Point-to-Point Motion Control of a Pendulum-like 3-Dof Underactuated Cable-Driven Robot

Simon Lefrançois and Clément Gosselin

Abstract—This paper presents a novel planar three-degree-of-freedom pendulum-like underactuated robot. The robot consists of an end-effector with an actuated arm suspended on a cable wound on a reel. The robot can achieve full planar point-to-point motion (position and orientation) with zero-velocity landing by swinging itself as children do on playground swings. The equations of motion of the underactuated cable-driven robot are first developed. Then, the actuated joint trajectory design for swing-up as well as an optimization technique used to control the behaviour of the passive joint are proposed. Finally, a prototype of the robot and its real-time controller are presented with experimental results for point-to-point trajectories. The proposed mechanism constitutes a low-cost solution for applications requiring large workspaces by combining the advantages of cable-driven systems and underactuation and, to the best of our knowledge, this is the first work presenting the real-time control of such a mechanism.

I. INTRODUCTION

Cable-driven robots are well-known solutions for applications requiring large workspaces since cables can be wound on reels, thereby providing large motion ranges. Moreover, replacing rigid links with cables greatly reduces the weight and actuation power. However, cable-driven robots are generally based on parallel architectures with more (or at least as much) actuators than degrees of freedom since cables can only pull and not push. Indeed, implementing such mechanisms requires installing numerous actuators at different locations in space. A support structure as well as some calibration [1] are thereby needed, which drastically increases the implementation costs.

On the other hand, underactuated mechanisms are systems with fewer actuators than degrees of freedom whose control has attracted significant attention. The motion of the free (unactuated) joints is generally related to that of the other joints by complex dynamics, which makes the control problem challenging [2]. Moreover, due to underactuation, only a subset of the kinematically possible global trajectories are achievable. Nevertheless, such mechanisms are very well-suited for point-to-point operations since the latter involve mainly getting from a point to another regardless of the path.

Combining the advantages of underactuation (few actuators, simplicity) with those of cable transmissions (agility, large workspace, low mass) is a promising avenue in order to reduce the cost of large workspace point-to-point operations.

Potential applications include dock loading, construction, field robotics, domotics, surveillance systems, automated greenhouses as well as entertainment and human-robot interactions. The latter is possible since underactuated cable-driven robots are lightweight and cannot lead to constrained motions that are a concern in the context of human/robot cooperation [3].

A first two-degree-of-freedom cable-driven pendulum-like robot, referred to as the Winch-Bot, was presented in [4]. The authors used off-line trajectory planning to control the position of the robot. However, the proposed technique requires specific initial conditions that may be incompatible with the architecture or the current configuration of the robot.

The objective of this paper is to present a simple, fully operational 3-dof planar cable-driven underactuated robot and to provide an effective and robust trajectory planning scheme that allows the real-time control of the robot. The planning and control scheme developed in this paper allows the performance of point-to-point trajectories and leads to a natural behaviour of the robot. The proposed mechanism is a 3-dof planar robot whose first R joint is passive, as illustrated in Fig. 1. It consists of an end-effector of length \( L \), mass \( m_2 \) and inertia \( I_2 \) driven by a motor of mass \( m_1 \) and inertia \( I_1 \). The centre of mass of the end-effector is located at a distance \( d_2 \) from the pivot while the centre of mass of the actuator is located at a distance \( d_1 \) from the same pivot. The actuator is maintained in line with the cable using a

![Fig. 1. The three-degree-of-freedom underactuated cable-driven robot.](image-url)
rigid section at the end of the latter. The angle of the cable
with respect to a vertical axis is noted $\theta_1$ while the angle
between the cable and the end-effector is noted $\theta_2$. Angle $\theta_2$
is associated with the suspended motor while angle $\theta_1$ is an
unactuated coordinate. The end-effector motor is suspended
to a cable of length $\rho$ passing through a pulley and wound
on a reel actuated by a second fixed motor. Therefore, the
length of the cable can be controlled using the latter actuator.
Globally, the mechanism can be thought of as a planar serial
RPR robot whose first R joint is not actuated.

Based on the physical properties of the pendulum system
formed by the robot, a trajectory planning scheme is develop-
ed and a real-time optimization algorithm is used to control
the behaviour of the unactuated joint. The robot can achieve
planar point-to-point motion (position and orientation) with
zero-velocity landing by swinging itself as children do on
playground swings. It uses both techniques pointed out by
Case [5],[6] for effective swinging, namely, i) leg-stretching
as in seated swinging, using end-effector swing and ii) centre
of mass motion as in standing swinging, using cable extension.
Moreover, the proposed control technique does not require
specific initial conditions.

The rest of this paper is structured as follows: first, the
equations of motion of the underactuated cable-driven robot
are developed. Then, the actuated joint trajectory design
for swing-up as well as an optimization technique used to
control the behaviour of the unactuated joint are proposed.
A prototype of the robot and its real-time controller are
also presented with experimental results for point-to-point
trajectories over half-periods of oscillation is appropriate since

\[ (I_1 + I_2 + m_1(\rho + d_1)^2 + m_2(\rho + d_2 \cos \theta_2)^2) \ddot{\theta}_1 \]
\[ -m_2 d_2 \sin \theta_2 \dot{\theta}_2 + (I_2 + m_2(d_2^2 + \rho \dot{d}_2 \cos \theta_2))(\ddot{\theta}_2 + 2(\dot{\theta}_1 + \dot{\theta}_2)) + m_2 g(\rho + d_1) \sin \theta_1 \]
\[ + m_2 g(\rho \sin \theta_1 + d_2 \sin (\theta_1 + \theta_2)) = 0 \tag{3} \]
\[ m_2 d_2 \sin \theta_2 (\dot{\theta}_1 + \dot{\theta}_2) - (m_1 + m_2) \ddot{\rho} \]
\[ + (m_1 (\rho + d_1) + m_2 \rho) \dot{\theta}_1^2 \]
\[ + m_2 d_2 \cos \theta_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + (m_1 + m_2) g \cos \theta_1 = T \tag{4} \]
\[ (I_2 + m_2(d_2^2 + \rho \dot{d}_2 \cos \theta_2))(\ddot{\theta}_1 - m_2 d_2 \sin \theta_2 \ddot{\rho}) \]
\[ + (m_2 d_2^2 + I_2) \ddot{\theta}_2 + 2m_2 d_2 \dot{\rho} \dot{\theta}_1 \cos \theta_2 \]
\[ + m_2 d_2 \dot{\rho}^2 \sin \theta_2 + m_2 d_2 g \sin (\theta_1 + \theta_2) = \tau \tag{5} \]

where $m_i$, $I_i$ and $d_i$ are respectively the mass, inertia and
location of the centre of mass of body $i$, $T$ is the tension
in the cable, $\tau$ is the torque applied by the motor mounted at
the end of the cable and $g$ is the gravitational acceleration.

Equation (3) is the most relevant motion equation of this
system since it does not include any control input as there
is no actuator at the first joint. Equations (4) and (5) will be
used to compute the actuator force and torque and to verify
that the tension is positive in the cable and that the actuator
torque is within its limits.

II. EQUATIONS OF MOTION

In this section, the kinematic and dynamic equations
governing the motion of the underactuated 3-dof cable-driven
robot are obtained.

A. Kinematics

Referring to Fig. 1, the direct kinematics of the robot can
be written as:
\[
x = \rho \cos \theta_1 + L \cos (\theta_1 + \theta_2)\\
y = \rho \sin \theta_1 + L \sin (\theta_1 + \theta_2)\\
\phi = \dot{\theta}_1 + \dot{\theta}_2.
\]
From these equations, the solution of the inverse kinematic
problem is readily obtained as:
\[
\theta_1 = \arctan\left(\frac{y - L \sin \phi}{x - L \sin \phi} \frac{\rho}{\rho}\right)\\
\rho = \sqrt{(x - L \cos \phi)^2 + (y - L \sin \phi)^2}\\
\theta_2 = \phi - \theta_1.
\]

B. Dynamics

In order to simplify the dynamic model, it is first assumed
that the cable is a massless rigid body. This implies that
tension $T$ in the cable is always sufficient to avoid sagging.

Moreover, friction in guides and pulleys as well as aerody-
namic effects are neglected compared to the other forces in
the system. Then, using Lagrangian dynamics, the equations
of motion can be obtained and written as:

From these equations, the solution of the inverse kinematic
problem is readily obtained as:
\[
\theta_1 = \arctan\left(\frac{y - L \sin \phi}{x - L \sin \phi} \frac{\rho}{\rho}\right)\\
\rho = \sqrt{(x - L \cos \phi)^2 + (y - L \sin \phi)^2}\\
\theta_2 = \phi - \theta_1.
\]

III. TRAJECTORY PLANNING

From the dynamic model presented in the above section,
and for given initial conditions, cable angle $\theta_1(t)$ can be
predicted for known trajectories of the actuated joints $\rho(t), \theta_2(t)
by integrating (3). However, for the prediction to be correct,
the prescribed trajectories must also satisfy the following
conditions:
1) Positive tension is maintained
2) Actuator torque and joint limits are satisfied
3) Joint trajectories and their derivatives are smooth, to
prevent shocks
4) Prediction time is small enough so that the computa-
tional burden and the model errors are bounded.

From the latter point, it is clear that planning entire trajecto-
ries from zero initial conditions to large cable angles would
be very difficult. On the other hand, since the motion of the
robot is governed by pendulum-like dynamics, planning tra-
jectories over half-periods of oscillation is appropriate since
the prediction horizon is relatively small and the dynamics
are similar in-between zero-velocity states $\left(\dot{\theta}_1 = 0\right)$.
A. Trajectory of the Unactuated Joint

Defining a goal \([x_g, y_g, \phi_g]\) to be reached with a zero final velocity, joint coordinates \(\theta_{1g}, \rho_g, \theta_{2g}\) are obtained from (2). Since the actuated joint goals are easily reachable, the objective is to find actuated joint trajectories that allow the robot to reach the desired cable angle or, at least, within half a period, a cable angle which is closer to the prescribed goal. These trajectories must also satisfy initial conditions to maintain continuity and final conditions that satisfy the objectives. Since the system is governed by complex dynamics, it is clear that optimal trajectories are not obtainable within acceptable computation time.

Thus, a technique similar to the one presented by Tzortzis and Papadopoulos [7] for underactuated manipulators will be used here. Indeed, trajectories defined using \(p + q\) parameters are used together with \(p\) boundary conditions on each actuated joint thereby leaving \(q\) free parameters. These free parameters can then be tuned through optimization in order to produce the desired underactuated joint behaviour. Here, there are \(p = 6\) boundary conditions (initial/final positions, velocities and accelerations) and, for simplicity reasons, \(q = 1\) free parameter is left for each actuated joint.

Since a zero velocity is desired at the target configuration (for smooth landing), extrema of \(\theta_1\) are considered as starting/ending point of trajectories. This leads to a cosine-like function for \(\theta_1(t)\) on a half-period. Moreover, since a large workspace is desired, large values of \(\theta_1\) must be reachable and the robot must be able to efficiently excite itself to achieve such motions. Special attention must then be given to trajectory planning and this will be addressed for each actuated joint independently.

B. Cable Extension

Considering a lumped end-effector (no actuator mounted at the end of the cable), the system becomes similar to a variable-length (Lorentz) pendulum whose swing-up motion was studied in [8],[9]. It was suggested by Burns [10] and proven optimal for instantaneous variation of length by Piccoli and Kulkarni [11] that lengthening the cable when the angular velocity of the cable is minimum and shortening it when the angular velocity is maximum amplifies the system’s energy.

Hence, given the mathematical form of \(\theta_1\) prescribed above, a sine-like function is chosen for the cable extension. As shown in Fig. (2) —which was obtained through numerical simulation—, exciting the system at twice its natural frequency with a \(\pm \pi/2\) phase from cable angle position is best-suited to increase (or decrease) the cable angle, which is consistent with Burns’ results [10]. Since the cable angle describes a cosine-like function between two zero-velocity states \(\theta_1 = 0\), the following function is chosen for the cable extension:

\[
\rho(t) = A_1 \sin(2\omega t) + B_1 \sin(3\omega t) + C_1 \sin(4\omega t)
\]

\[
+ \rho_i + \frac{(\rho_f - \rho_i) \omega t}{\pi}
\]

for \(0 \leq t \leq \pi/\omega\).

In (6), indices \(i\) and \(f\) refer to initial/final conditions, \(t\) is the time, \(\omega\) is the system’s natural frequency and \(A_1\) is a free parameter which is optimized for each half-period in order to control the behaviour of angle \(\theta_1\). Coefficients \(B_1\) and \(C_1\) are obtained by prescribing the time derivative of (6) at times \(t = 0\) and \(t = \pi/\omega\) to be equal to the initial/final velocities \(\dot{\rho}_i\) and \(\dot{\rho}_f\). This leads to:

\[
B_1 = \frac{\dot{\rho}_i - \dot{\rho}_f}{6\omega}
\]

\[
C_1 = -\frac{A_1}{2} + \frac{\rho_i - \rho_f}{4\pi} + \frac{\dot{\rho}_i + \dot{\rho}_f}{8\omega}.
\]

For stability reasons, the boundary condition on accelerations are chosen to be zero by definition of (6). Initial conditions are measured from encoders while final conditions are defined as follows:

\[
\rho_f = \begin{cases} 
\rho_g & \text{if } |\rho_g - \rho_i| < \Delta \rho_{\text{max}} \\
\rho_i + \Delta \rho_{\text{max}} & \text{else if } \rho_g > \rho_i \\
\rho_i - \Delta \rho_{\text{max}} & \text{else} 
\end{cases}
\]

\[
\dot{\rho}_f = \begin{cases} 
0 & \text{if } |\rho_g - \rho_i| < \Delta \rho_{\text{max}} \\
2\omega A_1 & \text{else}
\end{cases}
\]

where \(\rho_g\) is the target cable length and \(\Delta \rho_{\text{max}}\) is the maximal cable length variation allowed by the maximum velocity of the actuator. Hence, a target cable length is set at first and maintained until the target cable angle \(\theta_{1g}\) is reached. The final velocity is chosen to match the basic sine conditions in order to limit accelerations and is set to zero for smooth landing when the target point is reached.

C. End-Effector Swing

On the other hand, considering a fixed-length cable, the problem becomes similar to a double pendulum with a long first link. The swing-up problem of a double-pendulum
with a passive first joint was widely studied in the context of the Acrobot [12],[13],[14] and Brachiation Robots [15]. Spong [12] suggests that, in order to increase the system energy, the motion of the lower link must be “in-phase” with the upper link. Thus, a sine-like function is chosen for the end-effector swing since the upper link motion is of this form. As shown in Fig. (3) — which was obtained through numerical simulation — exciting the system at its natural frequency is best-suited to increase (or decrease) the cable angle, as pointed out by Spong. A phase of $-3\pi/4$ or $\pi/4$ from the cable angle is also preferable. However, in order to synchronize the goal-reaching with the cable extension and to impose zero boundary acceleration from the outset, a sine function with a phase of $\pm\pi/2$ is chosen since it produces almost the same amplitude of excitation. Therefore, the following function was chosen for the end-effector swing:

$$\theta_2(t) = A_2 \sin(\omega t) + B_2 \sin(2\omega t) + C_2 \sin(3\omega t) + \theta_{2i} + \frac{(\theta_{2f} - \theta_{2i})\omega t}{\pi},$$

for $0 \leq t \leq \pi/\omega$.

Similarly to $A_1$, $A_2$ is a free parameter used to adjust the behaviour of angle $\theta_1$. Also, coefficients $B_2$ and $C_2$ are obtained by prescribing the time derivative of (11) at times $t = 0$ and $t = \pi/\omega$ for initial/final velocities $\dot{\theta}_2 = \dot{\theta}_{2i}$ and $\dot{\theta}_2 = \dot{\theta}_{2f}$. This leads to:

$$B_2 = \frac{\theta_{2i} - \theta_{2f}}{2\pi} + \frac{\dot{\theta}_{2i} + \dot{\theta}_{2f}}{4\pi},$$

$$C_2 = \frac{A_2}{3} + \frac{\dot{\theta}_{2i} - \dot{\theta}_{2f}}{6\omega}.$$  

For stability reasons, the boundary conditions on accelerations are chosen to be zero by definition of (11). Initial conditions are measured from encoders while final conditions are defined as follows:

$$\begin{bmatrix} \theta_{2f} \\ \dot{\theta}_{2f} \end{bmatrix} = \begin{cases} [\theta_{2g}, 0]^T & \text{if } |\rho_g - \rho_i| < \Delta \rho_{\text{max}} \\
\quad \quad \quad \quad \text{and } \theta_{1p} = \theta_{1g} \\
\quad \quad \quad \quad [0, -\omega A_2]^T & \text{else} \end{cases}$$  

Similarly to the cable length, the final velocity is chosen here to match the basic sine conditions in order to limit accelerations and is set to zero for smooth landing when the target point is reach. Since $\theta_{2g}$ is always reachable within a half-period, $\theta_{2f}$ is kept to zero for symmetry reasons until the goal is reached.

IV. OPTIMIZATION OF THE FREE JOINT TRAJECTORY

As pointed out before, one free parameter $A_j$ is used for each actuated joint in order to adjust the behaviour of angle $\theta_1$. However, since the robot dynamics are complex, there is no analytical equation defining $\theta_1(t)$ from the actuated joint trajectories. Indeed, it must be predicted by integrating (3) in real-time.

Fig. 3. Average amplitude gain for $\theta_1$ by half-period for end-effector swing $\omega_n = 3.05(\text{rad/s}), \rho = 1(m), \dot{\theta}_2 = 0(\text{rad}), \Delta \theta_2 = \pi/2(\text{rad})$.

A. Optimization Function

The main objective is to minimize the difference between the cable angle prediction $\theta_{1p}$ and its desired value $\theta_{1g}$ using the free parameters $(A_1, A_2)$ whose values can be tuned without modifying the actuated joints’ final positions.

However, $A_1$ and $A_2$ affect the tension, the torque, the joint maximal positions as well as the velocities, which must be considered in the optimization. Indeed, fulfilling the associated constraints must also be included in the optimization function. This is accomplished here using penalty functions, which are chosen to be combinations of ramps and step-functions as follows:

$$P_j = \begin{cases} 0 & \text{if } \eta_j \leq \eta_{j,\text{max}} \\
K_1 + K_2 (\eta_j - \eta_{j,\text{max}}) & \text{else} \end{cases}$$

where $P_j$ is the penalty function associated to constraint $\eta_j$, $\eta_{j,\text{max}}$ is the maximal allowed value for this constraint and $K_1, K_2$ are positive constants tuned experimentally. Constraints include joint positions, velocities and accelerations as well as cable tension and end-effector torque. Thus, the optimization function is defined as:

$$\min_{A_1, A_2} \Delta, \text{ with } \Delta = (|\theta_{1g}| - |\theta_{1p}|)^2 + \sum_{j=1}^{c} P_j$$

where $c$ is the total number of constraints.

In this equation, the absolute value of the cable angle is used since, from the pendulum-like dynamics, values of $\theta_{1p}$ will vary from positive to negative within each half-period.

B. Optimization Algorithm

For each step of the optimization process, given $(A_1, A_2)$, the optimization function can be computed using results obtained by integrating (3). For solving in real-time, the Nelder-Mead algorithm [16] is used since it requires no derivative which, for our problem, can only be computed
through finite differences. Moreover, this algorithm is fast and convergence is guaranteed for strictly convex functions. The three starting points needed for the algorithm are chosen to be:

\[
(A_1, A_2)_1 = \left( -\rho_{\text{max}} \frac{4\omega}{2}, \frac{\theta_{2\text{max}}}{2} \right)
\]

\[
(A_1, A_2)_2 = \left( -\rho_{\text{max}} \frac{4\omega}{2}, -\frac{\theta_{2\text{max}}}{2} \right)
\]

\[
(A_1, A_2)_3 = \left( \frac{\rho_{\text{max}}}{4\omega}, \frac{\theta_{2\text{max}}}{2} \right).
\]

When included in (6) and (11) for standard initial/final conditions, these values correspond to reaching approximately half of maximum cable velocity and half of maximal end-effector angle which are associated with the most restrictive penalty functions. Thus, the starting solutions are located in the middle of the penalty-free zone which accelerates the algorithm’s convergence.

C. Frequency Determination

As described in section III, the trajectories are defined as functions of the system’s natural frequency. However, the natural frequency is also a function of the cable length \( \rho \). Hence, the natural frequency must be determined while performing optimization for the system to be well-synchronized, i.e., for the goals for each joint to be reached simultaneously.

Since no analytical function is available to define the system’s frequency, it must be computed while integrating equations of motion. Since the half-period on which optimization is performed is defined from zero-velocity states, the frequency is obtained by finding the time of the next zero velocity point \( \dot{\theta}_1 = 0 \). Then, the actuated joint trajectories are defined using the latter frequency and the optimization algorithm is started. Moreover, the frequency is re-computed throughout the algorithm since it is also a slow-varying function of \((A_1, A_2)\).

V. IMPLEMENTATION

A small prototype of the underactuated robot was built as shown in Fig. 4 with the specifications presented in Table I. The latter were obtained either by direct measurements or CAD analysis and the robot was designed using a parallelogram shape to ensure planar motion. DC motors with encoders are used to actuate and measure both the end-effector and the cables which are both wound on a single reel to ensure that they have the same length. The cable angle \( \theta_1 \) is measured using encoders mounted on the robot’s supporting pulleys.

A controller was implemented on a real-time QNX computer with a servo-rate of 500 Hz. Closed-loop PID were used to control the actuated joint positions and a 12-step fixed-step fourth order Runge-Kutta formula [17] was used to integrate the equations of motion. \( M = 4 \) iterations of the Nelder-Mead algorithm were computed for each step and a total of \( N = 6 \) steps (0.012s) were used to define the free-parameters \((A_1, A_2)\). Nevertheless, even through

<table>
<thead>
<tr>
<th>Description</th>
<th>( i = 1 )</th>
<th>( i = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass ( m_i ) (kg)</td>
<td>1.291</td>
<td>0.404</td>
</tr>
<tr>
<td>Inertia ( I_i ) (kgm(^2))</td>
<td>( 4.0 \times 10^{-3} )</td>
<td>( 4.8 \times 10^{-3} )</td>
</tr>
<tr>
<td>Location of CoM ( d_{ij} ) (m)</td>
<td>0.046</td>
<td>0.033</td>
</tr>
<tr>
<td>End-Effector Length ( L ) (m)</td>
<td>0.300</td>
<td>0.300</td>
</tr>
</tbody>
</table>

optimization, the closed-loop PID is operated at 500 Hz. The structure of the controller is shown in Fig. 5.

At each time step \( k \), if the number of steps performed \( n \) is less than maximum number \( N \), an \( M \)-iteration optimization algorithm including the definition of parameters \((A_1, A_2)\), the integration of the equations of motion, the frequency computation and the computation of the optimization function, is started using the previous values of \((A_1, A_2)\) as a
starting point. Then, the desired trajectories are sent to a PID controller driving the actuators. The joint coordinates are measured and the next time step is processed. The number of steps performed is reset to zero and starting points from (17) are used when zero-velocity is reached for the cable angle.

In practice, it was difficult to make predictions for small cable angles since the dynamics are slightly different [18]. Indeed, when $\theta_1$ is small, the system is more subject to cable flexion and vibrations. Therefore, pre-designed excitation trajectories with fixed amplitude and frequency were used for end-effector swing in order to initiate the motion.

### A. Experimental results

A series of end-effector Cartesian coordinates $[x_i, y_i, \phi_i]^T$, for $i = 1..r$ simulating point-to-point trajectories are used to evaluate performances. Points were chosen to include increasing and decreasing amplitudes of $\theta_1$ as well as shortening and lengthening of the cable.

Using the controller structure and the trajectory planning described above, actuated joint trajectories and cable angle predictions were computed in real-time for these objectives. Fig. 6 presents joint trajectories for point-to-point motion, Fig. 7 presents cable angle prediction vs. measurements for the goal region and Fig. 8 presents cable angle prediction errors. The robot motion from which these graphs were obtained can be seen in the accompanying video.

Cable angle predictions for the underactuated robot were generally sufficient for great controllability. Goal-reaching precision was excellent since joint motions were well-synchronized with the frequency estimation for pick-and-place actions. Predictions were slightly less accurate when decreasing the system’s energy since the tension in the cable was then reduced. This causes off-axis rotations which cannot be predicted by our model. Nevertheless, the experimental results were sufficient to confirm the accuracy of the method presented above.

### VI. Future Work

The next step in our project is to perform $x - y$ path-tracking trajectories. Even if the system is limited by underactuation, it is still overdetermined for 2-dof positioning tasks which gives great flexibility for complex operations. It is now planned to develop a method defining path-following trajectories for actuated and unactuated joints by choosing the best possible initial conditions and optimizing path control. Initial conditions can be obtained using the point-to-point controller presented above and path-tracking can be accomplished similarly by combining path points in a global objective.
VII. CONCLUSION

A three-degree-of-freedom planar robot combining the advantages of cable-driven actuation and underactuation was presented. The actuated joint trajectory design for swing-up as well as an optimization technique used to control the behaviour of the free joint were presented. Actuator and joint limitations as well as positive tension were also included in the problem. Finally, a small prototype of the robot and its real-time controller were presented. It was shown through experimentation that the strategy developed was successful at reaching objectives and that precision was sufficient for point-to-point trajectories. Further developments on $x - y$ path-tracking trajectories was also discussed.

REFERENCES