hence we can apply the result of Theorem 2 for the nonlinear observability analysis of the CMSLAM problem resulting in the following nonlinear observability matrix $\mathbf{O}_{n,m}$.

$$\mathbf{O}_{n,m}(\mathbf{d}_{n,m},\ \mathbf{f}(n,m),\ \mathbf{h}(n,m)) = \begin{bmatrix} \mathbf{d}_{n,m} L^0_{\mathbf{f}(n,m)} \mathbf{h}(n,m) \\ \mathbf{d}_{n,m} L^1_{\mathbf{f}(n,m)} \mathbf{h}(n,m) \\ \cdots\cdots\cdots\cdots \\ \mathbf{d}_{n,m} L^{3n+2m-1}_{\mathbf{f}(n,m)} \mathbf{h}(n,m) \end{bmatrix} \quad (16)$$

where $\mathbf{d}_{n,m}$ is the gradient operator with respect to $\mathbf{x}_{n,m}$ and $L^i_{\mathbf{f}(n,m)}$ denotes the Lie derivative of order *i* with respect to $\mathbf{f}(n,m)$. By definition for any positive integers *i* and *j*;

$$L^0_{\mathbf{f}(n,m)} \mathbf{h}_{i,j} = \mathbf{h}_{i,j} = \begin{bmatrix} h_{i,j,1,0}((x_j - x_{v,i}),(y_j - y_{v,i})) \\ h_{i,j,2,0}((x_j - x_{v,i}),(y_j - y_{v,i}),\theta_{v,i}) \end{bmatrix} \quad (17)$$

where $h_{i,j,1,k}$ and $h_{i,j,2,k}$ denote the k^{th} order Lie derivatives in functional form of the range measurement and the bearing measurement of the j^{th} landmark respectively observed by the i^{th} vehicle. Thus, $\mathbf{d}_{n,m} L^0_{\mathbf{f}(n,m)} \mathbf{h}_{i,j}$ has the structure below.

$$\mathbf{d}_{n,m} L^0_{\mathbf{f}(n,m)} \mathbf{h}_{i,j} = \left[\begin{array}{ccccc|cccc} \mathbf{0}_1 & h^{1,0}_{i,j} & h^{2,0}_{i,j} & h^{3,0}_{i,j} & \mathbf{0}_2 & \mathbf{0}_3 & h^{4,0}_{i,j} & h^{5,0}_{i,j} & \mathbf{0}_4 \\ \mathbf{0}_5 & h^{6,0}_{i,j} & h^{7,0}_{i,j} & h^{8,0}_{i,j} & \mathbf{0}_6 & \mathbf{0}_7 & h^{9,0}_{i,j} & h^{10,0}_{i,j} & \mathbf{0}_8 \end{array} \right] \quad (18)$$

where $\mathbf{0}_1 = \mathbf{0}_5 = \mathbf{0}_{1\times 3(i-1)}$ for $\forall\ i > 1$ and $\mathbf{0}_1$ and $\mathbf{0}_5$ do not exist for $i = 1$. $\mathbf{0}_2 = \mathbf{0}_6 = \mathbf{0}_{1\times 3(n-i)}$ for $\forall$ *i*<n and $\mathbf{0}_2$ and $\mathbf{0}_6$ do not exist for $i = n$. $\mathbf{0}_3 = \mathbf{0}_7 = \mathbf{0}_{1\times 2(j-1)}$ for $\forall\ j > 1$ and $\mathbf{0}_3$ and $\mathbf{0}_7$ do not exist for $j = 1$. $\mathbf{0}_4 = \mathbf{0}_8 = \mathbf{0}_{1\times 2(m-j)}$ for $\forall$ *j*<*m* and $\mathbf{0}_4$ and $\mathbf{0}_8$ do not exist for $j = m$. *l* which is an integer from 1 to 10 represents the position of the element $h^{l,k}_{i,j}$ in $\mathbf{d}_{n,m} L^k_{\mathbf{f}(n,m)} \mathbf{h}_{i,j}$ as shown in (18) and *k* is the order of the Lie derivative. Hence it follows that;

$$h^{4,0}_{i,j} = -h^{1,0}_{i,j} = -h^{1,0}_{i,j}((x_j - x_{v,i}),(y_j - y_{v,i})) \quad (19)$$

By similar reasoning and simplification it follows that;

$$h^{9,0}_{i,j} = -h^{6,0}_{i,j} = -h^{6,0}_{i,j}((x_j - x_{v,i}),(y_j - y_{v,i})) \quad (20)$$

$$h^{5,0}_{i,j} = -h^{2,0}_{i,j} = -h^{2,0}_{i,j}((x_j - x_{v,i}),(y_j - y_{v,i})) \quad (21)$$

$$h^{10,0}_{i,j} = -h^{7,0}_{i,j} = -h^{7,0}_{i,j}((x_j - x_{v,i}),(y_j - y_{v,i})) \quad (22)$$

$$h^{3,0}_{i,j} = 0 \quad (23)$$

$$h^{8,0}_{i,j} = -1 \quad (24)$$

Thus, when *q* is a positive integer, by recursion it follows that;

$$L^q_{\mathbf{f}(n,m)} \mathbf{h}_{i,j} = (\mathbf{d}_{n,m} L^{q-1}_{\mathbf{f}(n,m)} \mathbf{h}_{i,j}).\mathbf{f}(n,m) \quad (25)$$

$$L^q_{\mathbf{f}(n,m)} \mathbf{h}_{i,j} = \begin{bmatrix} h_{i,j,1,q}((x_j - x_{v,i}),(y_j - y_{v,i}),\theta_{v,i}) \\ h_{i,j,2,q}((x_j - x_{v,i}),(y_j - y_{v,i}),\theta_{v,i}) \end{bmatrix} \quad (26)$$

Hence, the gradient $\mathbf{d}_{n,m} L^q_{\mathbf{f}(n,m)} \mathbf{h}_{i,j}$ of (26) is;

$$\mathbf{d}_{n,m} L^q_{\mathbf{f}(n,m)} \mathbf{h}_{i,j} = \left[\begin{array}{ccccc|cccc} \mathbf{0}_1 & h^{1,q}_{i,j} & h^{2,q}_{i,j} & h^{3,q}_{i,j} & \mathbf{0}_2 & \mathbf{0}_3 & h^{4,q}_{i,j} & h^{5,q}_{i,j} & \mathbf{0}_4 \\ \mathbf{0}_5 & h^{6,q}_{i,j} & h^{7,q}_{i,j} & h^{8,q}_{i,j} & \mathbf{0}_6 & \mathbf{0}_7 & h^{9,q}_{i,j} & h^{10,q}_{i,j} & \mathbf{0}_8 \end{array} \right] \quad (27)$$

Hence it follows that;

$$h^{4,q}_{i,j} = -h^{1,q}_{i,j} = -h^{1,q}_{i,j}((x_j - x_{v,i}),(y_j - y_{v,i}),\theta_{v,i}) \quad (28)$$

$$h^{9,q}_{i,j} = -h^{6,q}_{i,j} = -h^{6,q}_{i,j}((x_j - x_{v,i}),(y_j - y_{v,i}),\theta_{v,i}) \quad (29)$$

$$h^{5,q}_{i,j} = -h^{2,q}_{i,j} = -h^{2,q}_{i,j}((x_j - x_{v,i}),(y_j - y_{v,i}),\theta_{v,i}) \quad (30)$$

$$h^{10,q}_{i,j} = -h^{7,q}_{i,j} = -h^{7,q}_{i,j}((x_j - x_{v,i}),(y_j - y_{v,i}),\theta_{v,i}) \quad (31)$$

Let the i^{th} column of $\mathbf{O}_{n,m}$ comprising order zero to $3n+2m-1$ Lie derivatives of all the observations be denoted by $\mathbf{C}^i(n,m)$.

B. Nonlinear Observability Matrix for $n \geq 2$ *and* $m > 1$

Result 1: Nonlinear observability matrix of the SLAM problem estimating $n \geq 2$ ***vehicle poses and*** $m > 1$ ***landmarks is rank deficient by 3 when every vehicle observes only the estimated landmarks.***

Proof: The nonlinear observability matrix for CMSLAM estimating 2 landmarks from 2 vehicles assuming both vehicles observe both landmarks is;

$$\mathbf{O}_{2,2} = \mathbf{O}_{2,2}(\mathbf{d}_{2,2},\ \mathbf{f}(2,2),\ \mathbf{h}(2,2)) \quad (32)$$

By row reduction and simplification of (32) it can be shown that the rank of $\mathbf{O}_{2,2}$ is 7. $\mathbf{O}_{2,2}$ is therefore rank deficient by 3. Hence Result 1 is true for *n*=2 and *m*=2. Consider $\mathbf{O}_{\lambda,\mu}$ where λ and μ are positive integers both greater than 2. Assume that Result 1 is true for $n = \lambda$ and $m = \mu$ or in other words the nonlinear observability matrix resulting from Theorem 2 is rank deficient by 3. From (16);

$$\mathbf{O}_{\lambda,\mu} = \mathbf{O}_{\lambda,\mu}(\mathbf{d}_{\lambda,\mu},\ \mathbf{f}(\lambda,\mu),\ \mathbf{h}(\lambda,\mu)) \quad (33)$$

Using the properties of the Jacobians of Lie derivatives of $\mathbf{d}_{n,m} L^q_{\mathbf{f}(n,m)} \mathbf{h}_{i,j}$ detailed in (16) to (31) and by reordering rows of $\mathbf{O}_{\lambda,\mu}$ we can also express (33) by (34)-(36).

$$\mathbf{O}_{\lambda,\mu} = \left[\begin{array}{cccc|c} \mathbf{H}^v(1) & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{H}^M(1) \\ \mathbf{0} & \mathbf{H}^v(2) & \cdots & \mathbf{0} & \mathbf{H}^M(2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{H}^v(\lambda) & \mathbf{H}^M(\lambda) \end{array} \right] \quad (34)$$

$$\mathbf{H}^M(i) = \begin{bmatrix} \mathbf{H}^M_{i,1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}^M_{i,2} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{H}^M_{i,\mu} \end{bmatrix} \quad (35)$$

$$\mathbf{H}^v(i) = \left[(\mathbf{H}^v_{i,1})^T \quad (\mathbf{H}^v_{i,2})^T \quad .. \quad (\mathbf{H}^v_{i,\mu})^T \right]^T \quad (36)$$

Here $\mathbf{H}^v(i)$ is the part of the three columns of $\mathbf{O}_{\lambda,\mu}$ corresponding to the i^{th} vehicle and associated with observing all the μ landmarks. $\mathbf{H}^M(i)$ is the part of the

2μ columns of $\mathbf{O}_{\lambda,\mu}$ corresponding to the μ landmark positions of the state vector associated with the i^{th} vehicle. $\mathbf{H}^M_{i,j}$ is the part of the two columns of $\mathbf{O}_{\lambda,\mu}$ corresponding to the j^{th} landmark position and associated with the i^{th} vehicle. $\mathbf{H}^v_{i,j}$ is the part of the three columns of $\mathbf{O}_{\lambda,\mu}$ corresponding to the i^{th} vehicle and associated with the j^{th} landmark. Consider now the addition of one vehicle state to the SLAM state vector. The resulting nonlinear observability matrix is;

$$\mathbf{O}_{\lambda+1,\mu} = \mathbf{O}_{\lambda+1,\mu}(\mathbf{d}_{\lambda+1,\mu},\ \mathbf{f}(\lambda+1,\mu),\ \mathbf{h}(\lambda+1,\mu)) \tag{37}$$

By definition

$$L^0_{\mathbf{f}(\lambda+1,\mu)}\mathbf{h}(\lambda+1,\mu) = \left[(\mathbf{h}(\lambda,\mu))^T \quad (\mathbf{h}_{\lambda+1})^T \right]^T \tag{38}$$

$$L^1_{\mathbf{f}(\lambda+1,\mu)}\mathbf{h}(\lambda+1,\mu) = \begin{bmatrix} L^1_{\mathbf{f}(\lambda,\mu)}\mathbf{h}(\lambda,\mu) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \multicolumn{4}{c}{L^1_{\mathbf{f}(\lambda+1,\mu)}\mathbf{h}_{\lambda+1}} \end{bmatrix} \tag{39}$$

$$\mathbf{d}_{\lambda+1,\mu}L^1_{\mathbf{f}(\lambda+1,\mu)}\mathbf{h}(\lambda+1,\mu) = \begin{bmatrix} \mathbf{d}_{\lambda,\mu}L^1_{\mathbf{f}(\lambda,\mu)}\mathbf{h}(\lambda,\mu) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \multicolumn{4}{c}{\mathbf{d}_{\lambda+1,\mu}L^1_{\mathbf{f}(\lambda+1,\mu)}\mathbf{h}_{\lambda+1}} \end{bmatrix} \tag{40}$$

Therefore, by recursion it can be shown that;

$$\mathbf{d}_{\lambda+1,\mu}L^l_{\mathbf{f}(\lambda+1,\mu)}\mathbf{h}(\lambda+1,\mu) = \begin{bmatrix} \mathbf{d}_{\lambda,\mu}L^l_{\mathbf{f}(\lambda,\mu)}\mathbf{h}(\lambda,\mu) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \multicolumn{4}{c}{\mathbf{d}_{\lambda+1,\mu}L^l_{\mathbf{f}(\lambda+1,\mu)}\mathbf{h}_{\lambda+1}} \end{bmatrix} \tag{41}$$

for any positive integer *l*. Thus, from (41)

$$\mathbf{O}_{\lambda+1,\mu} = \begin{bmatrix} \mathbf{O}_{\lambda,\mu} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{d}_{\lambda,\mu}L^{3\lambda+2\mu}_{\mathbf{f}(\lambda,\mu)}\mathbf{h}(\lambda,\mu) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{d}_{\lambda,\mu}L^{3\lambda+2\mu+1}_{\mathbf{f}(\lambda,\mu)}\mathbf{h}(\lambda,\mu) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{d}_{\lambda,\mu}L^{3\lambda+2\mu+2}_{\mathbf{f}(\lambda,\mu)}\mathbf{h}(\lambda,\mu) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \multicolumn{4}{c}{\mathbf{d}_{\lambda+1,\mu}L^0_{\mathbf{f}(\lambda+1,\mu)}\mathbf{h}_{\lambda+1}} \\ \multicolumn{4}{c}{\cdots\cdots} \\ \multicolumn{4}{c}{\mathbf{d}_{\lambda+1,\mu}L^{3\lambda+2\mu+2}_{\mathbf{f}(\lambda+1,\mu)}\mathbf{h}_{\lambda+1}} \end{bmatrix} \tag{42}$$

Using the notation in (34)-(36) $\mathbf{O}_{\lambda+1,\mu}$ can also be expressed as follows.

$$\mathbf{O}_{\lambda+1,\mu} = \left[\begin{array}{ccccc|c} \mathbf{H}^v(1) & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{H}^M(1) \\ \mathbf{0} & \mathbf{H}^v(2) & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{H}^M(2) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{H}^v(\lambda) & \mathbf{0} & \mathbf{H}^M(\lambda) \\ \hline \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{H}^v(\lambda+1) & \mathbf{H}^M(\lambda+1) \end{array}\right] \tag{43}$$

$\mathbf{O}_{\lambda,\mu}$ has three null columns according to the assumption on its rank. Any such null column of $\mathbf{O}_{\lambda,\mu}$ is;

$$\sum_{i=1}^{\lambda}\sum_{k=0}^{2} T^v_{3i+k-2}\mathbf{C}^{3i+k-2}(\lambda,\mu) + \sum_{j=1}^{\mu}\sum_{k=0}^{1} T^M_{3\lambda+2j+k-1}\mathbf{C}^{3\lambda+2j+k-1}(\lambda,\mu) = \mathbf{0} \tag{44}$$

where T^v_i for $i = 1, 2, ..., 3\lambda$ and $T^M_{3\lambda+j}$ for $j = 1, 2, ..., 2\mu$ are transformations on the columns of $\mathbf{O}_{\lambda,\mu}$. Let $\mathbf{C}^k_{i,j}(\lambda,\mu)$ denote the elements of the k^{th} column of $\mathbf{O}_{\lambda,\mu}$ comprising Lie derivatives of order zero to $3\lambda+2\mu-1$ corresponding to the i^{th} vehicle observing j^{th} landmark. By expanding (44) using (34) for all $i = 1, 2, ..., \lambda$ and $j = 1, 2, ..., \mu$ we have;

$$\sum_{k=0}^{2} T^v_{3i+k-2}\mathbf{C}^{3i+k-2}_{i,j}(\lambda,\mu) + \sum_{k=0}^{1} T^M_{3\lambda+2j+k-1}\mathbf{C}^{3\lambda+2j+k-1}_{i,j}(\lambda,\mu) = \mathbf{0} \tag{45}$$

From the converse of Theorem 2 and assumption of the rank condition of $\mathbf{O}_{\lambda,\mu}$ it can be concluded that terms $\mathbf{O}_{\lambda,\mu}$, $\mathbf{d}_{\lambda,\mu}L^{3\lambda+2\mu}_{\mathbf{f}(\lambda,\mu)}\mathbf{h}(\lambda,\mu)$, $\mathbf{d}_{\lambda,\mu}L^{3\lambda+2\mu+1}_{\mathbf{f}(\lambda,\mu)}\mathbf{h}(\lambda,\mu)$ and $\mathbf{d}_{\lambda,\mu}L^{3\lambda+2\mu+2}_{\mathbf{f}(\lambda,\mu)}\mathbf{h}(\lambda,\mu)$ in (42) when considered as a matrix have only 3 null columns. Hence we can extend (45) to Lie derivatives of order zero to $3\lambda+2\mu+2$. Thus, for all $i = 1, 2, ..., \lambda$ and $j = 1, 2, ..., \mu$ we have;

$$\sum_{k=0}^{2} T^v_{3i+k-2}\mathbf{C}^{3i+k-2}_{i,j}(\lambda+1,\mu) + \sum_{k=0}^{1} T^M_{3\lambda+2j+k-1}\mathbf{C}^{3\lambda+2j+k-1}_{i,j}(\lambda+1,\mu) = \mathbf{0} \tag{46}$$

Here (46) covers all the rows of (43) from $\mathbf{H}^v(1)$ to $\mathbf{H}^v(\lambda)$. Also note that $\mathbf{C}^{3i+k-2}_{i,j}(\lambda+1,\mu)$ for *k*=0, 1 and 2 and $\mathbf{C}^{3\lambda+2j+k-1}_{i,j}(\lambda+1,\mu)$ for *k*=0 and 1 are functions of $x_{v,i}$, $y_{v,i}$, $\theta_{v,i}$, x_j and y_j. Now for $j = 1, 2, ..., \mu$ let

$$\sum_{k=0}^{2} T^v_{3\lambda+k+1}\mathbf{C}^{3\lambda+k+1}_{\lambda+1,j}(\lambda+1,\mu) + \sum_{k=0}^{1} T^M_{3\lambda+2j+k+2}\mathbf{C}^{3\lambda+2j+k+2}_{\lambda+1,j}(\lambda+1,\mu) = \mathbf{n}_j \tag{47}$$

In (47) for all $j = 1, 2, ..., \mu$, $\mathbf{C}^{3\lambda+k+1}_{\lambda+1,j}(\lambda+1,\mu)$ for *k*=0, 1 and 2 and $\mathbf{C}^{3\lambda+2j+k+2}_{\lambda+1,j}(\lambda+1,\mu)$ for *k*=0 and 1 are functions of $x_{v,\lambda+1}$, $y_{v,\lambda+1}$, $\theta_{v,\lambda+1}$, x_j and y_j. It follows from (47) that the structure of $\mathbf{O}_{\lambda+1,\mu}$ is such that rows corresponding to the vehicle $\lambda+1$ can be interchanged with rows corresponding to any vehicle $i = 1, 2, ..., \lambda$. Therefore, it follows that $\mathbf{n}_j = 0$ for all $j = 1, 2, ..., \mu$ in (47). Hence by combining (46) and (47) we have;

$$\sum_{i=1}^{\lambda+1}\sum_{k=0}^{2} T^v_{3i+k-2}\mathbf{C}^{3i+k-2}(\lambda+1,\mu) + \sum_{j=1}^{\mu}\sum_{k=0}^{1} T^M_{3\lambda+2j+k-1}\mathbf{C}^{3\lambda+2j+k-1}(\lambda+1,\mu) = \mathbf{0} \tag{48}$$

The equation (48) results in a null column in $\mathbf{O}_{\lambda+1,\mu}$. Therefore, all three null columns of $\mathbf{O}_{\lambda,\mu}$ result in null columns in $\mathbf{O}_{\lambda+1,\mu}$. However, $\mathbf{O}_{\lambda,\mu}$ has the maximum of three null columns according to the assumption made on its rank. Hence from (43), any column operation done on columns 1 to $3\lambda+2\mu+3$ using all the columns of $\mathbf{O}_{\lambda+1,\mu}$, does not result in more than 3 null columns because it contradicts with (44)-(48) on the rank of the matrix $\mathbf{O}_{\lambda,\mu}$. Thus, if there are more than 3 null columns in $\mathbf{O}_{\lambda+1,\mu}$ they must be obtained only from column operations on the columns $3\lambda+1$, $3\lambda+2$ and $3\lambda+3$ of $\mathbf{O}_{\lambda+1,\mu}$ corresponding to the $\lambda+1^{\text{th}}$ vehicle state. Let the zero order Lie derivatives corresponding to observations of landmark 1 by vehicle $\lambda+1$ be given by the 4x3 matrix $\mathbf{M}^v_{\lambda+1}$.

$$rank\left(\mathbf{M}^v_{\lambda+1}\right) = 3 \tag{49}$$

Therefore, from (49) it follows that $\mathbf{O}_{\lambda+1,\mu}$ cannot have more than 3 null columns. Thus, Result 1 is true for $n = \lambda+1$ and hence for any number of vehicles by the Principle of Mathematical Induction. Consider now the addition of one landmark state to the CMSLAM state vector given by (3). The resulting nonlinear observability matrix is;

$$\mathbf{O}_{\lambda,\mu+1} = \mathbf{O}_{\lambda,\mu+1}(\mathbf{d}_{\lambda,\mu+1},\ \mathbf{f}(\lambda,\mu+1),\ \mathbf{h}(\lambda,\mu+1)) \quad (50)$$

By a similar simplification procedure to that of (38)-(42);

$$\mathbf{O}_{\lambda,\mu+1} = \begin{bmatrix} \mathbf{O}_{\lambda,\mu} & \mathbf{0} & \mathbf{0} \\ \mathbf{d}_{\lambda,\mu} L^{3\lambda+2\mu}_{\mathbf{f}(\lambda,\mu)} \mathbf{h}(\lambda,\mu) & \mathbf{0} & \mathbf{0} \\ \mathbf{d}_{\lambda,\mu} L^{3\lambda+2\mu+1}_{\mathbf{f}(\lambda,\mu)} \mathbf{h}(\lambda,\mu) & \mathbf{0} & \mathbf{0} \\ \mathbf{d}_{\lambda,\mu+1} L^{0}_{\mathbf{f}(\lambda,\mu+1)} \mathbf{h}^{\mu+1} & & \\ \ldots\ldots\ldots\ldots\ldots & & \\ \mathbf{d}_{\lambda,\mu+1} L^{3\lambda+2\mu+1}_{\mathbf{f}(\lambda,\mu+1)} \mathbf{h}^{\mu+1} & & \end{bmatrix} \quad (51)$$

Hence using the notation of (50)-(52) $\mathbf{O}_{\lambda,\mu+1}$ can also be expressed by (34) with $\mathbf{H}^M(i)$ given by the following;

$$\mathbf{H}^M(i) = \begin{bmatrix} \mathbf{H}^M_{i,1} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}^M_{i,2} & \ldots & \mathbf{0} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{H}^M_{i,\mu+1} \end{bmatrix} \quad (52)$$

Again according to the assumption of the rank of $\mathbf{O}_{\lambda,\mu}$, $\mathbf{O}_{\lambda,\mu}$ has three null columns. Any such null column of $\mathbf{O}_{\lambda,\mu}$ can be expressed by (44) using the same notation. As before by expanding (44) we obtain (45). From the converse of Theorem 2 and the assumption of the rank condition of $\mathbf{O}_{\lambda,\mu}$ it can be concluded that $\mathbf{O}_{\lambda,\mu}$, $\mathbf{d}_{\lambda,\mu} L^{3\lambda+2\mu}_{\mathbf{f}(\lambda,\mu)} \mathbf{h}(\lambda,\mu)$ and $\mathbf{d}_{\lambda,\mu} L^{3\lambda+2\mu+1}_{\mathbf{f}(\lambda,\mu)} \mathbf{h}(\lambda,\mu)$ terms in (51) when considered as a matrix have only 3 null columns. Hence we can extend (45) to Lie derivatives of order zero to $3\lambda+2\mu+1$. Thus, for all $i = 1,\ 2, \ldots, \lambda$ and $j = 1,\ 2, \ldots, \mu$ we have;

$$\sum_{k=0}^{2} T^{v}_{3i+k-2} \mathbf{C}^{3i+k-2}_{i,j}(\lambda,\mu+1) + \sum_{k=0}^{1} T^{M}_{3\lambda+2j+k-1} \mathbf{C}^{3\lambda+2j+k-1}_{i,j}(\lambda,\mu+1) = \mathbf{0} \quad (53)$$

Note that $\mathbf{C}^{3i+k-2}_{i,j}(\lambda,\mu+1)$ for k=0, 1 and 2 and $\mathbf{C}^{3\lambda+2j+k-1}_{i,j}(\lambda,\mu+1)$ for k=0 and 1 are functions of $x_{v,i}$, $y_{v,i}$, $\theta_{v,i}$, x_j and y_j. Let for $i = 1,\ 2, \ldots, \lambda$

$$\sum_{k=0}^{2} T^{v}_{3i+k-2} \mathbf{C}^{3i+k-2}_{i,\mu+1}(\lambda,\mu+1) + \sum_{k=0}^{1} T^{M}_{3\lambda+2\mu+k+1} \mathbf{C}^{3\lambda+2\mu+k+1}_{i,\mu+1}(\lambda,\mu+1) = \mathbf{n}_i \quad (54)$$

The structure of $\mathbf{O}_{\lambda,\mu+1}$ is such that rows corresponding to the landmark $\mu+1$ can be interchanged with rows corresponding to any landmark $j = 1,\ 2, \ldots, \mu$. Hence it follows that $\mathbf{n}_i = \mathbf{0}$ for all $i = 1,\ 2, \ldots, \lambda$ in (54). Hence by combining (53) and (54) we have;

$$\sum_{i=1}^{\lambda}\sum_{k=0}^{2}\{T^{v}_{3i+k-2}\mathbf{C}^{3i+k-2}(\lambda,\mu+1)\} + \sum_{j=1}^{\mu+1}\sum_{k=0}^{1}\{T^{M}_{3\lambda+2j+k-2}\mathbf{C}^{3\lambda+2j+k-1}(\lambda,\mu+1)\} = \mathbf{0} \quad (55)$$

The equation (55) results in a null column in $\mathbf{O}_{\lambda,\mu+1}$. Therefore, all three null columns of $\mathbf{O}_{\lambda,\mu}$ result in null columns in $\mathbf{O}_{\lambda,\mu+1}$. However, $\mathbf{O}_{\lambda,\mu}$ has the maximum of three null columns according to the assumption made on its rank. Hence from (52), any column operation on columns 1 to $3\lambda+2\mu+2$ using all the columns of $\mathbf{O}_{\lambda,\mu+1}$ does not result in more than 3 null columns (because it contradicts with (44), (45), (53), (54), (55) and on the rank of $\mathbf{O}_{\lambda,\mu}$). Thus, if there are more than 3 null columns in $\mathbf{O}_{\lambda,\mu+1}$ they must be obtained only from column operations on the columns $3\lambda+2\mu+1$ and $3\lambda+2\mu+2$ of $\mathbf{O}_{\lambda,\mu+1}$ corresponding to the newly added landmark state. Let the zero order Lie derivatives corresponding to the observing of landmark $\mu+1$ by vehicles 1 and 2 be given by $\mathbf{M}^M_{\mu+1}$. The rank of $\mathbf{M}^M_{\mu+1}$ is 2. Therefore, $\mathbf{O}_{\lambda,\mu+1}$ cannot have more than 3 null columns. Thus, the Result 1 is true for $m = \mu+1$ and hence for any number of landmarks by the Principle of Mathematical Induction. Thus, by the Principle of Mathematical Induction Result 1 is true for any number of landmarks or vehicles in the CMSLAM state vector.

Result 2: When observing two known landmarks and all the estimated landmarks by all the vehicles, the CMSLAM is locally weakly observable.

Proof: Let x^*_j and y^*_j denote the longitudinal and lateral coordinates of the j^{th} known landmark. Suppose a known landmark *j* be observed from vehicle *i*. Let

$$L^0_{\mathbf{f}(n,m)}\mathbf{h}^*_{i,j} = \mathbf{h}^*_{i,j} = \begin{bmatrix} h^*_{i,j,1,0}((x^*_j - x_{v,i}),(y^*_j - y_{v,i})) \\ h^*_{i,j,2,0}((x^*_j - x_{v,i}),(y^*_j - y_{v,i}),\theta_{v,i}) \end{bmatrix} \quad (56)$$

where $\mathbf{h}^*_{i,j}$ is the measurement model when the range and bearing of the known landmark *j* are observed by vehicle *i*, $h^*_{i,j,1,k}$ and $h^*_{i,j,2,k}$ are the k^{th} order Lie derivative of the measurement model corresponding to observing range and bearing of the j^{th} known landmark. Thus, using the same recursive simplification procedure given by (17)-(31) we can obtain the Jacobians of higher order Lie derivatives of order *q* of $\mathbf{h}^*_{i,j}$ as follows.

$$\mathbf{d}_{n,m} L^q_{\mathbf{f}(n,m)}\mathbf{h}^*_{i,j} = \left[\begin{array}{cccccc|c} \mathbf{0}_1 & \bar{h}^{1,q}_{i,j} & \bar{h}^{2,q}_{i,j} & \bar{h}^{3,q}_{i,j} & \mathbf{0}_2 & & \mathbf{0}_3 \\ \mathbf{0}_4 & \bar{h}^{6,q}_{i,j} & \bar{h}^{7,q}_{i,j} & \bar{h}^{8,q}_{i,j} & \mathbf{0}_5 & & \mathbf{0}_6 \end{array}\right] \quad (57)$$

where $\mathbf{0}_1 = \mathbf{0}_4 = \mathbf{0}_{1\times3(i-1)}$ for $\forall\ i > 1$ and $\mathbf{0}_1$ and $\mathbf{0}_4$ do not exist for $i = 1$, $\mathbf{0}_2 = \mathbf{0}_5 = \mathbf{0}_{1\times3(n-i)}$ for $\forall\ i < n$, and $\mathbf{0}_2$ and $\mathbf{0}_5$ do not exist for $i = n$, and $\mathbf{0}_3 = \mathbf{0}_6 = \mathbf{0}_{1\times2m}$. When all the vehicles observe two known landmarks and all the estimated landmarks the observation model is given by (6) where $\mathbf{h}_i$ is given below.

$$\mathbf{h}_i = \begin{bmatrix} \mathbf{h}^T_{i,1} & \mathbf{h}^T_{i,2} & .. & \mathbf{h}^T_{i,m} & (\mathbf{h}^*_{i,1})^T & (\mathbf{h}^*_{i,2})^T \end{bmatrix}^T \quad (58)$$

Let $\bar{\mathbf{C}}^k_{i,j}(n,m)$ denote the elements of the k^{th} column of $\mathbf{O}_{n,m}$ comprising Lie derivatives of order zero to $3n+2m-1$ corresponding to the i^{th} vehicle observing j^{th} known landmark. Inspired by the notation used in (34)-(36) we use a compact notation of $\bar{\mathbf{H}}^M_{i,j}$ for the elements of the columns of $\mathbf{O}_{n,m}$ corresponding to the map and associated with j^{th}

known landmark and i^{th} vehicle. We use $\bar{\mathbf{H}}_{i,j}^{v}$ for the elements of the columns of $\mathbf{O}_{n,m}$ corresponding to i^{th} vehicle and associated with j^{th} known landmark. Hence from (57) it follows that $\bar{\mathbf{H}}_{i,j}^{M} = \mathbf{0}$ for all i and j. From (34) $\mathbf{O}_{n,m}$ is given by $\lambda = n$ and $\mu = m$. Let the L.H.S. of (35) and (36) be $\bar{\mathbf{H}}^{M}(i)$ and $\bar{\mathbf{H}}^{v}(i)$ respectively.

$$\bar{\mathbf{H}}^{M}(i) = \left[(\mathbf{H}^{M}(i))^{T} \quad \mathbf{0} \quad \mathbf{0} \right]^{T} \tag{59}$$

$$\bar{\mathbf{H}}^{v}(i) = \left[(\mathbf{H}_{i,1}^{v})^{T} \quad (\mathbf{H}_{i,2}^{v})^{T} \quad .. \quad (\mathbf{H}_{i,m}^{v})^{T} \quad (\bar{\mathbf{H}}_{i,1}^{v})^{T} \quad (\bar{\mathbf{H}}_{i,2}^{v})^{T} \right]^{T} \tag{60}$$

From the structure of $\mathbf{O}_{n,m}$ it can be observed that $\bar{\mathbf{C}}_{i,1}^{k}(n,m)$ and $\bar{\mathbf{C}}_{i,2}^{k}(n,m)$ are independent from any other vehicle state in the CMSLAM state other than the i^{th} vehicle state. Consider the Assumption (a) given below.

Assumption (a): Assume now that there are transformations T_i^1, T_i^2, and T_i^3 that operate on the three columns of $\mathbf{O}_{n,m}$ corresponding to vehicle i that results in a null column in $\bar{\mathbf{C}}_{i,1}^{k}(n,m)$ and $\bar{\mathbf{C}}_{i,2}^{k}(n,m)$. Solving for T_i^1, T_i^2, and T_i^3 it follows that the assumption (a) is a contradiction. Hence the columns $3i-2$, $3i-1$, and $3i$ of $\mathbf{O}_{n,m}$ are linearly independent. Therefore, $\mathbf{O}_{n,m}$ has $3n$ linearly independent columns at columns corresponding to n vehicles. Similarly, $\mathbf{O}_{n,m}$ has $2m$ null columns corresponding to m landmark positions. Hence, by the structure of $\mathbf{O}_{n,m}$ it follows that $\mathbf{O}_{n,m}$ has $3n+2m$ linearly independent columns. Hence, $\mathbf{O}_{n,m}$ is full rank. Thus, the CMSLAM is locally weakly observable.

C. Nonlinear Observability Matrix for $n \geq 2$ and $m = 1$

Result 3: The n vehicle one landmark CMSLAM problem, in which the estimated landmark is observed by all the vehicles, has a nonlinear observability matrix with n null column vectors such that.

$$(y_1 - y_{v,i})\mathbf{C}^{3i-2}(n,1) + (-x_1 + x_{v,i})\mathbf{C}^{3i-1}(n,1) + \mathbf{C}^{3i}(n,1) = \mathbf{0} \tag{61}$$

where $i = 1,2,...,n.$

Proof: When n=2, it can be shown for i=1 and 2 that;

$$(y_1 - y_{v,i})\mathbf{C}^{3i-2}(2,1) + (-x_1 + x_{v,i})\mathbf{C}^{3i-1}(2,1) + \mathbf{C}^{3i}(2,1) = \mathbf{0} \tag{62}$$

Thus, Result 3 is true for n=2. Assume that Result 3 is true for $n = \lambda$. By expanding (61), for all $i = 1,\ 2,....,\ \lambda$;

$$(y_1 - y_{v,i})\mathbf{C}_{i,1}^{3i-2}(\lambda,1) + (-x_1 + x_{v,i})\mathbf{C}_{i,1}^{3i-1}(\lambda,1) + \mathbf{C}_{i,1}^{3i}(\lambda,1) = \mathbf{0} \tag{63}$$

Consider now the addition of a new vehicle state vector to the CMSLAM problem. By expanding (63) for all $i = 1,\ 2,....,\ \lambda$

$$(y_1 - y_{v,i})\mathbf{C}_{i,1}^{3i-2}(\lambda+1,1) + (-x_1 + x_{v,i})\mathbf{C}_{i,1}^{3i-1}(\lambda+1,1) + \mathbf{C}_{i,1}^{3i}(\lambda+1,1) = \mathbf{0} \tag{64}$$

Also note that $\mathbf{C}_{i,1}^{3i-2}(\lambda+1,1)$, $\mathbf{C}_{i,1}^{3i-1}(\lambda+1,1)$ and $\mathbf{C}_{i,1}^{3i}(\lambda+1,1)$ are functions of $x_{v,i}$, $\mathrm{y}_{v,i}$, $\theta_{v,i}$, x_1 and y_1. Now let

$$(y_1 - y_{v,i})\mathbf{C}_{\lambda+1,1}^{3\lambda+1}(\lambda+1,1) + (-x_1 + x_{v,i})\mathbf{C}_{\lambda+1,1}^{3\lambda+2}(\lambda+1,1) + \mathbf{C}_{\lambda+1,j}^{3\lambda+3}(\lambda+1,1) = \mathbf{n} \tag{65}$$

In (65), $\mathbf{C}_{\lambda+1,1}^{3\lambda+1}(\lambda+1,1)$, $\mathbf{C}_{\lambda+1,1}^{3\lambda+2}(\lambda+1,1)$ and $\mathbf{C}_{\lambda+1,1}^{3\lambda+3}(\lambda+1,1)$ are functions of $x_{v,\lambda+1}$, $\mathrm{y}_{v,\lambda+1}$, $\theta_{v,\lambda+1}$, x_1 and y_1. Thus, the structure of $\mathbf{O}_{\lambda+1,1}$ is such that rows corresponding to the vehicle $\lambda+1$ can be interchanged with the rows corresponding to any vehicle $i = 1,\ 2,....,\ \lambda$. Hence it follows that $\mathbf{n} = \mathbf{0}$. Hence the null vector is true for $n = \lambda + 1$. Therefore, by the Principle of Mathematical Induction Result 3 is true for any number of vehicles.

Result 4: If the n vehicle one landmark CMSLAM problem in which the estimated landmark and one known landmark are observed by all the vehicles has a nonlinear observability matrix with a null column vector, the null vector is a linear combination of all the columns corresponding to n vehicles and the estimated landmark of the nonlinear observability matrix.

Proof: Let the nonlinear observability matrix of the n vehicle 1 landmark CMSLAM be as follows;

$$\mathbf{O}_{n,1} = \begin{bmatrix} \mathbf{H}^{v}(1) & \mathbf{0} & .... & \mathbf{0} & \mathbf{H}^{M}(1) \\ \bar{\mathbf{H}}^{v}(1) & \mathbf{0} & .... & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}^{v}(2) & .... & \mathbf{0} & \mathbf{H}^{M}(2) \\ \mathbf{0} & \bar{\mathbf{H}}^{v}(2) & .... & \mathbf{0} & \mathbf{0} \\ .... & .... & .... & .... & .... \\ \mathbf{0} & \mathbf{0} & .... & \mathbf{H}^{v}(n) & \mathbf{H}^{M}(n) \\ \mathbf{0} & \mathbf{0} & .... & \bar{\mathbf{H}}^{v}(n) & \mathbf{0} \end{bmatrix} \tag{66}$$

where $\bar{\mathbf{H}}^{v}(i)$ for all $i = 1,2,...,n$ corresponds to the vehicle i observing known landmark 1. Let there be column operations T_1, T_2 or T_3 on columns corresponding to vehicle i that results in null elements.

$$\bar{h}_{i,1}^{1,0}T_1 + \bar{h}_{i,1}^{2,0}T_2 + \bar{h}_{i,1}^{3,0}T_3 = 0 \tag{67}$$

$$\bar{h}_{i,1}^{6,0}T_1 + \bar{h}_{i,1}^{7,0}T_2 + \bar{h}_{i,1}^{8,0}T_3 = 0 \tag{68}$$

When any one of T_1, T_2 or T_3 is zero and another is equal to one (67) and (68) result in two inconsistent equations. Hence we can't find operations on two columns that will result in a null column. If we assume now that $T_3 = 1$ we obtain $T_1 = \left(-y_{v,i} + y_1^*\right)$ and $T_2 = \left(x_{v,i} - x_1^*\right)$. However, it follows that

$$(-y_1^* + y_{v,i})h_{i,1}^{1,0} + (x_1^* - x_{v,i})h_{i,1}^{2,0} + h_{i,1}^{3,0} \neq 0 \tag{69}$$

$$(-y_1^* + y_{v,i})h_{i,1}^{6,0} + (x_1^* - x_{v,i})h_{i,1}^{7,0} + h_{i,1}^{8,0} \neq 0 \tag{70}$$

Extending operations to the columns $3n+1$ and $3n+2$ let;

$$(-y_1^* + y_{v,i})h_{i,1}^{1,0} + (x_1^* - x_{v,i})h_{i,1}^{2,0} + h_{i,1}^{3,0} + h_{i,1}^{4,0}T^{3n+1} + h_{i,1}^{5,0}T^{3n+2} = 0 \tag{71}$$

$$(-y_1^* + y_{v,i})h_{i,1}^{6,0} + (x_1^* - x_{v,i})h_{i,1}^{7,0} + h_{i,1}^{8,0} + h_{i,1}^{9,0}T^{3n+1} + h_{i,1}^{10,0}T^{3n+2} = 0 \tag{72}$$

From (71) and (72) $T^{3n+1} = (-y_1^* + y_1)$ and $T^{3n+2} = (x_1^* - x_1)$. Therefore, operations on all five columns are required to create a null column. However, operations on columns of $\mathbf{O}_{n,1}$ corresponding to landmark 1 result in all vehicle states being included in the equations. Hence by the same logic of equations (67)-(72) it follows that a null column in $\mathbf{O}_{n,1}$ is a linear combination of all the columns corresponding to all the vehicles and the landmark.

Result 5: The n vehicle one landmark CMSLAM problem, in which the estimated landmark and one known landmark are observed by all the vehicles, is locally weakly observable when (a) One known landmark distinct from the one which is observed by all the vehicles is observed by

at least one vehicle or (b) At least one vehicle's longitudinal and lateral coordinates are observed

Proof: From Result 4 it follows that operations on all the columns are required to generate a null column when one known landmark is observed by all the vehicles.

When (a) is also true, from Result 2 it follows that one can't have zero elements on rows corresponding to vehicles that observe two known distinct landmarks by any column operation in the nonlinear observability matrix $\mathbf{O}_{n,m}$. Hence $\mathbf{O}_{n,m}$ is full rank. Therefore, CMSLAM is locally weakly observable.

When (b) is also true and since gradients of zero order Lie derivatives at the first two columns corresponding to a vehicle state result in an identity matrix, they cannot be made zero by any column operation on $\mathbf{O}_{n,m}$. Hence $\mathbf{O}_{n,m}$ is full rank. Therefore, CMSLAM is locally weakly observable.

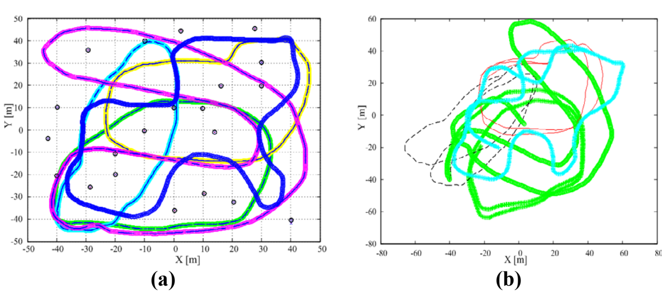
V. USE OF A CAR-LIKE VEHICLE MODEL

When a car-like vehicle model ((9) and (10)) is used, the heading of the vehicle cannot be controlled independently from the speed input. Therefore, the CMSLAM state vector cannot be represented in control affine form. In this context, application of Theorem 2 is not possible. However, it is interesting to note that the proofs of Results 1-5 used the properties of only the zero order Lie derivatives of the measurement models. Hence, the order of the Lie derivatives used in the observability analysis is independent of the proofs of Results 1-5. This means we can have the same proofs if we assume that the number of the Lie derivatives used in the nonlinear observability matrix is independent from the number of vehicles and the number of estimated landmarks.

Thus, it follows from the proofs of Results 1-5 that we can arbitrarily increase the order of the Lie derivatives of the measurement model used in the calculation of the nonlinear observability matrix. Hence the proofs of Results 1-5 satisfy nonlinear observability conditions given by Theorem 1. It is also observed that the subsequent analysis for nonlinear observability of the car-like vehicle models is similar to those with the unicycle model if the substitution $\omega_i = v_i \tan(\gamma_i)/W_i$ is used in the analysis because the composition of ω_i does not affect the nonlinear observability analysis of Section IV. Therefore, Results 1-5 are true even if we use car like vehicle models in CMSLAM.

VI. SIMULATIONS AND EXPERIMENTS

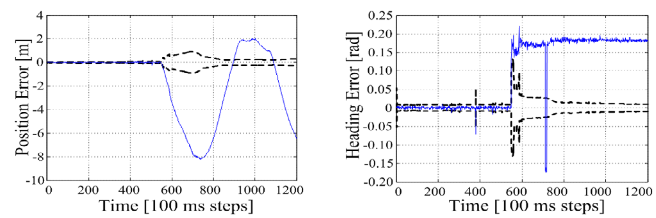
A. Simulations



(a) (b)

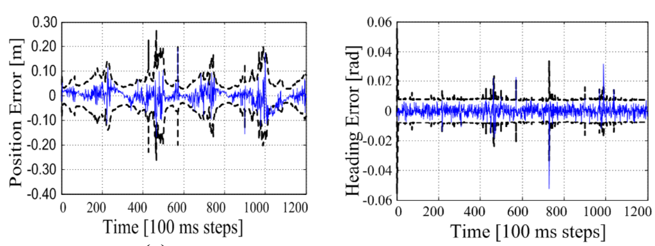
Fig. 1 CMSLAM simulation. Estimated vehicle trajectories are shown by the thick lines, while the dotted lines show the true trajectories. Estimated landmarks are shown by crosses and the true landmarks are shown by the circles. Fig. 1 (a) CMSLAM simulations, Fig 1 (b) Odometry based vehicle paths.

Results of CMSLAM proved in the previous sections are verified by CMSLAM simulations assuming a 2D environment (Fig. 1). It is assumed that five car-like mobile robots are moving in a 2D environment of 100x100 m^2 area according to specified trajectories while observing point landmarks in the environment using range and bearing sensors. An extended Kalman filter based approach to CMSLAM was used to compare the performance of CMSLAM in the context of Results 1-5. A nearest neighbor data association method [2] and a map management method based on the number of times a landmark is observed were also used in the simulation of CMSLAM. It is assumed in the simulations that all the vehicles observe all the estimated landmarks and two a priori known landmarks. The estimated vehicle path and the map are consistent (Fig. 1 (a)) when the nonlinear observability conditions stipulated in the Results are satisfied. Fig. 1 (b) shows the vehicle paths if odometry only is used in vehicle path estimation. Fig. 2 shows how the localization error of a vehicle diverges in CMSLAM when nonlinear observability conditions are not satisfied. The same CMSLAM simulation when repeated with nonlinear observability conditions are satisfied shows consistent vehicle localization errors bounded by the 95% confidence limits as shown in Fig. 3. Furthermore, extensive Monti-Carlo simulations show that CMSLAM (for $n \geq 2$ and $m = 1$ or $m > 1$) is consistent only when nonlinear observability conditions are satisfied. In particular, simulations also show that nonlinear observability conditions enable CMSLAM to be consistent even when initialized with large errors.



(a) (b)

Fig. 2 CMSLAM with nonlinear observability conditions not satisfied. Thin continuous line shows the localization error and the thick dotted line shows the 95% confidence limit of the uncertainty. Fig. 2. (a)- Lateral position error and Fig. 2 (b)- Heading error of a vehicle.



(a) (b)

Fig. 3 CMSLAM with nonlinear observability conditions are satisfied. Thin continuous line shows the localization error and the thick dotted line shows the 95% confidence limit of the uncertainty. Fig. 3. (a)- Lateral position error and Fig. 3 (b)- Heading error of a vehicle.

B. Experiments

Experiments are performed with a part of the Victoria Park dataset of the University of Sydney. The dataset was obtained by driving a utility vehicle equipped with GPS, wheel and steering encoders and a laser range finder. A dataset for the second vehicle was emulated by adding noise deliberately to the measurements of the original data set. The estimated trajectories using the odometry are shown in Fig. 4 (a). CMSLAM is implemented using the data set obtained ensuring at least two a priori known landmarks are visible to the vehicle at all times. Several landmark locations (trees) were estimated using the GPS prior to the CMSLAM experiment to be used as known landmarks. It is noted that the nonlinear observability guaranteed full recovery of estimated state variables consistently and therefore during the experiments all the landmark locations were not estimated throughout. When the landmarks are visible they are estimated. When the landmark visibility is poor (estimated via a normalized measurement innovation as in [2]) they were removed from the state vector and kept for future use. Removal of state variables from the CMSLAM state vector is theoretically sound as long as the full nonlinear observability is maintained. Hence, map management is performed by making sure CMSLAM is locally weakly observable at all times. Fig. 4 (b) shows the estimated map and the vehicle paths of the two vehicles. It can be observed that the estimated vehicle paths and the landmarks are consistent with the true vehicle path and the landmark locations even without maintaining the full map vehicle state correlations all the times when nonlinear observability of CMSLAM is ensured.

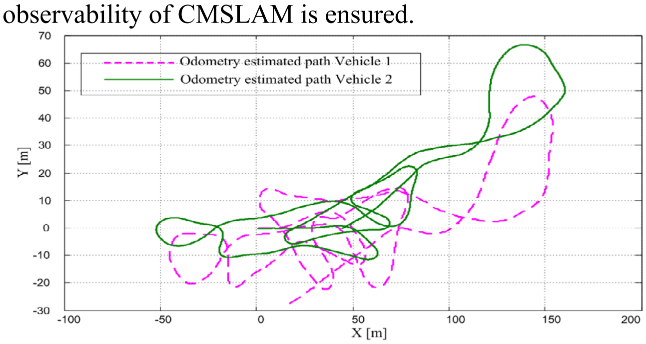


Fig. 4 (a)

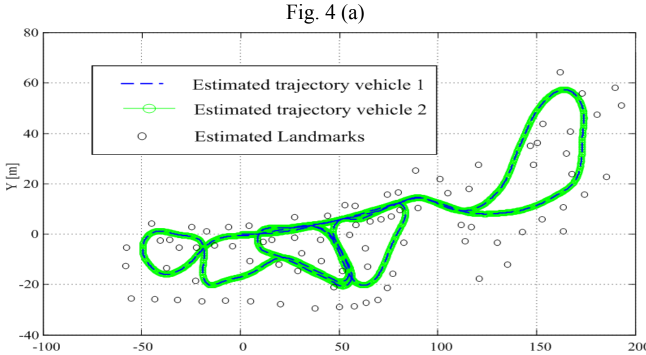


Fig. 4 (b)

Fig. 4 CMSLAM with Victoria Park data set. Fig. 4 (a)-Odometry only vehicle path estimation. Fig. 4 (b)- CMSLAM with 2 vehicles using Victoria Park data set.

VII. Conclusion

The work described in this paper gives a useful insight into the nonlinear observability properties of the CMSLAM problem. The properties are different from the single vehicle SLAM and vary over the number of landmarks estimated. It is shown that all the vehicles must observe all the estimated landmarks and at least two a *priori* known landmarks all the time to maintain local weak observability of the CMSLAM problem when at least 2 unknown landmarks are estimated. It is also shown that the nonlinear observability conditions change when CMSLAM estimates only one unknown landmark.

It is shown that all the vehicles must observe a known landmark and at least one vehicle must observe two distinct known landmarks for the nonlinear observability when CMSLAM estimates only one unknown landmark. The same nonlinear observability conditions are proved to be satisfied when one known landmark is observed by all the vehicles and longitudinal and lateral coordinates of a vehicle position are observed by at least one vehicle thus verifying that the conditions for nonlinear observability of CMSLAM is not equal to those required for *n* copies of independent single vehicle SLAM problems [4].

The properties of the nonlinear observability matrix are vital in understanding the nonlinear observability of CMSLAM in greater detail, providing a greater insight to the nonlinear properties of the CMSLAM problem and for designing efficient nonlinear observers for CMSLAM, localization and mapping. The results established in this paper is therefore invaluable in designing computationally feasible solutions for critical fleet location estimation problems in which vehicle positions are estimated from a central location such as in open pit mining and military deployment applications.

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