A Discrete-Time Filter for the On-Line Generation of Trajectories with Bounded Velocity, Acceleration, and Jerk

Oscar Gerelli and Corrado Guarino Lo Bianco, IEEE Member

Abstract—The performances of controlled systems can be improved by driving them with smooth reference signals. In case of rough signals, smoothness can be achieved with the help of appropriate dynamic filters. To this purpose a novel discrete-time filter is proposed in the paper. It has been appositely designed for real-time motion applications like those that can be encountered in robotic or mechatronic contexts. The filter generates output signals which are continuous together with their first and second time derivatives. Simultaneously, the first, the second, and the third time derivatives are bounded within freely assignable limits. If such limits are changed on-the-fly, the filter hangs the new bounds in minimum-time. An example case shows the filter while tracking steps, ramps and parabolas by means of bounded-dynamic transients.

I. INTRODUCTION

Controllers for robotic and mechatronic applications are commonly driven by means of trajectory generators whose output signals must typically satisfy several requirements [1]. Smoothness is probably the most important: in order to avoid unnecessary mechanical solicitations and to improve the controller performances, the continuity on the position reference signal and on its first derivative, i.e., the velocity, are commonly required.

Other requirements are taken into account as well. For example, the actuators electromechanical limits are often considered by constraining the maximum velocity or acceleration. Furthermore, when possible, also dynamic bounds are accounted for. Trajectories are typically the outcome of off-line algorithms which do not only fulfill the assigned requirements, but also optimize proper performance indexes. A typical robotic application has been proposed by Lin et al. in [2]: minimum time trajectories were generated by considering bounds on the joint velocities, accelerations and jerks. Other approaches, like those proposed in [3], [4], also consider the existence of dynamic bounds.

Above mentioned planning techniques, due to their computational burden, can only be applied off-line. As a consequence, if constraints change during transients, replanning must be handled by means of alternative approaches that can be roughly divided into two categories: methods based on decision trees, which efficiently evaluate new feasible trajectories depending on the current system state and on the new constraints [5], [6], [7], [8], and methods based on feedback dynamic systems, which produce feasible trajectories by filtering the originally unfeasible inputs [9], [10]. In all the cases, minimum time transients subject to kinematic constraints are obtained.

This paper proposes an approach based on the latter technique: the filter input and output signals coincide when the formers satisfy the given constraints, otherwise the filter determines new feasible signals which at best emulate the original ones. Continuous-time second-order implementations able to generate outputs which are continuous together with their first derivative, and characterized by bounded velocities and accelerations, were originally considered in [9], [10]. Discrete-time versions, showing similar performances, have been later proposed in [11], [12], [13], [14].

A third-order evolution of the continuous-time filter has been proposed in [15]. It is characterized by the same basic properties of its precursors, but also guarantees continuous acceleration and bounded jerk. Subsequently, a third-order discrete-time version has appeared in [16]. It generates continuous position, velocity, and acceleration signals, but only the jerk is bounded: constraints on the maximum velocity and acceleration are not taken into account.

This paper aims to fill this lacuna. Still in a discrete-time framework, a third-order filter with enhanced capabilities is proposed: it possesses the same characteristics of that described in [16], but it can also handle freely assignable bounds on the velocity and the acceleration.

The paper is organized as follows. In §II the optimal trajectory scaling problem is proposed and solved by means of a new discrete-time filter. The convergence properties of the filter are deeply investigated in §III. A test case is proposed in §IV, while final conclusions are reported in §V.

II. THE OPTIMAL TRAJECTORY SCALING PROBLEM AND THE DISCRETE-TIME FILTER

The reference-scaling problem considered in this paper is solved by means of a discrete-time filter. In the following, subscript $i \in \mathbb{Z}$ is used to indicate sampled variables acquired at time $t = iT$, where $T$ is the system sampling time. Let us consider the following problem:

**Problem 1:** Design a nonlinear discrete-time filter whose output $x_i$ tracks at best a given reference signal $r_i$ which is known together with its first and second time derivatives, while $\dot{r}_i = 0$. The filter must fulfill the following requirements:

1) the first, the second, and the third time derivatives of $x_i$ must be bounded:

$$\dot{x}_i \leq \dot{x}_i \leq \dot{x}_i, \quad \ddot{x}_i \leq \ddot{x}_i \leq \ddot{x}_i, \quad -U \leq \dddot{x}_i \leq U,$$

where $\dot{x}_i, \dddot{x}_i, \dddot{x}_i, \dddot{x}_i \in \mathbb{R}$, and $U \in \mathbb{R}^+$. 

This work is supported in part by the AER-TECH LAB, Emilia Romagna, Italy.

The authors are with the Dip. di Ingegneria dell’Informazione, University of Parma, Italy, email: {guarino, gerelli}@ce.unipr.it.

978-1-4244-5040-4/10/$26.00 ©2010 IEEE
2) bounds (1) can be time-varying and can also change during transients;
3) if (1) are not satisfied, due to the filter initial conditions or to a sudden change of the bounds, \( \dot{x}_i \) must be forced in a single step within the given limits, while \( \dot{x}_i \) and \( x_i \) must reach the assigned bounds in minimum time;
4) when a reference signal \( r_i \) satisfying (1) is applied, the tracking condition \( x_i = r_i \) is reached in minimum time and, compatibly with (1), without overshoot;
5) when a discontinuous reference signal is applied (or the reference signal has time derivatives larger than the bound values), the tracking is lost. As soon as the reference signal newly satisfies (1), tracking is achieved in minimum time;
6) the time derivatives \( \dot{x}_i, \ddot{x}_i, \) and \( \dddot{x}_i \) of bounded output \( x_i \) must be available for the generation of feedforward actions.

The solution of Problem 1 represents an interesting challenge: an optimal minimum-time reference tracking problem subject to constraints on the output dynamics. Roughly speaking, given a reference \( r_i \), which could possibly not fulfill bounds (1), the filter must generate a feasible output \( x_i \) which tracks \( r_i \) at best, compatibly with the constraints. This implies that feasibility is a priority for the filter, so that \( r_i \) is voluntarily lost any time it becomes unfeasible. It is worth noticing that constraints on the maximum velocity and acceleration could also be asymmetric.

The solution proposed in the following is based on a discrete time filter whose scheme is shown in Fig. 1. It is made of a chain of three integrators driven by an algebraic control law (ACL). The filter outputs coincide with the filter states \( x_i, \dot{x}_i, \) and \( \ddot{x}_i \).

The integrators’ chain can be posed into a state-space form leading to the following discrete-time system

\[
x_{i+1} = A x_i + b u_i ,
\]

where

\[
A = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & \frac{T}{2} \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} \frac{T^2}{2} \\ \frac{T}{2} \\ T \end{bmatrix},
\]

and \( x_i = [x_i \, \dot{x}_i \, \ddot{x}_i]^T \) is the system state.

The hypothesis \( \dot{r}_i = 0 \) implies that reference signal \( r_i \) can be a constant, a ramp or a parabola. Indeed, due to such hypothesis, \( r_i \) generally evolves like a parabola according to the following expressions

\[
\dot{\dot{r}}_{i+1} := 0, \quad \dddot{r}_{i+1} := \dot{\dot{r}}_i, \quad \dot{\dot{r}}_{i+1} := \dot{\dot{r}}_i + T \dddot{r}_i, \quad r_{i+1} := r_i + T \dot{\dot{r}}_i + \frac{T^2}{2} \dddot{r}_i,
\]

but a linear trend can be achieved by further imposing \( \dddot{r}_i = 0 \) or, finally, a constant signal is obtained if also \( \dot{\dot{r}}_i = 0 \).

Consider the following change of coordinates \( y_i := x_i - r_i, \quad \dot{y}_i := \dot{x}_i - \dot{r}_i, \quad \ddot{y}_i := \ddot{x}_i - \ddot{r}_i \), which allocates the system origin on the trajectory to be tracked. Due to the change of coordinates, and bearing in mind (4)–(7), system (2) becomes

\[
y_{i+1} = A y_i + b u_i ,
\]

where \( A \) and \( b \) coincide with (3), while \( y_i := [y_i \, \dot{y}_i \, \ddot{y}_i]^T \).

Matrices \( A \) and \( b \) depend on sampling time \( T \). In order to drop such dependence, a further transformation \( y_i = W z_i \) is proposed, where

\[
W = TU \begin{bmatrix} T^2 & -T^2 & \frac{T^2}{2} \\ 0 & T & -\frac{T}{2} \\ 0 & 0 & 1 \end{bmatrix},
\]

and \( z_i := [z_{i,1} \, z_{i,2} \, z_{i,3}]^T \). The transformed system is

\[
z_{i+1} = A_d z_i + b_d u_i ,
\]

with

\[
A_d = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad b_d = \frac{1}{U} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
\]

Matrix \( W \) is non singular, so that the inverse transformation \( z_i = W^{-1} y_i \) exists with

\[
W^{-1} = \frac{1}{TU} \begin{bmatrix} \frac{T^2}{2} & \frac{T}{2} & \frac{T}{2} \\ 0 & \frac{T}{2} & \frac{T}{2} \\ 0 & 0 & 1 \end{bmatrix}.
\]

The solution proposed in this paper for Problem 1 is obtained by driving (2) with the following ACL (subscript \( i \) has been dropped for conciseness):

\[
z_{i,1}^+ := \frac{\dot{x} - \dot{r}}{T^2 U}, \quad z_{i,2}^- := \frac{\ddot{x} - \ddot{r}}{T^2 U}, \quad z_{i,3}^+ := \frac{\dddot{x} - \dddot{r}}{T U}, \quad z_{i,3}^- := \frac{\dddot{x} - \dddot{r}}{T U},
\]

\[
z_{i,1}^+ := -\left[ z_{i,1}^+ \right], \quad z_{i,2}^- := -\left[ z_{i,2}^- \right] - \frac{1}{2}, \quad z_{i,2}^+ := -\left[ z_{i,2}^+ \right] - \frac{1}{2}, \quad d_1 := z_{i,2}^+ - z_{i,1}^+.
\]
\[ d_2 := z_2 - z_2^-, \quad (20) \]

for \( n=1,2 \):

\[
\gamma_n := \begin{cases} 
    z_+^1 & \text{if } d_n < z_+^1 \\
    d_n & \text{if } z_+^1 \leq d_n \leq z_-^2 \\
    z_-^2 & \text{if } d_n > z_-^2 
\end{cases}, \quad (21) 
\]

\[
m_n := \frac{1 + \sqrt{1 + 8|\gamma_n|}}{2}, \quad (22) 
\]

\[
\sigma_n := -m_n - \frac{1}{2} \text{sgn}(\gamma_n) - \frac{\gamma_n}{m_n}, \quad (23) 
\]

\[ \sigma := \begin{cases} 
    \sigma_1 & \text{if } \sigma_1 < \sigma_3 \\
    \sigma_2 & \text{if } \sigma_2 \leq \sigma_3 \leq \sigma_1 \\
    \sigma_3 & \text{if } \sigma_3 < \sigma_2
\end{cases}, \quad (25) 
\]

\[
\alpha := z_3 - \sigma, \quad (26) 
\]

\[
u := -U \text{sat}(\alpha), \quad (27) 
\]

where \( z_1, z_2 \) and \( z_3 \) are obtained by means of (12), while integers \( h, k, \) and \( \eta \) are functions of \( z_1 \) and \( z_2 \). For details on \( h, k, \) and \( \eta \) the interested reader can refer to [16]. In the next section the ACL characteristics will be deeply analyzed and, in particular, it will be proved that it solves Problem 1 given that \( z_2^1, z_3^1 \in \mathbb{R}^+ \) and \( z_2^3, z_3^2 \in \mathbb{R}^- \). The two operators \( \lfloor \cdot \rfloor \) and \( \lceil \cdot \rceil \) respectively evaluate the floor and the ceiling of a real number. Function \( \text{sat}(\cdot) \) saturates its argument to \( \pm 1 \).

The ACL is a variable structure controller [17] which switches among three different sliding-mode controllers depending on the current system state. Associated with each controller is one of the three sliding surfaces (SSs) \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) with its corresponding boundary layer (BL). It is evident from (19)–(23) that \( \sigma_1 \) and \( \sigma_2 \) only depend on \( z_2 \): functions \( \sigma_1(z_2) \) and \( \sigma_2(z_2) \) are shown in Figs. 2a and 2b respectively, together with the corresponding system trajectories. Conversely, due to (24), surface \( \sigma_3 \) depends on both \( z_1 \) and \( z_2 \). As a consequence, the position of \( \sigma_3 \) is not constant when projected onto the \((z_2, z_3)\)-plane: given \( z_2 \), it is possible to have several values of \( \sigma_3 \) depending on \( z_1 \), as evidenced by Fig. 3. Surface \( \sigma_3 \) is the same proposed in [16] and guarantees the optimal minimum-time convergence toward the origin in absence of constraints on the velocity and acceleration. In this paper, the attention is mainly focused on \( \sigma_1 \) and \( \sigma_2 \), which are appositely introduced to fulfill the velocity and the acceleration constraints.

where \( z_1, z_2 \) and \( z_3 \) are obtained by means of (12), while integers \( h, k, \) and \( \eta \) are functions of \( z_1 \) and \( z_2 \). For details on \( h, k, \) and \( \eta \) the interested reader can refer to [16]. In the next section the ACL characteristics will be deeply analyzed and, in particular, it will be proved that it solves Problem 1 given that \( z_2^1, z_3^1 \in \mathbb{R}^+ \) and \( z_2^3, z_3^2 \in \mathbb{R}^- \). The two operators \( \lfloor \cdot \rfloor \) and \( \lceil \cdot \rceil \) respectively evaluate the floor and the ceiling of a real number. Function \( \text{sat}(\cdot) \) saturates its argument to \( \pm 1 \).

The ACL is a variable structure controller [17] which switches among three different sliding-mode controllers depending on the current system state. Associated with each controller is one of the three sliding surfaces (SSs) \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) with its corresponding boundary layer (BL). It is evident from (19)–(23) that \( \sigma_1 \) and \( \sigma_2 \) only depend on \( z_2 \): functions \( \sigma_1(z_2) \) and \( \sigma_2(z_2) \) are shown in Figs. 2a and 2b respectively, together with the corresponding system trajectories. Conversely, due to (24), surface \( \sigma_3 \) depends on both \( z_1 \) and \( z_2 \). As a consequence, the position of \( \sigma_3 \) is not constant when projected onto the \((z_2, z_3)\)-plane: given \( z_2 \), it is possible to have several values of \( \sigma_3 \) depending on \( z_1 \), as evidenced by Fig. 3. Surface \( \sigma_3 \) is the same proposed in [16] and guarantees the optimal minimum-time convergence toward the origin in absence of constraints on the velocity and acceleration. In this paper, the attention is mainly focused on \( \sigma_1 \) and \( \sigma_2 \), which are appositely introduced to fulfill the velocity and the acceleration constraints.

Fig. 2. System trajectories in the \((z_2, z_3)\) phase plane for \( \sigma = \sigma_1 \) (figure a) and for \( \sigma = \sigma_2 \) (figure b). SSs \( \sigma_1 \) and \( \sigma_2 \) are indicated by means of dashed lines and are surrounded by their BLs (dash dotted lines). The dotted quadrangle contours the feasible area.

Fig. 3. The three SSs \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) which characterize the filter ACL. The projection of \( \sigma_3 \) on the \((z_2,z_3)\)-plane depends on the current value of \( z_1 \): three different cases are shown for \( z_2 = -20 \) and \( z_1 = 1500, 150, -100 \).

III. THE FILTER CONVERGENCE PROPERTIES

Some preliminary considerations are instrumental to better understand the filter behavior. Velocity bounds \( \dot{x}^-, \dot{x}^+ \) and acceleration bounds \( \ddot{x}^-, \ddot{x}^+ \) can be converted into constraints on \( \dot{y} \) and \( \ddot{y} \) according to \( \dot{y}^+ := \dot{x}^+ - \dot{r}, \dot{y}^- := \dot{x}^- - \dot{r}, \ddot{y}^+ := \ddot{x}^+ - \dot{r}, \ddot{y}^- := \ddot{x}^- - \dot{r} \). Subsequently, such constraints are transformed by means of (12) into equivalent bounds surrounding the feasible area in the \((z_2, z_3)\)-plane. Bearing in mind (13)–(16), such bounds can be expressed as follows: \( z_3 = z_3^1 \); \( z_3 = z_3^3 \); \( z_2 - z_2^3 = z_2^2 \); \( z_2 - z_2^2 = z_2^3 \). The feasible zone is highlighted in Figs. 2, 3, and 4 by means of a dotted quadrangle: until the state remains inside the quadrangle, velocity and acceleration constraints are fulfilled. The control laws associated with \( \sigma_1 \)
and \( \sigma_2 \) are designed to force the system, in minimum-time, inside the feasible area. It is worth noting that such area is independent from \( z_1 \), and for this reason the discussion will essentially focus on the system behavior on the \((z_2, z_3)\)-plane.

The system evolution in the \((z_2, z_3)\)-plane can be deduced from (10) and (11)

\[
\begin{bmatrix}
    z_{2,i+1} \\
    z_{3,i+1}
\end{bmatrix} = \begin{bmatrix}
    1 & 1 \\
    0 & 1
\end{bmatrix} \begin{bmatrix}
    z_{2,i} \\
    z_{3,i}
\end{bmatrix} + \frac{1}{U} \begin{bmatrix}
    1 \\
    1
\end{bmatrix} u_i. \tag{28}
\]

It is straightforward to verify that the system dynamics coincides with that of the system considered in [11, 12], with the sole difference that the role of the pair \( z_1 \) and \( z_2 \) is now played by \( z_2 \) and \( z_3 \), while the role of \( y \) and \( \bar{y} \) is played by \( \bar{y} \) and \( y \). \( \sigma_1 \) and \( \sigma_2 \) are similar to the SS proposed in [12], but \( \sigma_1 \) is obtained by right shifting the original SS by \( z_2^+ \), while \( \sigma_2 \) is obtained through a left shift equal to \( z_2^- \).

Let us subdivide the \((z_2, z_3)\)-plane into the five regions \( R_i \), \( i = 1, 2, \ldots, 5 \) shown in Fig. 4.

**Property 1:** For any point \((z_2, z_3)\) lying in \( R_1 \), the ACL returns \( u = -U \). Conversely, for any point \((z_2, z_3)\) lying in \( R_2 \), the ACL returns \( u = U \).

**Proof:** Let us suppose that point \((z_2, z_3)\) is lying in \( R_1 \). Three cases could arise depending on the relationship occurring between \( \sigma_1, \sigma_2, \) and \( \sigma_3 \). If \( \sigma_3(z_1, z_2) > \sigma_1(z_2) \), due to (25) the SS is \( \sigma = \sigma_1(z_2) \). Since point \((z_2, z_3)\) is outside the BL and \( z_3 > \sigma_1 \), (26) and (27) return \( u = -U \). Similarly, if \( \sigma_1(z_2) > \sigma_3(z_1, z_2) \), then the SS is \( \sigma = \sigma_3(z_1, z_2) \). Even in this case the point is located outside the BL, so that \( \alpha > 1 \) and, in turn, \( u = -U \). Finally, also when \( \sigma = \sigma_2(z_2) \) the situation does not change: \( \alpha > 1 \) and \( u = -U \).

Similar considerations hold if \((z_2, z_3)\) is initially located in \( R_2 \): in this case the command signal becomes \( u = U \).

**Remark 1:** Property 1 asserts that the maximum command is always applied in \( R_1 \) and \( R_2 \), so that trajectories are shaped as shown in Fig. 4 and the system converges in minimum time toward one of the regions \( R_3, R_4 \), and \( R_5 \). As a consequence, the area where the acceleration constraint is satisfied, i.e., \( z_3^* \leq z_3 \leq z_3^+ \), is reached in minimum-time.

**Property 2:** Any point \((z_2, z_3)\) lying in \( R_3 \), is forced in a single step on \( \sigma_1 \) and then it slides toward \( R_5 \) with command signal \( u = 0 \). Conversely, any point in \( R_4 \) is forced in a single step on \( \sigma_2 \) and then it slides toward \( R_5 \) with \( u = 0 \).

**Proof:** Consider a point \((z_2, z_3)\) belonging to \( R_3 \). Region \( R_3 \) coincides with the common BL of \( \sigma_1 \) and \( \sigma_2 \). It is easy to verify that in such region (25) always returns, independently from the position of \( \sigma_3 \), \( \sigma = \sigma_1 = \sigma_2 = z_3^* \geq 0 \). According to (28), \( z_3 \) evolves as follows

\[
z_{3,i+1} = z_{3,i} + \frac{u_i}{U}. \tag{29}
\]

Due to (26) and (27), the command signal is \( u_i = -U(z_{3,i} - \sigma_1) = -U(z_{3,i} - z_3^*) \), so that (29) returns \( z_{3,i+1} = z_3^+ \) independently from \( z_{3,i} \), i.e., the SS is reached in a single step. Successively, \( u \) becomes equal to zero since the point lies on the SS and, again due to (29), \( z_3 \) remains constant and positive. The evolution of \( z_2 \) can be deduced from (28) as well and, in particular, it is equal to

\[
z_{2,i+1} = z_{2,i} + z_{3,i} + \frac{u_i}{U}. \tag{30}
\]

For any point lying on the SS \( u_i = 0 \) and \( z_{3,i} = z_3^* \), so that \( z_{2,i+1} = z_{2,i} + z_3^* \), i.e., \( z_2 \) slides right with increments equal to \( z_3^* \).

A similar transient occurs in \( R_4 \) but, due to symmetry, the state slides left.

**Remark 2:** Property 2 asserts that if the system enters in \( R_3 \) or in \( R_4 \), it is “captured” and pushed toward \( R_5 \). The constraint on the maximum acceleration is not violated since \( z_3 = z_3^* \) in \( R_3 \), while \( z_3 = z_3^* \) in \( R_4 \). In any case, the movement along \( z_2 \) occurs with the maximum admissible acceleration and, consequently, \( R_5 \) is reached in minimum time.

Therefore, due to Properties 1 and 2 and independently from the initial conditions, the state reaches with certainty region \( R_5 \) with the minimum number of steps and compatibly with the assigned constraints. Next properties will show that the state cannot abandon region \( R_5 \) once it has been reached and, furthermore, it converges toward the origin.

The system evolution of any point lying in \( R_5 \) depends, due to (25), on the relationship existing between the SSs.

**Property 3:** Consider a starting point \((z_2, z_3)\) belonging to \( R_5 \). The system remains inside \( R_5 \) and converges toward the origin or one of the two points \((z_2^*, 0)\), \((z_2^*, 0)\).

**Proof:** Three cases could arise depending on \( \sigma_3 \).

- Case \( \sigma_3(z_1, z_2) > \sigma_1 \)

The SS, according to (25), is \( \sigma_1 \). Trajectories within \( R_5 \) assume the shape shown in Fig. 2a. It is evident that all the trajectories starting from points located inside \( R_5 \) remain inside that region and converge toward \( \sigma_1 \). With the same reasonings reported in [12], it is possible to prove that the state first joins the BL of \( \sigma_1 \) and then it slides toward \((z_2^*, 0)\), which is reached in minimum-time compatibly with the constraint on \( z_3 \).

- Case \( \sigma_3(z_1, z_2) < \sigma_2 \)

The situation is similar to the previous one but, according to Fig. 2b, the system is first attracted by \( \sigma_2 \) and then it slides toward \((z_2^*, 0)\).

- Case \( \sigma_3(z_2) < \sigma_3(z_1, z_2) < \sigma_1 \)

The SS, according to (25), is \( \sigma_3 \). It is worth remembering that \( \sigma_3 \) is designed to force the system toward the origin in minimum-time. Differently from the other two SSs, \( \sigma_3 \) could
abandon $R_5$, consequently driving the system state outside the feasible zone: immediately the ACL switches to surface $\sigma_1$ or $\sigma_2$, so that the state remains inside $R_5$.

The ACL strategy inside $R_5$ can be easily explained. Surface $\sigma_3$ forces the system toward the origin in minimum-time. Consequently, inside $R_5$, $\sigma_3$ is the preferred SS unless it would drive the system outside the feasible area. In such eventuality, the state is “parked” in $(z_2^+, 0)$ or $(z_2^-, 0)$, which are both feasible, waiting until $\sigma_3$ newly enters region $R_5$.

**Property 4:** The two points $(z_2^+, 0)$ and $(z_2^-, 0)$ are left with certainty in finite time.

**Proof:** Let us first suppose that $\sigma_3 > \sigma_1$. Bearing in mind **Property 3**, it is possible to assert that the system converges to $(z_2^+, 0)$, which lays on $\sigma_1$, and by virtue of (27) we have $u = 0$. Once $(z_2^+, 0)$ has been reached, due to (10) and (11) the system evolves as follows

\[
\begin{align*}
\sigma_3_{i+1} & = \sigma_3_i = 0, \quad (31) \\
\sigma_2_{i+1} & = \sigma_2_i + \sigma_3_i = z_2^+, \quad (32)
\end{align*}
\]

i.e., owing to (31) and (32) it remains in $(z_2^+, 0)$, but, in the meanwhile, according to (33), $z_1$ increases with steps equal to $z_2 > 0$. The system behaviour can be better understood by observing the phase plane from a different point of view. Fig. 5 shows the trend of $\sigma_3$ in the $(z_1, z_3)$-plane when $z_2 = z_2^+$. The figure highlights that $\sigma_3$ is a monotonically decreasing function of $z_1$. This is a structural characteristic of $\sigma_3$, which applies for any value of $z_2$. In $(z_2^+, 0)$, $\sigma_1 = 0$, which implies that $\sigma_3 > \sigma_1$. A consequence of the $\sigma_3$ positivity is that the current value of $z_1$ is certainly located on the left of $z_{1c}$, where $z_{1c}$ is the solution of the equation $\sigma_3(z_1, z_2^+) = 0$ (see also Fig. 5). Due to (33), $z_1$ increases at the maximum velocity allowed by the feasibility conditions and the corresponding value of $\sigma_3$ decreases. As soon as $\sigma_3$ becomes negative the control law switches, the convergence point $(z_2^+, 0)$ is abandoned and the system starts following $\sigma_3$ with control law $u = -U$ [16].

Analogous considerations hold when $\sigma_3 < \sigma_2$ and the system is initially locked in $(z_2^-, 0)$.
IV. A TEST CASE

The proposed filter has been tested by means of a discontinuous signal made of a step, a ramp and a parabola. Kinematic bounds have been taken into account by initially assuming $x^+ = 0.65$, $\dot{x}^+ = -1$, $\ddot{x}^+ = 1.6$, $\dddot{x}^+ = -2$, $U = 5$. Fig. 7 shows a comparison between the original rough signal and the filter output: $x$ tracks at best reference $r$, compatibly with the given constraints, which are never exceeded. Fig. 7 also shows that a $t = 1.2$ s, the filter bounds are changed as follows: $x^+ = 0.56$, $\dot{x}^- = -1$, $\ddot{x}^- = 1.3$, $\dddot{x}^- = -1.8$, $U = 7$. As required, jerk constraint is immediately recovered, while acceleration and velocity constraints are newly satisfied in minimum-time compatible with the jerk constraint itself.

One remark concerns the possible presence of overshoots at the end of an hanging transient. Every time constraints $\dot{x}^+$, $\dot{x}^-$, $\ddot{x}^+$, $\ddot{x}^-$ are touched the minimum-time sliding surface $\sigma_3$ is lost: the system cannot be driven toward the origin until $\sigma_3$ is newly reached. If $\sigma_3$ is lost during the final transient toward $r$, an overshoot will appear. This property also characterizes the continuous-time filter proposed in [15]. Very stringent values for $\dot{x}^+$, $\dot{x}^-$, $\ddot{x}^+$, $\dddot{x}^-$ have been selected for the example case in order to highlight these overshoots.

A second problem could arise if $r$ is not hanged with a deadbeat approach: overshoots could appear even when saturations are not involved, and the jerk signal could chatter after the reference signal has been reached. The proposed filter is not affected by this problems, which conversely could influence, e.g., the filter in [1] obtained by discretizing [15].

V. CONCLUSIONS

The discrete-time filter proposed in the paper is able to generate smooth reference signal which tracks at the best rough inputs, while fulfilling assigned kinematic constraints. The proposed filter enhances the performances of analogous schemes proposed in the literature since it takes into account, for the first time in a discrete-time context, the existence of bounds on the first and the second time derivatives. The filter is suited for motion control applications which require the online generation of smooth trajectories.

REFERENCES