

Vision-based Robot Control with Omnidirectional Cameras and Conformal Geometric Algebra

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Abstract—Traditional cameras have a narrow field of view, to enlarge the field of view omnidirectional cameras can be used. In this work, we propose, a simple an elegant solution to the image formation model for omnidirectional cameras with parabolic mirrors. We propose the use of conformal geometric algebra (CGA), since the involved transformation operations in the model can be represented as an special group of multivectors. This representation is advantageous since the inversions are linearized, furthermore the transformation can be applied to all the geometric objects of the CGA. In consequence, the paracatadioptric image formation can be simplified, since the procedure is the same for points, *point-pairs*, lines, or circles. As an application example the control of a nonholonomic mobile robot using paracatadioptric line images and the proposed framework is described.

I. INTRODUCTION

Conventional cameras suffer from a limited field of view. One effective way to increase the field of view is to use mirrors in combination with conventional cameras. The approach of combining mirrors with conventional cameras to enhance sensor field of view is referred as catadioptric image formation.

In order to be able to model the catadioptric sensor geometrically, it must satisfy the restriction that all the measurements of light intensity pass through only one point in the space (effective viewpoint). The complete class of mirrors that satisfy such restriction where analyzed by Baker and Nayar [1]. In [2] the authors deal with the epipolar geometry of two catadioptric sensors. Later, in [3] a general model for central catadioptric image formation was given. Also, a representation of this general model using the CGA was shown in [4]. In contrast with previous works where the paracatadioptric projection is defined for points or a parametric representation of geometric entities, the present work introduces a model which can handle the paracatadioptric projection of points, *point-pairs*, lines and circles analytically.

Vision based servoing schemes are effective methods to control robot motion from camera observations [5], [6]. Visual servoing applications can benefit from sensors providing large fields of view. The present work is mainly concerned with the use of projected lines extracted from central catadioptric images as input of a visual servoing control loop. The paracatadioptric image of a line is in

general a circle but sometimes it can be a line. This is something that should be taken into account to avoid a singularity in the visual servoing task.

The rest of this paper is organized as follows: The next section will give a brief introduction to the conformal geometric algebra. In section III we show the equivalence between inversions on the sphere and the parabolic projections. In section IV a paracatadioptric image formation model using CGA is proposed. In section V the experimental results are given. Finally, the conclusions are in section VI.

II. CONFORMAL GEOMETRIC ALGEBRA

In general, a geometric algebra $\mathcal{G}_{p,q,r}$ is a linear space of dimension 2^n , $n = p+q+r$, with a subspace structure, called blades, to represent multivectors. A multivector is a higher grade algebraic entity in comparison to vectors of a vector space as first grade entities, or scalars as grade zero entities. The geometric algebra is generated from a n -dimensional vector space \mathcal{V}^n by defining the *geometric product* as an associative and multilinear product satisfying the contraction rule $\mathbf{a}^2 = \epsilon|\mathbf{a}|^2$, for $\mathbf{a} \in (\mathcal{V})^n$, where ϵ is $-1, 0$ or 1 and is called the signature of \mathbf{a} . When $\mathbf{a} \neq 0$ but its magnitude $|\mathbf{a}|$ is equal to zero, \mathbf{a} is said to be a *null vector*.

The geometric product of two entities is denoted by the juxtaposition of the entities, just as in matrix algebra, where the matrix product of two matrices is represented by juxtaposition of two matrix symbols. The geometric product of two basis vectors, e_i and e_j , is

$$e_i e_j = \begin{cases} 1 & \text{for } i = j \in \{1, 2, \dots, p\} \\ -1 & \text{for } i = j \in \{p+1, \dots, p+q\} \\ 0 & \text{for } i = j \in \{p+q+1, \dots, n\} \\ e_i e_j = e_i \wedge e_j = -e_j \wedge e_i & \text{for } i \neq j \end{cases} \quad (1)$$

The geometric algebra of \mathcal{R}^3 is denoted by $\mathcal{G}(\mathcal{R}^3)$, or simply \mathcal{G}_3 . It has three basis vectors e_1, e_2, e_3 where the geometric product of $e_1 e_1, e_2 e_2$ and $e_3 e_3$ is 1 , and the geometric product of any other combinations of basis vectors is 0 .

The geometric product (denoted by juxtaposition) of vectors \mathbf{a} and \mathbf{b} , is defined as

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} . \quad (2)$$

From the geometric product two new products can be defined, the *inner product*

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) = \mathbf{b} \cdot \mathbf{a} \quad (3)$$

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which in the case of vectors entities coincides with the standard scalar product of linear algebra, but in general, when both entities are not vectors, it represents an operation which does not result in a scalar. The other product is called the *outer product*

$$\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba}) = -\mathbf{b} \wedge \mathbf{a} \quad (4)$$

which in the vector case results in a *bivector*, a subspace of grade two. The outer product of r vectors can be defined as the anti-symmetric part of the geometric product of the r -vectors, which is called an *r-blade*. A linear combination of *r-blades* is called an *r-vector*. The set of *r-vectors* is an $\binom{n}{r}$ -dimensional subspace of \mathcal{G}_n , denoted by \mathcal{G}_n^r . The whole of \mathcal{G}_n is given by the subspace sum of \mathcal{G}_n^i , for $i = 0, 1, \dots, n$. A generic element in \mathcal{G}_n is called a *multivector*, which can be written in the expanded form

$$M = \sum_{i=0}^m \langle M \rangle_i, \quad (5)$$

where $\langle M \rangle_i$ denotes the *i-vector* part. An element M in \mathcal{G}_n is invertible if there exists an element N in \mathcal{G}_n such that $MN = NM = 1$. The element N , if it exists, is unique and it is called the inverse of M , which is denoted by M^{-1} . Every non null vector \mathbf{u} is invertible with $\mathbf{u}^{-1} = 1/\mathbf{u} = \mathbf{u}/\mathbf{u}^2$. The concept of magnitude is extended from vectors to any multivector by

$$\|M\| = \sqrt{\sum_{i=0}^m \|\langle M \rangle_i\|^2}, \quad (6)$$

where $\|\langle M \rangle_i\|^2 = \sqrt{\|\langle M \rangle_i \cdot \langle M \rangle_i\|}$.

The CGA [7] is the geometric algebra over an homogeneous conformal space. This framework extends the functionality of projective geometry to include circles and spheres. Furthermore, it includes operations like dilations, inversions, rotations and translations, which can be applied to points, lines, planes, point pairs, circles and spheres.

The CGA \mathcal{G}_n adds two extra vector basis (\mathbf{e}_+ and \mathbf{e}_-) to the Euclidean space \mathcal{R}^3 . For example, for the 3D space we have the following basis vectors: $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_-, \mathbf{e}_+$, where $\mathbf{e}_+^2 = 1$ and $\mathbf{e}_-^2 = -1$. With this extra basis two null vectors can be defined

$$\mathbf{e}_0 = \frac{\mathbf{e}_- - \mathbf{e}_+}{2} \quad \text{and} \quad \mathbf{e}_\infty = \mathbf{e}_- + \mathbf{e}_+, \quad (7)$$

The vector \mathbf{e}_0 can be interpreted as the origin of the coordinate system, and the vector \mathbf{e}_∞ as the point at infinity. The outer product of \mathbf{e}_+ and \mathbf{e}_- produces a special bivector called the *E-plane*, $E = \mathbf{e}_+ \mathbf{e}_-$, which represents the Minkowski plane.

To specify a 3-dimensional Euclidean point in a unique form in this 5-dimensional space, we require the definition of two constraints. The first constraint is that the representation must be homogeneous, that is λX and X represent the same point in Euclidean space. The second constraint requires that

the vector X be a null vector (*i.e.* $X^2 = 0$, but $X \neq 0$). The equation that satisfies these constraints is

$$X = \mathbf{x} + \frac{1}{2}\mathbf{x}^2\mathbf{e}_\infty + \mathbf{e}_0 \quad (8)$$

where $\mathbf{x} \in \mathcal{R}^n$ and $X \in \mathcal{R}^{n+1,1}$. Note that this is a bijective mapping. From now and in the rest of the paper the points X are named conformal points and the points \mathbf{x} are named Euclidean points. To recover the Euclidean point from a conformal point we can use $\mathbf{x} = (X \wedge E)E$.

The outer product of conformal points can be used to define geometric entities. For example the outer product of four points defines a sphere (3D sphere) containing the four points $S = A \wedge B \wedge C \wedge D$. The outer product of three points defines a circle (2D sphere), $C = A \wedge B \wedge C$. Similarly the outer product of two points defines a *point-pair* (1D sphere) $Q = A \wedge B$. A plane is defined with three points and the point at infinity $P = A \wedge B \wedge C \wedge \mathbf{e}_\infty$. A line is defined with two points and the point at infinity $L = A \wedge B \wedge \mathbf{e}_\infty$. Similarly, the outer product of a point and the point at infinity \mathbf{e}_∞ defines an entity called a *flat-point*, $S = A \wedge \mathbf{e}_\infty$, which can be the result of the intersection of two lines. The entities constructed in this way are defined in what is called the outer product null space (OPNS). In the OPNS we can test for incidence of a point in an entity by simply computing the outer product of the point with the entity. For example, if the point X lies on the sphere S then we have $X \wedge S = 0$.

The dual representation of the OPNS is the inner product null space (IPNS). In the IPNS a sphere is defined as

$$S = \mathbf{c} + \frac{1}{2}(\mathbf{c}^2 - \rho^2)\mathbf{e}_\infty + \mathbf{e}_0, \quad (9)$$

which is very similar to (8). The main difference is the radius ρ of the sphere. Thus, a point can be considered as a sphere with zero radius. In the IPNS the plane is defined as

$$\Pi = \mathbf{n} + \delta\mathbf{e}_\infty, \quad (10)$$

which denotes the Hesse normal form representation of the plane, where \mathbf{n} is the normal vector and δ the distance from the origin to the plane.

As we already mention, the OPNS and IPNS representations are dual to each other, to change from one representation to another we simply multiply the entity by the *pseudoscalar*, e.g. XI^{-1} . Where I denotes the *pseudoscalar* of the algebra, and is defined as the outer product of all the basis vectors of the algebra, e.g. the *pseudoscalar* for the Euclidean space is $I_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_{123}$. The *pseudoscalar* for the CGA which embeds the 3D space is defined as $I = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_+\mathbf{e}_- = \mathbf{e}_{123+-}$. To distinguish between the OPNS and the IPNS representations, we will use the notation M^* to denote that the *multivector* M is defined in the OPNS, and we will omit the superscript to denote a *multivector* defined in the IPNS.

A. Conformal Transformations

In CGA the conformal transformations are linearized using the fact that the conformal group on \mathcal{R}^n is isomorphic to the Lorentz group on \mathcal{R}^{n+1} . Hence, nonlinear conformal transformations on \mathcal{R}^n can be linearized by representing them as Lorentz transformations and, thereby, further simplified as versor representation. These versors can be applied not only to points but also to all the CGA entities (spheres, planes, circles, lines and point-pairs).

In CGA the rotations are performed by means of an entity called rotor which is defined by $R = \exp\left(\frac{\theta}{2}\mathbf{I}\right)$ where \mathbf{I} is the bivector representing the dual of the rotation axis. To rotate an entity, we simply multiply it by the rotor R from the left and the reverse of the rotor \tilde{R} from the right, $Z = RY\tilde{R}$. The \tilde{R} denotes the reversion of the rotor, and it is defined as

$$\langle M \rangle_i (-1)^{\frac{i(i-1)}{2}} \langle M \rangle_i, \text{ for } M \in \mathcal{G}_n, 0 \leq i \leq n. \quad (11)$$

The translations can be carried out by an entity called a translator which is defined as $T = \left(1 + \frac{\mathbf{e}_\infty \mathbf{t}}{2}\right) = \exp\left(\frac{\mathbf{e}_\infty \mathbf{t}}{2}\right)$. With this representation the translator can be applied multiplicatively to an entity similarly to the rotor, by multiplying the entity from the left by the translator and from the right with the reverse of the translator: $Z = TY\tilde{T}$.

Finally, the rigid motion can be expressed using a *motor* which is the combination of a rotor and a translator: $M = TR$. The rigid body motion of an entity Y is described with $Z = MY\tilde{M}$. For more details on the geometric algebra and CGA, the interested reader is referred to view [7], [8], [9].

III. PARABOLIC PROJECTION AND SPHERE PROJECTION

In this section we will show the equivalence between the parabolic projection and the sphere projection, followed by a sphere inversion.

A. Parabolic Projection

The projection induced by a parabolic mirror to an image plane is called parabolic projection. The parabolic projection of a point $\mathbf{x} = xe_1 + ye_2 + ze_3 \in \mathcal{G}_3$ is defined as the projection \mathbf{x}_p onto the mirror surface, followed by an orthographic projection, which leads to the point \mathbf{x}_c .

Assume that a parabola is placed such that its axis is the e_3 axis, with focal length p and its focus is located at the origin. The equation of this parabola is

$$\frac{x^2 + y^2}{4p} - p = z. \quad (12)$$

The projection of the point \mathbf{x} onto the mirror is

$$\mathbf{x}_p = \lambda \mathbf{x}, \quad (13)$$

where λ is defined as

$$\lambda = \frac{2p}{\|\mathbf{x}\| - z}. \quad (14)$$

Finally, the point \mathbf{x}_p is projected onto a plane perpendicular to the axis of the parabola. The reason for this is that any incident ray on the mirror is reflected such that it is perpendicular to the image plane.

B. Relationship between inversion and parabolic projection

Let \mathbf{x} be a point in \mathcal{G}_3 and, also let, \mathbf{x}' be a point in \mathcal{G}_3 representing the inversion of the point \mathbf{x} with respect to sphere centered at the origin, and with radius r , then we have the following equation

$$\mathbf{x}\mathbf{x}' = r^2, \quad (15)$$

where $\mathbf{x}' = r^2/\mathbf{x}$, and where $1/\mathbf{x} = \mathbf{x}/\mathbf{x}^2$.

When the sphere is centered at a point $\mathbf{c} \in \mathcal{G}_3$, the inversion of the point \mathbf{x} with such a sphere is defined as

$$\mathbf{x}' = r^2 \frac{1}{\mathbf{x} - \mathbf{c}} + \mathbf{c}. \quad (16)$$

Now, given a sphere centered at the origin with radius p and a point $\mathbf{x} \in \mathcal{G}_3$, the projection of the point \mathbf{x} onto the sphere is simply

$$\mathbf{x}_s = p \frac{\mathbf{x}}{\|\mathbf{x}\|}. \quad (17)$$

The inversion of the point \mathbf{x}_s with respect to a second sphere of radius $2p$ and centered at point \mathbf{n} , can be computed using using (16), thus we have the following equation

$$\alpha \|\mathbf{x}_s - \mathbf{n}\|^2 = (2p)^2. \quad (18)$$

The term $\|\mathbf{x}_s - \mathbf{n}\|^2$ in (18) is equivalent to

$$(\mathbf{x}_s - \mathbf{n})(\mathbf{x}_s - \mathbf{n}) = \mathbf{x}_s^2 - \mathbf{x}_s \mathbf{n} - \mathbf{n} \mathbf{x}_s + \mathbf{n}^2, \quad (19)$$

note that $-\mathbf{x}_s \mathbf{n} - \mathbf{n} \mathbf{x}_s$ is equivalent to the inner product (3), thus we have that

$$\|\mathbf{x}_s - \mathbf{n}\|^2 = \mathbf{x}_s^2 - 2(\mathbf{x}_s \cdot \mathbf{n}) + \mathbf{n}^2. \quad (20)$$

Substituting (20) in (18) and rewriting it

$$\alpha = \frac{(2p)^2}{\mathbf{x}_s^2 - 2(\mathbf{x}_s \cdot \mathbf{n}) + \mathbf{n}^2}. \quad (21)$$

With out lose of generality, let us define a point \mathbf{n} which lies on the mirror axis, and at distance p from the focus of the mirror. Now, substituting $\mathbf{n} = pe_3$ and (17) in the previous equation

$$\alpha = \frac{(2p)^2}{p^2 \frac{\mathbf{x}^2}{\|\mathbf{x}\|^2} - 2(p \frac{\mathbf{x}}{\|\mathbf{x}\|} \cdot pe_3) + (pe_3)^2} = \frac{2}{1 - \frac{1}{\|\mathbf{x}\|}(\mathbf{x} \cdot e_3)}, \quad (22)$$

where the following equivalences had been used: $\mathbf{x}^2 = \mathbf{x} \cdot \mathbf{x}$, $\|\mathbf{x}\|^2 = (\sqrt{\mathbf{x} \cdot \mathbf{x}})^2 = \mathbf{x} \cdot \mathbf{x}$. The paracatadioptric projection of the point \mathbf{x} is defined as $\mathbf{x}_c = \alpha \mathbf{x}_s = \alpha \frac{p}{\|\mathbf{x}\|} \mathbf{x}$, where

$$\alpha \frac{p}{\|\mathbf{x}\|} = \frac{2}{1 - \frac{1}{\|\mathbf{x}\|}(\mathbf{x} \cdot e_3)} \frac{p}{\|\mathbf{x}\|} = \frac{2p}{\|\mathbf{x}\| - z}, \quad (23)$$

which is exactly the same value of the scalar λ from the parabolic projection (14), note that $\mathbf{x} \cdot e_3 = z$. Therefore, we can conclude that the spherical projection of a point in space followed by the inversion of the resulting point is equivalent to the parabolic projection, see Fig. 1.

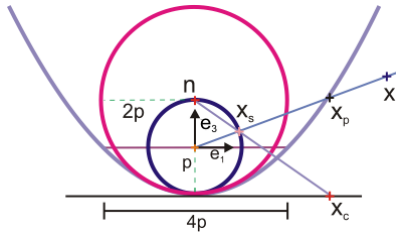


Fig. 1. Equivalence between parabolic projection and inversion

C. Inversion and the conformal geometric algebra

In CGA, the conformal transformations are represented as versors [9]. In particular, the versor of the inversion is a sphere, and it is applied in the same way as the rotor, or the translator. Given a sphere of radius r centered at \mathbf{c} represented by the vector

$$S = \mathbf{c} + \frac{1}{2}(\mathbf{c}^2 - r^2)\mathbf{e}_\infty + \mathbf{e}_0 \quad (24)$$

the inversion of a point X with respect to S is

$$X' = SX\tilde{S} \quad (25)$$

To clarify the above equation and without loss of generality, let us analyze the special case when S is a unit sphere, centered at the origin. Then S reduces to

$$S = -\frac{1}{2}\mathbf{e}_\infty + \mathbf{e}_0 = -\frac{1}{2}(\mathbf{e}_- + \mathbf{e}_+) + \frac{1}{2}(\mathbf{e}_- - \mathbf{e}_+) = -\mathbf{e}_+ \quad (26)$$

and, thus, (25) becomes

$$\begin{aligned} (-\mathbf{e}_+)X(-\mathbf{e}_+) &= (-\mathbf{e}_+)(\mathbf{x} + \frac{1}{2}\mathbf{x}^2\mathbf{e}_\infty + \mathbf{e}_0)(-\mathbf{e}_+) \quad (27) \\ &= \mathbf{e}_+\mathbf{x}\mathbf{e}_+ + \frac{1}{2}\mathbf{x}^2\mathbf{e}_+\mathbf{e}_\infty\mathbf{e}_+ + \mathbf{e}_+\mathbf{e}_0\mathbf{e}_+ . \end{aligned}$$

The term $\mathbf{e}_+\mathbf{x}\mathbf{e}_+$ is equal to

$$x\mathbf{e}_+\mathbf{e}_1\mathbf{e}_+ + y\mathbf{e}_+\mathbf{e}_2\mathbf{e}_+ + z\mathbf{e}_+\mathbf{e}_3\mathbf{e}_+ = -x\mathbf{e}_1 - y\mathbf{e}_2 - z\mathbf{e}_3 = -\mathbf{x} . \quad (28)$$

Substituting (7) in the term $\mathbf{e}_+\mathbf{e}_\infty\mathbf{e}_+$ we get

$$\mathbf{e}_+(\mathbf{e}_- + \mathbf{e}_+)\mathbf{e}_+ = (\mathbf{e}_+\mathbf{e}_- + 1)\mathbf{e}_+ = -\mathbf{e}_- + \mathbf{e}_+ = -2\mathbf{e}_0 . \quad (29)$$

Substituting (7) in $\mathbf{e}_+\mathbf{e}_0\mathbf{e}_+$ we get

$$\mathbf{e}_+\frac{\mathbf{e}_- - \mathbf{e}_+}{2}\mathbf{e}_+ = \frac{\mathbf{e}_+\mathbf{e}_- - 1}{2}\mathbf{e}_+ = \frac{-\mathbf{e}_- - \mathbf{e}_+}{2} = -\frac{\mathbf{e}_\infty}{2} . \quad (30)$$

From equations (29) and (30) we can conclude that the inversion of the point at infinity is the point at the origin and the inversion of the point at the origin is the point at infinity. Finally, rewriting (27) we have

$$X' = -\mathbf{x} - \frac{1}{2}\mathbf{e}_\infty - \mathbf{x}^2\mathbf{e}_0 = \frac{1}{\mathbf{x}} + \frac{1}{2}\left(\frac{1}{\mathbf{x}}\right)^2\mathbf{e}_\infty + \mathbf{e}_0 . \quad (31)$$

From the previous equation we recognize the Euclidean point

$$\frac{1}{\mathbf{x}} = \frac{\mathbf{x}}{\mathbf{x}^2} = \frac{\mathbf{x}}{\|\mathbf{x}\|^2} , \quad (32)$$

which represents the inversion of the point \mathbf{x} . The case of the inversion with respect to an arbitrary sphere is

$$\sigma X' = SX\tilde{S} = \left(\frac{\mathbf{x} - \mathbf{c}}{r}\right)^2 \left(f(\mathbf{x}) + \frac{1}{2}f(\mathbf{x})^2\mathbf{e}_\infty + \mathbf{e}_0\right) , \quad (33)$$

where $f(\mathbf{x})$ is equal to (16), the inversion in \mathcal{R}^n . The value σ represents the scalar factor of the homogeneous point.

The interesting thing about the inversion in the CGA is that it can be applied not only to points, but also to any other entity of CGA. In the following section we will see how the paracatadioptric image formation can be described in terms of CGA.

IV. PARACATADIOPTRIC IMAGE FORMATION AND CONFORMAL GEOMETRIC ALGEBRA

In the previous section we saw the equivalence between the parabolic projection and the inversion. We also saw how to compute the inversion in the CGA using a versor, in this case the versor is simply the sphere where the inversion will be carried out. In this section we will define the paracatadioptric image formation using CGA.

Given a parabolic mirror with a focal length p , the projection of a point in the space through the mirror followed by an orthographic projection can be handled by two spheres. Where the first sphere is centered at the focus of the mirror and its radius is p . This sphere can be defined as

$$S = \mathbf{c} + \frac{1}{2}(\mathbf{c}^2 - p^2)\mathbf{e}_\infty + \mathbf{e}_0 . \quad (34)$$

The second sphere S_0 can be defined in several ways, but we prefer to define it with respect to a point N on the sphere S (i.e. $N \cdot S = 0$). If we compare the point equation (8) with the sphere equation (9), we can observe that the sphere has an extra term $-\frac{1}{2}r^2\mathbf{e}_\infty$. If we extract that term to the point N we get a sphere centered at N with a radius r . Thus, the sphere S_0 is defined as

$$S_0 = N - \frac{1}{2}(2p)^2\mathbf{e}_\infty = n + \frac{1}{2}(n^2 - 4p^2)\mathbf{e}_\infty + \mathbf{e}_0 , \quad (35)$$

where $2p$ is the radius of the sphere. With these two spheres the image formation of points, circles and lines will be showed in the next subsections.

A. Point Images

Let X be a point in the space, its projection onto the sphere can be found by finding the line passing through it and the sphere center, that is

$$L^* = S \wedge X \wedge \mathbf{e}_\infty , \quad (36)$$

Then, this line is intersected with the sphere S

$$Q = S \cdot L^* . \quad (37)$$

Where Q denotes a *point-pair* ($Q^* = X_1 \wedge X_2$). The point closest to X can be found with

$$X_s = \frac{Q^* - |Q^*|}{Q^* \cdot e_\infty} . \quad (38)$$

Finally, the projection onto the paracatadioptric image plane is simply

$$X_c = S_0 X_s \widetilde{S}_0 . \quad (39)$$

The point X_c is the projection of the point X onto the catadioptric image plane, which is exactly the same point obtained through the parabolic projection (Fig. 2).

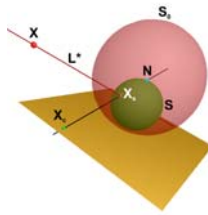


Fig. 2. Point projection onto the catadioptric image plane

B. Back Projection of Point Images

Given a point X_c on the catadioptric image (Fig. 2), its projection to the sphere is simply $X_s = \widetilde{S}_0 X_c S_0$. The point X_s lies on a line that passes through the sphere center, that is $L^* = P_1 \wedge S \wedge e_\infty$. The original point X , also, lies on this line, but since we have a single image the depth can not be determined and thus the point X can not be calculated.

C. Circle Images

The circle images can be found in the same way as for the points images. To see that, let X_1, X_2, X_3 be three points on the sphere S , the circle defined by them is $C^* = X_1 \wedge X_2 \wedge X_3$, which can be a great or a small circle. The projection of the circle onto the catadioptric image is carried out as in (39) with $C_2^* = S_0 C^* \widetilde{S}_0$. Where C_2^* could be a line, but this is not a problem in CGA, since it is represented as a circle with one point at infinity. The back projection of a circle (or line) C_2^* that lies on the catadioptric image plane, can be found easily with $C^* = \widetilde{S}_0 C_2^* S_0$.

In Fig. 3 the projection of circles on the sphere, and their projection onto the catadioptric image plane is shown.

D. Line Images

The paracatadioptric projection of a line L^* in the 3D space (Fig. 4) can be found by defining a plane where the line L^* and the center of the sphere S lie, that is $\Pi^* = L^* \wedge S$. Then, the plane Π^* is intersected with the sphere to obtain a great circle as $C_s^* = S \cdot \Pi^*$. Finally, the circle C_s is projected onto the image plane using the inversion $C_c^* = S_0 C_s^* \widetilde{S}_0$.

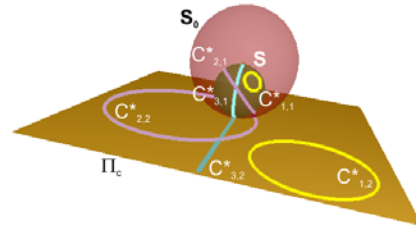


Fig. 3. Projection of circles on the sphere

V. EXPERIMENTAL RESULTS

The task to achieve consists of driving a mobile robot parallel to a given straight line. The mobile robot is a nonholonomic system with a paracatadioptric sensor. We assume that the camera optical axis is superposed with the rotation axis of the mobile robot. Thus, the kinematic screw is only composed by the linear velocity v along the e_1 axis and the angular velocity ω .

The problem will be solved using a paracatadioptric image of lines, where one of those lines is the paracatadioptric image of the desired line C_d and the other one is the current paracatadioptric image of the current line C . These lines will be projected onto the sphere and then onto a virtual perspective plane Π_p , in this plane the image projection are straight lines. Finally, with the lines on the perspective plane we will compute the angular and lateral deviations.

Consider the paracatadioptric image C_d of the desired 3D line L_d^* , its back-projection onto the sphere is defined as

$$C_s^* = \widetilde{S}_0 C_d^* S_0 . \quad (40)$$

Then, the plane where the circle lies is defined as

$$\Pi_d^* = C_s^* \wedge e_\infty . \quad (41)$$

Finally, the intersection of the plane Π_d^* with the perspective plane Π_p^* is

$$L_{pd}^* = \Pi_d^* \cdot \Pi_p^* . \quad (42)$$

This line is the projection of the paracatadioptric image line C_d^* onto the perspective plane Π_p^* . The perspective plane can be defined as $\Pi_p = \hat{\mathbf{n}} + \delta e_\infty$, where $\hat{\mathbf{n}} = \mathbf{n}/|\mathbf{n}|$ and $\mathbf{n} = (S \wedge S_\wedge e_\infty) \cdot -E$. Note that the expression $S \wedge S_\wedge e_\infty$ represents the line passing through the centers of both spheres (S and S_0). The value of the scalar δ can be defined arbitrarily.

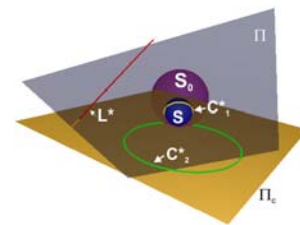


Fig. 4. Projection of a line in the space

The current paracatadioptric line C can be projected into the line L in the perspective plane in similar way using the above equations, see Fig. 5.

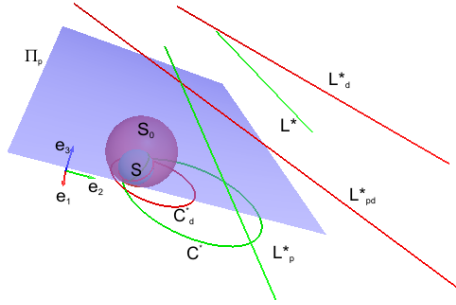


Fig. 5. Projection of the paracatadioptric lines into the perspective plane.

The lines L_p^* and L_{pd}^* , on the perspective plane, define a rotor which can be computed with

$$R = 1 + L_p^* L_{pd}^* . \quad (43)$$

Where $L_p^* L_{pd}^*$ represents the geometric product of the two lines. The angle between the lines is then

$$\theta = (R \cdot e_{12}) \langle R \rangle_0 , \quad (44)$$

which represents the angular deviation. The lateral deviation can be found with the signed distance between the lines, the signed distance between the lines is

$$d = (L^* \cdot e_{12} e_0) - (L_d^* \cdot e_{12} e_0) . \quad (45)$$

The angular and lateral deviations are used in a dynamic controller, proposed in [10], to generate the robot's angular velocity. The dynamic controller is

$$\omega = -k_2 v d \frac{\sin \theta}{\theta} - k_3 |v| \theta , \quad (46)$$

where the control gains are defined as $k_2 = \alpha^2$ and $k_3 = 2\xi\alpha^2$. The value of α is left free to specify faster or slower systems, and where ξ is usually set to $1/\sqrt{2}$. The trajectories of the paracatadioptric images and the current paracatadioptric line are shown in Fig. 6. These trajectories confirm that the task is correctly realized. In Fig. 7 the angular and lateral deviations of the current paracatadioptric image with respect to the desired image are shown. These figures show that both deviations are well regulated to zero.

VI. CONCLUSIONS

In this work a comprehensive geometric model for paracatadioptric sensors has been presented. The model is based on the equivalence between paracatadioptric projection and the spherical projection followed by an inversion. The main reason for the use of CGA is due to its capability to represent inversion as versors (i.e. a special group of multivectors). The advantage of this representation is that it can be applied not only to points but also to point-pairs, lines, circles, spheres and planes in the same way. This will allow an

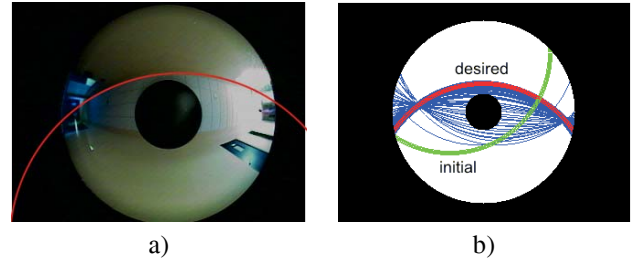


Fig. 6. a) Tracked line in the paracatadioptric image, b) Trajectory of the projected lines in the paracatadioptric image

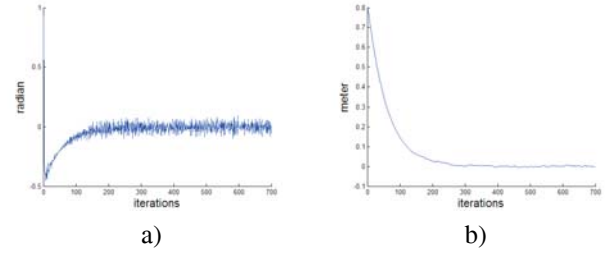


Fig. 7. a) Angular deviation, b) Lateral Deviation

easier implementation of paracatadioptric sensors in more complex applications. The proposed framework has been used to control a nonholonomic robot, with a paracatadioptric sensor. The input to the control scheme were paracatadioptric images of the desired and current lines.

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