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Graph Theory Roots of Spatial Operators for Kinematics and Dynamics

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Abstract— **Spatial operators have been used to analyze the dynamics of robotic multibody systems and to develop novel computational dynamics algorithms for them. Mass matrix factorization, inversion, diagonalization, linearization are among the several techniques developed using operators. These techniques have been shown to apply broadly to systems ranging from serial, tree, to closed topology systems, as well as to systems with rigid and flexible links/joints. This paper uses concepts from graph theory to obtain a deeper understanding of the mathematical foundations of spatial operators. We show that spatial kernel operators are instances of block weighted adjacency matrices for the underlying multibody topology graphs, and that spatial propagation operators are 1-resolvents of these matrices. We explore at an abstract level the properties of such 1-resolvents in order to understand the precise requirements on and the range of applicability of spatial operators to the broad class of dynamics problems.**

I. Introduction

The rich mathematical properties of the kinematics and dynamics of robotic multibody systems have been an area of strong research interest. Understanding the dynamical properties is needed to study the inherent physical behavior of systems, for system stability and control analysis, for the development of computational algorithms, and for the accurate modeling of such systems.

System-level mathematical operator techniques have been developed by researchers to analyze and obtain a deeper understanding of the underlying structure of the system kinematics and dynamics. *Spatial operators* were first developed by Rodriguez [1], [2], [3] for the study of the dynamics of serial chain rigid body manipulators. The operators were inspired by the strong mathematical parallels between the structure of the dynamical equations of motion and the time-domain problem of optimal estimation and smoothing. The covariance factorization and optimal filtering techniques behind the seminal Kalman filtering techniques were shown to be applicable to the dynamics domain. Spatial operators were used to obtain the following analytical factorization and inversion results for serial-chain, rigid body system mass matrices [1], [2]:

$$
\begin{aligned}
\mathcal{M} &= \mathrm{H}\phi\mathbf{M}\phi^*\mathrm{H}^* \\
\mathcal{M} &= [\mathbf{I}+\mathrm{H}\phi\mathcal{K}]\,\mathcal{D}\,[\mathbf{I}+\mathrm{H}\phi\mathcal{K}]^* \\
[\mathbf{I}+\mathrm{H}\phi\mathcal{K}]^{-1} &= [\mathbf{I}-\mathrm{H}\psi\mathcal{K}] \\
\mathcal{M}^{-1} &= [\mathbf{I}-\mathrm{H}\psi\mathcal{K}]^*\,\mathcal{D}^{-1}\,[\mathbf{I}-\mathrm{H}\psi\mathcal{K}]
\end{aligned}
\tag{1}
$$

In the above $\mathcal{M}$ denotes the configuration dependent system mass matrix, and H, ϕ etc. are examples of spatial operators. The expressions in Eq. (1) have had important applications in the development of efficient computational algorithms including the well known $O(n)$ articulated body forward dynamics algorithms, operational space dynamics, sensitivity analysis, study of under-actuated system dynamics etc.

Researchers [4], [5], [6] have used system-level matrices/operators to analyze and exploit the sparsity structure of the mass matrix. This research has led to the development efficient computational algorithms for the inverse and forward dynamics of these systems. Other researchers have explored using mass matrix factorization techniques towards the development of system-level global transforms to simplify the coupled equations of motion into diagonalized forms [7], [8], [9], [10], [11], [12]. One common feature to most of these techniques has been the use of *relative* instead of *absolute* [13] coordinates. While absolute coordinate models are arguably easier to assemble, relative coordinate models use minimal coordinates (for tree-topology systems), and are more suitable from a control perspective. While more complex, due to their superior computational and numeric performance, we adopt the relative coordinates approach throughout this paper.

The broad applicability of spatial operators - and the recurring mathematical patterns - across a diverse range of systems is the focus of this paper. For instance, it has been seen that while the details of the elements of the H, ϕ and other spatial operators in Eq. (1) change when generalizing to systems with tree-topology, flexible links, geared hinges, the mass matrix factorization and inversion relations in Eq. (1) remarkably continue to hold across the full range of these systems. This paper is motivated by the desire to identify at an abstract level the core properties and requirement that enable the broad applicability of spatial operators. The expected benefit is that such abstract level insights will enable the broader use of spatial operator techniques to complex kinematics and dynamics problems.

Towards this, we use graph theory techniques to explore the structure of spatial operators. Graph theory techniques have been applied to multibody techniques in the past to help systematically formulate and organize the complex equations of motion [14], [15]. A key contribution of this paper is to show that *spatial propagation operators are the 1-resolvents of a special type of weighted adjacency matrices associated with the multibody system tree graph topology*. We show that this equivalency holds quite generally independent of the specific branching structure of the system topology, the rigid/flexible nature of the component bodies, the body indexing schemes etc. We also study the intimate relationship

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between the structure of the operators and corresponding efficient, recursive computational algorithms. We use some basic examples to illustrate these insights. We also show how shift transformations of spatial operators alter their mathematical structure.

We begin by reviewing the properties of general graphs and trees in Section II. We examine the properties of adjacency matrices used to describe graph connectivity and review the nilpotency property of such matrices for directed trees. We generalize the adjacency matrices to the notion of *block weighted adjacency (BWA)* matrices with block matrix elements. Then Section III develops the notion of *Spatial Kernel Operators (SKO)* kernels and the related *Spatial Propagation Operators (SPO)*. In this paper, we focus primarily on tree topology multi-link systems because most kinematics and dynamics techniques for systems with general graph topology in fact rely on techniques applied to the underlying tree-topology system.

II. Directed Graphs and Trees

We begin with an overview of terminology and concepts from graph theory [16]. A **graph** is a collection of **nodes** and **edges** connecting pairs of nodes. A **directed graph** (also commonly known as a **digraph**), is one where the edges have direction, i.e. an edge from one node to another is not the same as an edge in the reverse direction. Each edge defines a parent/child relationship between the node pair for the edge. The node from which the edge emanates is referred to as the **parent node** and the destination node is said to be the **child node**. Figure 1 illustrates some examples of directed graphs. In the first graph, we see that nodes 4 and 6 are both parents of node 3. A node is said to be the **ancestor** of another node if there is a directed path from the ancestor node to the latter node. We use the notation $i \prec j$ (or equivalently $j \succ i$) to indicate that node j is the ancestor of node i, i.e. that there is a directed path from node j to i. The notation $i \not\prec j$ is used to denote that node j is not an ancestor of node i. The set of parent nodes of the k^{th} node is denoted $\wp(k)$ and the set of its immediate children nodes by $\mathbb{C}(k)$. Nodes with no parent nodes are referred to as **root nodes**. In general directed graphs can have, zero, one or multiple root nodes.

A **rooted digraph** is a digraph that has a single root node that is the ancestor node for all the other nodes in the system. All edges connected to the root node are directed away from the root node. Thus, with r denoting the root node, we have $r \succeq k$ for all nodes k in a rooted digraph. As we will see later, all graphs for multibody topologies are rooted digraphs with the inertial frame serving as the single root node.

A **tree** is a rooted digraph where each node (except the root node) has a single parent node, i.e. $\wp(k)$ contains at most one node member for any node k. One attribute of trees is that they have no loops, i.e. there are no directed paths that start from a node and return to the same node. In other words, a node cannot be its own ancestor. Hence $k \not\prec k$ for any node k. Another attribute of directed trees is that there are no node pairs that have more than one path connecting them, i.e. if

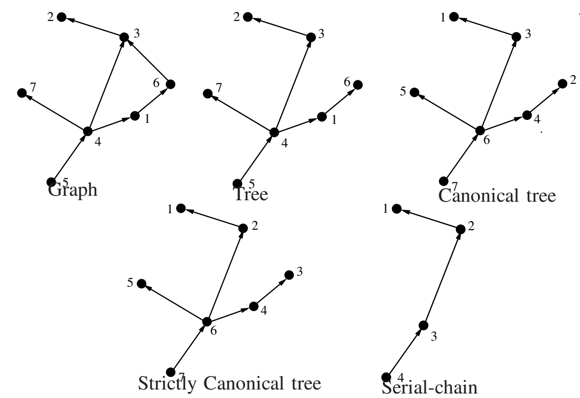

Fig. 1. Illustrative examples of A digraph with a loop, a general tree, and a tree with canonical node numbering

nodes i and j are both ancestors of node k, then one of the i and j nodes must be the ancestor of the other. Notationally, this condition states that if $k \prec i$ and $k \prec j$, then either $i \preceq j$ or $j \preceq i$ The second graph in Figure 1 is a tree. Note that while a node can have at most one parent in a tree, there is no restriction on the number of children nodes. In graph theory, a tree is also referred to as an **arborescence**. Another noteworthy fact is that while $j \succ i \implies i \not\succ j$, the converse is not true in general. In other words, the branching structure implies that the nodes are only partially ordered, and so we can have nodes on different branches with no paths between them. It is well known in graph theory that every rooted digraph has a **spanning tree**, i.e. a tree that contains all the nodes in the digraph and whose edges belong to the digraph. The set of edges removed to convert a rooted digraph into its spanning tree are also referred to as **cut edges**.

A **canonical tree** is a tree where the node numbering is such that a parent node's index is always greater than its child node's index, i.e. $\wp(k) > k$ for any node k. The third graph in Figure 1 illustrates a canonical tree. Any tree can be converted into a canonical tree with a suitable renumbering of the nodes. The node numbering for a canonical tree is not unique in general, since the canonical tree requirement imposes only a partial ordering on the node indices.

A **strictly canonical tree** is a canonical tree where the nodes within a serial-chain segment are numbered sequentially. The fourth graph in Figure 1 illustrates a strictly canonical tree. Once again, any rooted tree can be converted into a strictly canonical tree with a suitable renumbering of the nodes.

A **serial-chain** is a a tree where each node has at most one child and is illustrated in the last graph in Figure 1. The canonical numbering is unique for connected serial-chain systems.

A. Adjacency Matrices for Graphs

One way of representing the node/edge connectivity of a digraph is through an **adjacency matrix**, denoted $\mathbb{S}$, for the digraph. The adjacency matrix, $\mathbb{S}$, is a square $n \times n$ matrix for a graph with n nodes. The $(i, j)^{th}$ element of $\mathbb{S}$ is 1 only if the j^{th} node is a child of the i^{th} node and is 0 otherwise. Adjacency matrices for the graphs in Figure 1 are as follows:

$$\underset{\text{Graph}}{\begin{pmatrix} 0&0&0&0&0&\mathbf{1}&0\\ 0&0&0&0&0&0&0\\ 0&\mathbf{1}&0&0&0&0&0\\ \mathbf{1}&0&\mathbf{1}&0&0&0&\mathbf{1}\\ 0&0&0&\mathbf{1}&0&0&0\\ 0&0&\mathbf{1}&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}} \underset{\text{Tree}}{\begin{pmatrix} 0&0&0&0&0&\mathbf{1}&0\\ 0&0&0&0&0&0&0\\ 0&\mathbf{1}&0&0&0&0&0\\ \mathbf{1}&0&\mathbf{1}&0&0&0&\mathbf{1}\\ 0&0&0&\mathbf{1}&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}} \underset{\text{Canonical Tree}}{\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ \mathbf{1}&0&0&0&0&0&0\\ 0&\mathbf{1}&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&\mathbf{1}&0&\mathbf{1}&0&0&0\\ 0&0&0&0&0&\mathbf{1}&0 \end{pmatrix}}$$

$$\underset{\text{Strictly Canonical Tree}}{\begin{pmatrix} 0&0&0&0&0&0&0\\ \mathbf{1}&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&\mathbf{1}&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&\mathbf{1}&0&\mathbf{1}&0&0&0\\ 0&0&0&0&0&\mathbf{1}&0 \end{pmatrix}} \qquad \underset{\text{Serial-Chain}}{\begin{pmatrix} 0&0&0&0\\ \mathbf{1}&0&0&0\\ 0&\mathbf{1}&0&0\\ 0&0&\mathbf{1}&0 \end{pmatrix}}$$

Any adjacency matrix $\mathbb{S}$ can be expressed as

$$\mathbb{S} = \sum_k e_{\wp(k)} e_k^* \tag{2}$$

where e_i denotes a vector of length n with all zeros except for the i^{th} element which is 1. With $\delta_{j=k}$ denoting the Dirac delta function, note that

$$e_j^* e_k = \delta_{j,k} \tag{3}$$

B. Block Weighted Adjacency Matrices for Trees

The adjacency matrices we have looked at above have had 1 or 0 scalar elements. We now generalize these matrices to define *block weighted adjacency (BWA)* matrices, $\mathbb{S}_W$, whose block elements are weight matrices assigned to each graph edge. To make this precise, we assume that to each tree node j is assigned a positive weight dimension m_j and that each edge has assigned to it a weight matrix, $w(\wp(j), j)$, of dimension $m_{\wp(j)} \times m_j$. Then the BWA matrix, $\mathbb{S}_W$, is defined as

$$\mathbb{S}_W = \sum_k e_{\wp(k)} w(\wp(k), k) e_k^* \tag{4}$$

Eq. (4) is a generalization of Eq. (2) to work with block-weight matrices. e_k is now a column matrix whose i^{th} block element has dimension $m_i \times m_k$, with the only non-zero block element being the identity k^{th} matrix element. It is easy to verify that Eq. (4) is well-defined, and that $\mathbb{S}_W$ is a square matrix with dimension

$$N \stackrel{\triangle}{=} \sum_k m_k \tag{5}$$

The standard adjacency matrix, $\mathbb{S}$ in Eq. (2) is thus a special case of a BWA matrix where all the node dimensions are 1 and the edge weights are the 0 or 1 scalars.

Eq. (3) continues to hold for the new e_k block-element matrices, but with the product resulting in identity or zero matrices depending on whether j is equal to k or not respectively. Another observation that is easy to verify is that the higher powers, $\mathbb{S}_W^k$ have the same block-element structure as the original $\mathbb{S}_W$ matrix. The following lemma provides a specific expression for the block elements of $\mathbb{S}_W^k$.

Lemma 1: The $\mathbb{S}_W^k$ k^{th} power of the BWA matrix contains non-zero block-elements only for paths of length k connecting the nodes. The value of its block-element is the product of the k weights for all the edges along the path, i.e. the non-zero elements are of the form

$$w\left(\wp^k(i), i\right) \stackrel{\triangle}{=} w\left(\wp^k(i), \wp^{k-1}(i)\right) * \cdots * w\left(\wp(i), i\right) \in \mathcal{R}^{m_{\wp^k(i)} \times m_i} \tag{6}$$

Here $\wp^k(i)$ denotes the k^{th} upstream ancestor of the i^{th} node. The above generalizes the definition of $w(i,j)$ to all node pairs where node i is an ancestor of node j.

Proof: Let us illustrate the proof for $k = 2$. $\mathbb{S}_W^2$ is given by

$$\begin{aligned} &\left(\sum_j e_{\wp(j)} w\left(\wp(j), j\right) e_j^*\right) * \left(\sum_k e_{\wp(k)} w\left(\wp(k), k\right) e_k^*\right) \\ &\stackrel{3}{=} \sum_j \sum_k e_{\wp(j)} w\left(\wp(j), j\right) \delta_{j,\wp(k)} w\left(\wp(k), k\right) e_k^* \\ &= \sum_k e_{\wp^2(k)} w\left(\wp^2(k), \wp(k)\right) w\left(\wp(k), k\right) e_k^* \\ &\stackrel{6}{=} \sum_k e_{\wp^2(k)} w\left(\wp^2(k), k\right) e_k^* \end{aligned}$$

In a similar vein, Eq. (6) can be established 5for arbitrary k. ∎

With the above lemma, the weight matrices are well defined for all nodes i and j except for the case where $i = j$. By assigning the identity matrix as the weight for this case, we have the following generalized definition of weights that applies to all node pairs:

$$w(i,j) = \begin{cases} \mathbf{I} & \text{for } i = j \\ \mathbf{0} & \text{for } i \not\succ j \\ w\left(\wp^k(j), \wp^{k-1}(j)\right) * \cdots * w\left(\wp(j), j\right) & \text{for } i = \wp^k(j) \end{cases} \tag{7}$$

The following lemma establishes the nilpotency of $\mathbb{S}_W$.

Lemma 2: The n^{th} power of a tree BWA matrix $\mathbb{S}_W^n$ is zero, where n denotes the number of nodes in the system. *Hence, the BWA matrix for a tree is a nilpotent matrix.*

Proof: This is direct consequence of Lemma 1 since all node to node paths can have at most n edges in a tree with n nodes (since $\wp^n(k)$ is null for all nodes k). ∎

The following lemma establishes an important property of nilpotent matrices.

Lemma 3: If U is a nilpotent matrix such that $U^n = \mathbf{0}$, then it's *1-resolvent*[1], $W \stackrel{\triangle}{=} (\mathbf{I} - U)^{-1}$ is given by

$$W = \mathbf{I} + U + U^2 + \cdots + U^{n-1} \tag{8}$$

Proof: For W as defined in Eq. (8), we have that

$$\begin{aligned} UW = WU &= U + U^2 + \cdots + U^n \\ &= U + U^2 + \cdots + U^{n-1} = W - \mathbf{I} \end{aligned}$$

Rearranging terms, we have

$$\mathbf{I} = W - UW = (\mathbf{I} - U)W \implies (\mathbf{I} - U)^{-1} = W$$

∎

[1]The resolvent of a matrix A is $(\lambda\mathbf{I} - A)^{-1}$ for some scalar λ. We use the term 1-resolvent for the resolvent with $\lambda = 1$.

Lemma 4: The 1-resolvent of a tree adjacency matrix is given by:

$$(\mathbf{I} - \mathbb{S}_W)^{-1} = \mathbf{I} + \mathbb{S}_W + \mathbb{S}_W{}^2 + \cdots + \mathbb{S}_W{}^{n-1} \tag{9}$$

Proof: The result follows from the observation in Lemma 2 that a tree's adjacency matrix is nilpotent together with the 1-resolvent result from Lemma 3. ∎

As a side note, for a (non-tree) graph with *directed loops*, the 1-resolvent is not defined because $(\mathbf{I} - \mathbb{S}_W)$ is a singular-matrix. Now that we have introduced the BWA matrix, we introduce some new notation that better reflects its intimate relationship with its 1-resolvent matrix. With $\mathbb{A}$ denoting such a 1-resolvent matrix, we will denote its associated BWA matrix as $\mathcal{E}_{\mathbb{A}}$, i.e. in the notation from above, we have

$$\mathcal{E}_{\mathbb{A}} = \mathbb{S}_W \quad \text{and} \quad \mathbb{A} = (\mathbf{I} - \mathbb{S}_W)^{-1} \tag{10}$$

The following Lemma describes the expression for the block elements of a 1-resolvent matrix.

Lemma 5: The (k, j) element of a tree's 1-resolvent matrix $\mathbb{A} = (\mathbf{I} - \mathbb{S}_W)^{-1}$ is simply $w(k, j)$, that is

$$\mathbb{A}(k, j) = w(k, j) \tag{11}$$

Thus, $\mathbb{A}$ has identity along the diagonal, and non-zero (k, j) element only if the k^{th} node is an ancestor of the j^{th} node.

Proof: Eq. (11) follows from the expression for $(\mathbf{I} - \mathbb{S}_W)^{-1}$ in Eq. (9) as well as Lemma 1 together with Eq. (7) that describe the elements of the powers of $\mathbb{S}_W$. ∎

In view of Eq. (11), we start referring to the generalized weight matrices $w(j, k)$ for a 1-resolvent $\mathbb{A}$ as the more intuitive $\mathbb{A}(j, k)$ elements. The following Lemma highlights the semi-group properties of the generalized weight matrices.

Lemma 6: For a tree, let i, j and k be nodes where the i^{th} node is an ancestor of the j^{th} node, and the k^{th} node is on the path connecting them. Then,

$$\mathbb{A}(i, j) = \mathbb{A}(i, k)\mathbb{A}(k, j) \quad \text{for} \quad i \succeq k \succeq j \tag{12}$$

This property is also known as the **semi-group** property for the elements of a 1-resolvent matrix.

Proof: Eq. (12) follows simply from the generalized Eq. (7) expression for weight matrices. ∎

We now introduce the $\tilde{\mathbb{A}}$ matrix that is closely related to a $\mathbb{A}$ 1-resolvent matrix.

$$\tilde{\mathbb{A}} \triangleq \mathbb{A} - \mathbf{I} \tag{13}$$

Note that $\tilde{\mathbb{A}}$ is strictly lower-triangular for canonical trees. The following exercise establishes some basic properties of $\tilde{\mathbb{A}}$.

Lemma 7: For a 1-resolvent $\mathbb{A}$, we have the following identities:

$$\tilde{\mathbb{A}} = \mathbb{A} - \mathbf{I} = \mathcal{E}_{\mathbb{A}}\mathbb{A} = \mathbb{A}\mathcal{E}_{\mathbb{A}} \tag{14}$$

Proof: For any matrix X such that $(\mathbf{I} - X)$ is invertible, the following matrix identity holds:

$$X(\mathbf{I} - X)^{-1} = (\mathbf{I} - X)^{-1}X = (\mathbf{I} - X)^{-1} - \mathbf{I}$$

With $X = \mathcal{E}_{\mathbb{A}}$, the above equation directly leads to Eq. (14). ∎

III. SKO and SPO Operators for Tree Multibody Systems

Graphs and trees provide natural mathematical constructs to describe the topology and connectivity of bodies in a multi-link system where we designate the links as nodes, and the connecting hinges as edges. Choosing the inertial frame as the root node, and the edge directionality as going from inboard to outboard bodies across hinges, we obtain a *rooted digraph* representation for the system's topology. Thus an n link system has in principle a graph with $n+1$ nodes (with the inertial frame root node included in). However, we will in fact work with the n-node subgraph that excludes the inertial frame root node. There is no loss of information because in this subgraph, nodes with no parent are implicitly assumed to be the children of the inertial frame. This n-node sub-graph offers the benefit of allowing us to work with adjacency and related matrices of dimensions related to n instead of $n+1$. We refer to a multibody system as having a tree topology if and only if its extended $(n+1)$ node rooted digraph is a rooted tree. Such tree topology multibody systems will be the focus of the following sections.

A. Velocity mapping for a canonical serial-chain

Let us begin by considering the basic example of a n-link canonical rigid body serial-chain system with single degree of freedom hinges. For this serial chain system, $\wp(k) = k+1$. Using coordinate-free notation, the **rigid body transformation operator**, $\phi(k, k-1)$, associated with the k^{th} link is defined as

$$\phi(k, k-1) \triangleq \begin{pmatrix} \mathbf{I} & \widetilde{l}(k, k-1) \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \in \mathcal{R}^{6\times 6} \tag{15}$$

where $l(k, k-1)$ is the vector from the k^{th} joint to the $(k-1)^{th}$ joint. Define the BWA matrix, $\mathcal{E}_\phi \in \mathcal{R}^{6n\times 6n}$, for the n-node serial-chain tree graph with edge weight matrices chosen as $w(k+1, k) \triangleq \phi(k+1, k)$ with weight dimension $m_k = 6$. Then $\mathcal{E}_\phi$ has the form

$$\mathcal{E}_\phi \triangleq \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \phi(2,1) & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \phi(3,2) & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \phi(n, n-1) & \mathbf{0} \end{pmatrix} \tag{16}$$

Note that $\mathcal{E}_\phi$ is strictly lower triangular as would be expected of a canonical tree system. From Lemma 4 we know that the 1-resolvent of $\mathcal{E}_\phi$ is well defined and we denote it as follows:

$$\phi \triangleq (\mathbf{I} - \mathcal{E}_\phi)^{-1} \in \mathcal{R}^{6n\times 6n} \tag{17}$$

Using Eq. (7), we see that the $\phi(k, j)$ block matrix element is defined as

$$\phi(k, j) = \phi(k, k-1) \cdots \phi(j+1, j) \tag{18}$$

With $H^*(k)$ denoting the k^{th} hinge map matrix, define the block-diagonal matrix, $H \in \mathcal{R}^{n\times 6n}$

$$H \stackrel{\triangle}{=} \text{diag}\{H(k)\} \tag{19}$$

and the system-level **stacked** vectors $\mathcal{V} \in \mathcal{R}^{6n}$ and $\theta \in \mathcal{R}^{n}$

$$\mathcal{V} \stackrel{\triangle}{=} \text{col}\left\{\mathcal{V}(k)\right\} \quad \text{and} \quad \theta \stackrel{\triangle}{=} \text{col}\left\{\theta(k)\right\}$$

In the above $\mathcal{V}(k)$ denotes the 6-dimensional spatial velocity and $\theta(k)$ the generalized coordinate for the k^{th} link. The following Lemma describes the relationship between the $\dot{\theta}$ hinge velocities and the $\mathcal{V}$ link spatial velocities.

Lemma 8: $\mathcal{V}$ and $\dot{\theta}$ are related by the following expression:

$$\mathcal{V} = \phi^* H^* \dot{\theta} \tag{20}$$

Proof: We know that the k^{th} body's spatial velocity $\mathcal{V}(k)$ is related to that of the $(k+1)^{th}$ body's via the following::

$$\mathcal{V}(k) = \phi^*(k+1,k)\mathcal{V}(k+1) + H^*(k)\dot{\theta}(k) \tag{21}$$

$H^*(k)\dot{\theta}(k)$ is the spatial velocity contribution from the k^{th} hinge velocity, while $\phi^*(k{+}1,k)\mathcal{V}(k+1)$ is the contribution from the spatial velocity of the parent body. Stacking up the component level relationships in Eq. (21) for all the links we obtain the following equivalent relationship between the stacked vectors:

$$\mathcal{V} = \mathcal{E}_\phi^* \mathcal{V} + H^*\dot{\theta} \tag{22}$$

The above relationship is an implicit one with $\mathcal{V}$ appearing on both sides of the equation. Collecting the $\mathcal{V}$ terms and using Eq. (17) leads to Eq. (20). ∎

So we have now seen our first example of using the ϕ 1-resolvent matrix to express the basic, but important, kinematical relationship in Eq. (20). $\mathcal{E}_\phi$, ϕ and H are in fact precisely the *spatial operators* that have been previously used for this purpose in [1], [2], [17]. Indeed, these references go well beyond the velocity relationship and use these operators to develop the equations of motion and the mass matrix factorization and inversion expressions in Eq. (1). The reader is referred to these references for details. The important fact we have established for the above serial-chain system is that *the $\mathcal{E}_\phi$ BWA matrix for the serial chain is indeed a spatial operator as is its 1-resolvent ϕ, where the edge weights are those defined by Eq. (15).* Indeed, in the context of multibody systems, we refer to the BWA spatial operators such as $\mathcal{E}_\phi$ as **spatial kernel operators (SKO)** and the associated ϕ 1-resolvent matrices as **spatial propagation operators (SPO)**. For canonical serial-chains, ϕ has the following lower-triangular structure:

$$\phi = \begin{pmatrix} \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \phi(2,1) & \mathbf{I} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \phi(n,1) & \phi(n,2) & \dots & \mathbf{I} \end{pmatrix} \tag{23}$$

B. Velocity mapping for a general serial-chain

Consider the generalization of Section III-A to a non-canonical serial chain system. For such a system the parent node of the k^{th} node is node $\wp(k)$ instead of the $(k+1)^{th}$ node. Eq. (21) generalizes to

$$\mathcal{V}(k) = \phi^*(\wp(k),k)\mathcal{V}(\wp(k)) + H^*(k)\dot{\theta}(k) \tag{24}$$

Apart from this change, the notion of the $\mathcal{E}_\phi$ SKO is well-defined as that of its 1-resolvent ϕ. However for non-canonical systems, neither of these operators may be lower-triangular. The stacked velocity relationship in Lemma 8 and Eq. (20) continue to hold for non-canonical serial chains as well.

C. Velocity mapping for tree topology systems

Departing from a serial chain multibody system to a tree-topology one, we see that Eq. (24) continues to define the spatial velocity relationship between a body and its child body. The only thing that changes is that a body may have multiple child bodies in a tree-topology system. Once again, apart from this change, the notions of the $\mathcal{E}_\phi$ SKO and its 1-resolvent ϕ are well defined and the velocity relationship in Lemma 8 and Eq. (20) continue to be hold for general tree systems. The internal structure of the ϕ operator will change to reflect the branch topology of the system, but the operator level expression remains unaffected!

D. Generalizations

The above sections have shown that SKO operators are BWA matrices for the multibody system graph and that the their 1-resolvents are SPO operators. The key properties of the spatial operators are inherited from the properties of BWA matrices for the underlying graphs. The properties we described for the illustrative systems in the previous sections hold true broadly for more general systems. We list below some of the key areas where generalizations hold:

- the SKO and SPO operators play a role in not just velocity kinematic relationships but also in dynamics ones
- the SKO and SPO are not limited to canonical trees - any body numbering is permissible
- the weight matrices are not limited to the $\phi(k+1,k)$ matrices in Eq. (15)
- the weight matrices can be of non-uniform size
- the weight matrices do not have to be square
- the weight matrices do not have to be invertible
- the SKO and SPO techniques can also be used with non-tree topology systems

These generalization hold for SKO and SPO operators due to the fact that they hold generally for BWA matrices. Due to space limitations, we simply summarize some of the specific examples of spatial operators where the above generalizations apply and leave the details to the references:

- The SPO operators show up not only for velocity relationships but also in the dual inter-body force relationships and in the expression for the mass matrix in Eq. (1) for multibody systems [17].

- The $\mathcal{E}_\psi$ and ψ [17] are examples of SKO and SPO spatial operators that arise in the articulated body recursions for the $O(n)$ forward dynamics recursions as well as the mass matrix factorization and inversion results in Eq. (1). The weight matrices for these operators are not invertible.
- When generalizing to non-rigid links, the size of the weight matrices is 6 plus the number of deformation modes for the body [18]. Not only are the size of the weights different from 6, but since the number of modes can vary from body to body, the weight matrices are not square in general.
- Another example is that of geared motors and flexible hinges where the size of the weight matrices includes the additional hinge flexible degrees of freedom and so the weight matrices are not square in general.
- Recently developed constraint embedding techniques [19] have shown how to transform graph-topology systems with constraints into tree-topology systems with equivalent SKO and SPO spatial operators.

IV. Conclusions

Motivated by the wide success of spatial operators in tackling a broad range of kinematics and dynamics problems, we have sought in this paper to identify some of the basic principles and requirements underlying spatial operators. Towards this objective, we have identified key connections between spatial operators and graph theory concepts to show that BWA matrices associated with multibody system graphs are precisely the key SKO and SPO spatial operators. Our definition of BWA matrices is a weighted matrix generalization of adjacency matrices for directed graph systems. We have shown that such BWA matrices are nilpotent and have developed expressions for their 1-resolvent. We have established several key properties at the BWA matrix level of abstraction so that the key requirements for the use of spatial operators within different system contexts is clear. Thus it is easy to see how the spatial operators can be generally used for cases where the weight matrices are non-square, of non-uniform size and non-invertible, as well for general branched topology multibody systems.

This paper has developed abstract insights into the graph theory underpinnings of spatial operators. By identifying basic principles that are independent of system topology and specific properties, we expect in future work to exploit these connections to advance the broader application of spatial operator techniques to the kinematics and dynamics arena.

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