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General parameterization of holonomic kinematic inversion algorithms for redundant manipulators

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***Abstract*— Redundant robotic manipulators under kinematic control may exhibit unpredictable behaviours at joint level. When the end-effector describes a closed trajectory, the joint angles may not return to their initial values and final configuration in the joint space may depend on the trajectory followed by the end-effector.**

In this paper, a complete parameterization of holonomic local control strategies that avoid these problems is proposed. Only a basis of the null-space of the Jacobian matrix is required in order to design all the possible holonomic control strategies. The effectiveness of the proposed approach is verified on a simple case study and on a real industrial manipulator.

I. Introduction

A robotic manipulator is said to be kinematically redundant when it has more degrees of freedom than those strictly necessary to perform a given task. Since a general task consists in following an end-effector trajectory with a specified orientation, and thus requires only six degrees of freedom, it follows that a manipulator with seven or more joints is redundant. More in general, let m be the number of required degrees of freedom of the task and n be the number of joints of the robot: the robot is thus redundant if $n > m$.

The controller design of a redundant robot is definitely more involved than the case of a common non-redundant manipulator. Besides the issues related to sensing and actuation systems, the most challenging aspect related to redundant robots concerns motion planning. In fact, a redundant manipulator is able to perform a prescribed end-effector motion in infinite ways, which implies that the inverse kinematic problem has infinite solutions. This fact can be used in order to optimize some additional criteria such as singularity [1] or obstacle avoidance [2], torque minimization, [3] and [4], and others. Since the early 80's, several studies have been published on redundant manipulators, some of which will be reviewed in Section II. Further research in this field is strongly motivated by the renewed interest of industries in redundant manipulators, due to their increased dexterity. Robot manufactures are in fact putting on the market kinematically redundant robots, which naturally fosters research in the area.

Side effects of the adoption of kinematic redundancy are that the motion of the robot can be to some extent unpredictable. During a positioning task, the final configuration of the robot may depend on the planned end-effector trajectory even when the motion of the robot starts from the same initial joint configuration. Moreover, under a kinematic control strategy a closed end-effector trajectory can be mapped into an open trajectory on the configuration space. These facts are highly undesirable and may represent a limitation in using redundant manipulators.

This paper contributes presenting a full parameterization of kinematic control strategies that avoid these problems. By a proper choice of a certain matrix, it will be possible to design all the control strategies that guarantee repeatability of the method for a given manipulator. Moreover, it will be shown that the parameterization does not depend on the minimal representation of the orientation of the end-effector.

The remainder of this paper is organised as follows. In Section II some material on the kinematics of redundant manipulators is reviewed; in Section III the main results of this work are discussed. In Sections IV and V such results are verified on a simple case study and on a real industrial manipulator, respectively.

II. Mathematical background and state of the art

A manipulator consists of a series of rigid bodies connected by joints. If q_i $(i = 1,\ldots,n)$ denotes the variable characterising the position of the i-th joint, the posture of the entire chain is uniquely defined once the vector $q = \begin{bmatrix} q_1 & q_2 & \ldots & q_n \end{bmatrix}^T$ is given. The position of the end-effector (or TCP) is usually characterised by the vector $x = \begin{bmatrix} x_1 & x_2 & \ldots & x_m \end{bmatrix}^T$ which describes its position and/or orientation. The direct kinematic mapping associated to a manipulator is thus a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$:

$$x = f(q) \tag{1}$$

The kinematic inversion problem is to find q for a given x such that previous equation holds. Usually, this problem is addressed at the velocity level. In other words, the first time derivative of (1) is taken into account:

$$\dot{x} = \frac{\partial f}{\partial q}\dot{q} \equiv J(q)\,\dot{q} \tag{2}$$

where J is a $m \times n$ matrix called task-Jacobian, or simply Jacobian[1].

In order to study the kinematic inversion problem, the

[1]From now on, the dependence on q will be omitted.

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concepts of local control strategy and involutive distribution are introduced now.

Local control strategy Let $\mathbb{S}$ be a simply-connected open subset of $\mathbb{R}^n$ where the Jacobian matrix J is full rank and let G be a $n \times m$ matrix such that $JG = I_m$. Then G is called a local control strategy.

Considering the local control strategy G, a solution of the inverse kinematic problem is the following one:

$$\dot{q} = G\dot{x} \tag{3}$$

As first noticed in [5], differently from non-redundant manipulators, the motion of redundant manipulators under a kinematic local control strategy can be unpredictable [6]. More precisely, it has been shown that a closed trajectory in the end-effector task can be mapped, through a local control strategy, into an open trajectory in the configuration space. On the opposite side, it may exist a local control strategy able to map every closed end-effector trajectory into a closed trajectory in the configuration space. According to the literature, the latter control strategy is called *cyclic* or *repeatable*, see Fig. 1. Moreover, the final configuration

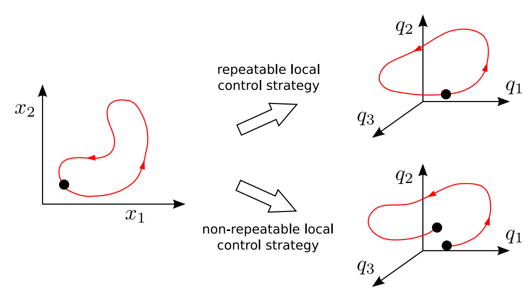


Fig. 1. Repeatable vs. non-repeatable local control strategies

of the robot during a positioning task may depend on the planned end-effector trajectory even when the motion of the robot starts from the same initial configuration [7], see Fig. 2. In classical mechanics this property is referred to

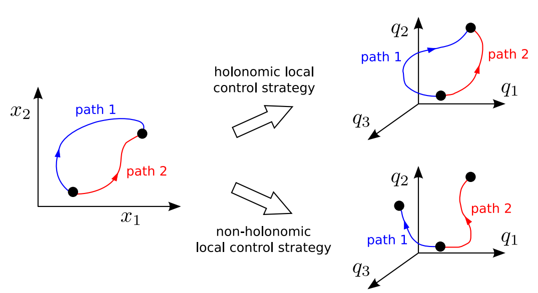


Fig. 2. Holonomic vs. non-holonomic local control strategies

as holonomy. More precisely, the final configuration of the robot may depend on the planned end-effector trajectory if and only if the local control strategy is non-holonomic [8].
The possible non-repeatable or non-holonomic behaviour of a redundant manipulator is highly undesirable. In fact, in many industrial applications, the robot is asked to perform repetitive end-effector motions. Using a repeatable local control strategy may result in a predictable joint motion and a simplified programming: many characteristics of the motion (e.g. joint and velocity limits avoidance, singularity avoidance, etc.) can be verified by simulating only the first cycle.
Moreover, since in the near future robots will be able to cooperate with humans co-workers, their holonomic behaviour is needed in order to avoid any inconvenience (like fear or unease) to the humans.

Distribution The distribution associated to the local control strategy $G = \begin{bmatrix} G_1 & \dots & G_m \end{bmatrix}$ (where G_i denotes the i-th column of G) is $span\,(G_1, \dots, G_m) = range\,(G)$.

Involutivity The distribution associated to the local control strategy G is said to be involutive if and only if

$$\forall\,(i,j) : [G_i, G_j] \in range\,(G) \tag{4}$$

where

$$[A,B] = \frac{\partial B}{\partial q}A - \frac{\partial A}{\partial q}B \tag{5}$$

denotes the Lie bracket operation [9].

In [10] a necessary and sufficient condition to check whether a local control strategy is cyclic or not is presented. In particular, it has been proven that a local control strategy G is cyclic if and only if its associated distribution is involutive. In other words, a local control strategy is holonomic if the underlying distribution is involutive (closed with respect to the Lie bracket operation). In addition, it can be shown that if the distribution associated to a local control strategy is involutive, then every Lie Bracket vanishes [7]. Finally, as stated in [11], the holonomy property, the cyclic property and the involutivity of the distribution associated to a local control strategy are equivalent.
For the sake of completeness, it must be noticed that if the Jacobian matrix J is square (i.e. the manipulator is non-redundant) and non singular, the local control strategy $G = J^{-1}$ is always involutive. It follows that only a local control strategy for redundant manipulators may be non-involutive and, thus, non-holonomic.
Considering the following class of local control strategies (weighted Moore-Penrose pseudo-inverse):

$$G_W = WJ^T \left(JWJ^T\right)^{-1} \tag{6}$$

where W is a symmetric positive definite matrix, some further results exist. In [12] a simplified criterion to check whether the distribution associated to G_W is involutive or not is presented. In particular, it can be proven that the distribution associated to G_W is involutive if and only if the distribution associated to WJ^T is involutive, as well.
The criteria discussed so far can be used only to check whether a local control strategy is holonomic or not, but they are difficult to apply in order to design a holonomic local control strategy.
Many researchers have tried to develop methodologies in order to design an cyclic local control strategy. For instance,

in [12] a repeatable control strategy based on optimization is proposed, while in [13] an asymptotic cyclic control strategy is presented. A solution based on impedance control has been developed in [14]. A further method recently developed can be found in [15].However, many of these solutions may suffer from algorithmic singularities (the matrix G is rank-deficient even when the Jacobian matrix J is full rank) or cannot be simply implemented in an industrial controller due to their extreme complexity.

III. MAIN RESULTS

Based on the following Lemma, we are now in position to prove our main results.

Lemma III.1 *Let J be a full rank Jacobian matrix and N any null space basis of J. Then, the distribution associated to N is involutive.*

Proof: See the Appendix. ∎

Theorem III.2 *Let $\mathbb{S} \subset \mathbb{R}^n$ be a simply-connected open subset where the Jacobian matrix J is full rank and consider a local control strategy G. Then the following statements are equivalent:*

1) *the local control strategy G generates a holonomic behaviour*
2) *there exists a non singular matrix $H = \begin{bmatrix} N_G & N \end{bmatrix}$, where the distribution associated to N_G is involutive and N is a basis of the null space of J, such that $G = HH^TJ^T\left(JHH^TJ^T\right)^{-1}$.*

Proof: [Proof of 2 ⇒ 1] consider the following local control strategy

$$G = HH^TJ^T\left(JHH^TJ^T\right)^{-1} \tag{7}$$

By hypothesis, H is non singular and J is full rank, thus, since HH^T is positive definite, thanks to Sylvester theorem the inversion of JHH^TJ^T is well-posed.
For the Proposition 1 in [12] the distribution associated to method (7) is involutive if and only if the distribution associated to HH^TJ^T is involutive. It can be seen that

$$HH^TJ^T = N_GN_G^TJ^T \tag{8}$$

Then, since the distribution associated to N_G is involutive and $N_G^TJ^T$ is a square matrix, for the same proposition, the distribution associated to (8) is involutive if and only if $N_G^TJ^T$ (or equivalently JN_G) is non singular. Since H is non singular, then

$$Range\left(N_G\right) \cap Range\left(N\right) = Range\left(N_G\right) \cap Null\left(J\right) = \{0\} \tag{9}$$

which means that JN_G is non singular. ∎

Proof: [Proof of 1 ⇒ 2] by hypothesis, the distribution associated to the method G is involutive, therefore the motion of the robot under the local control strategy G is constrained by an additional set of integrable Pfaffian constraints. In other words, there exists a function $g(\cdot)$ with $g : \mathbb{R}^n \to \mathbb{R}^{n-m}$ such that

$$J_G\dot{q} = 0 \tag{10}$$

where $J_G = \partial g / \partial q$ is full rank (see [11], [16] and [17]), the extended Jacobian

$$J_A = \begin{bmatrix} J^T & J_G^T \end{bmatrix}^T \tag{11}$$

is non singular and $J_GG \equiv 0$.
Let N_G be a basis of the null space of J_G: from Lemma III.1 the distribution associated to N_G is involutive. Let N be a basis of the null space of J and consider matrix $H = \begin{bmatrix} N_G & N \end{bmatrix}$. Since (11) is non singular:

$$Null\left(J\right) \cap Null\left(J_G\right) = \{0\} \tag{12}$$

It follows that $Range\left(N\right) \cap Null\left(J_G\right) = \{0\}$ and, similarly, $Null\left(J\right) \cap Range\left(N_G\right) = \{0\}$. Then, the matrix

$$J_AH = \begin{bmatrix} JN_G & 0 \\ 0 & J_GN \end{bmatrix} \tag{13}$$

is non singular which implies that H is non singular, as well. Finally, it is now easy to verify that

$$J_AG = J_AHH^TJ^T\left(JHH^TJ^T\right)^{-1} = \begin{bmatrix} I_m \\ 0 \end{bmatrix} \tag{14}$$

Since J_A is a square and non singular matrix, it follows that $G = HH^TJ^T\left(JHH^TJ^T\right)^{-1}$. ∎

Remark Theorem III.2 represents a full parameterization of the set of all holonomic local control strategies. In other words, it presents a necessary and sufficient condition for a local control strategy to be holonomic. Moreover, it shows that any cyclic control strategy can be obtained by choosing a proper positive definite weight matrix $W = HH^T$.
Using this result, it is then possible to design a holonomic local control strategy. Notice that, in order to design a holonomic local control strategy for a given redundant manipulator, namely in order to select the matrix N_G, only the knowledge of a null-space basis N of the Jacobian matrix J is required.

Theorem III.3 *Let $\mathbb{S} \subset \mathbb{R}^n$ be a simply-connected open subset where the Jacobian matrix J is full rank and consider the repeatable local control strategy $G = HH^TJ^T\left(JHH^TJ^T\right)^{-1}$ consistent with the formulation in Theorem III.2. Consider a new local control strategy $\hat{G}$, obtained by replacing J with $\hat{J}$, where the latter is the Jacobian matrix with respect to a different, yet non singular, description of the orientation. Then the distribution associated to $\hat{G}$ is involutive (i.e. $\hat{G}$ is cyclic, as well).*

Proof: Since the distribution associated to G is involutive, then HH^TJ^T has an underlying involutive distribution too (see [12]). Let J_ω be the geometrical Jacobian of the manipulator. The relationship between J and J_ω is well-known, [18]:

$$J_\omega = T_\phi J \tag{15}$$

where T_ϕ is a non singular matrix. It follows that

$$J^T = J_\omega^T\left(T_\phi^T\right)^{-1} \tag{16}$$

From the Proposition 1 in [12], since T_ϕ is non singular, it follows that $HH^T J_\omega^T$ has an underlying involutive distribution as well.
Now, consider the new description of the orientation such that:

$$J_\omega = T_\psi \hat{J} \tag{17}$$

It is straightforward to show that the distribution associated to $HH^T J_\omega^T \left(T_\psi^T\right)^{-1} = HH^T \hat{J}^T$ is involutive. Therefore the distribution associated to $\hat{G}$ is involutive, too. ∎

Remark Theorem III.3 states that the holonomy property of a local control strategy depends only on the weight matrix HH^T and on the geometrical Jacobian of a manipulator J_ω. Changing the minimal representation of the orientation of the end-effector does not affect the holonomy. In other words, matrix H can be selected regardless of the choice of the representation of the orientation of the end-effector.

IV. A CASE STUDY

Consider the planar PPR manipulator sketched in Fig. 3. The robot is redundant for the task of positioning the end-effector (point p, in Fig. 3) with unspecified orientation ($n = 3 > m = 2$). The Jacobian matrix of this manipulator (the length of the third link is unitary) is expressed as:

$$J = \begin{bmatrix} 1 & 0 & -s_3 \\ 0 & 1 & c_3 \end{bmatrix} \tag{18}$$

where $c_3 = \cos(q_3)$, $s_3 = \sin(q_3)$, while a basis of the null space of J is:

$$N = \begin{bmatrix} s_3 & -c_3 & 1 \end{bmatrix}^T \tag{19}$$

Consider the following matrix, found by inspection, which is associated to an involutive distribution (it can be simply verified by computing the Lie brackets of its columns):

$$N_G = \begin{bmatrix} c_3^2 & 0 \\ 0 & 1 \\ -s_3 & 0 \end{bmatrix} \tag{20}$$

It is easy to verify that $H = \begin{bmatrix} N_G & N \end{bmatrix}$ is non singular for every $q \in \mathbb{R}^n$, in fact:

$$det(H) = det \begin{bmatrix} c_3^2 & 0 & s_3 \\ 0 & 1 & -c_3 \\ -s_3 & 0 & 1 \end{bmatrix} = 1, \forall q \in \mathbb{R}^n \tag{21}$$

Therefore, for the Theorem III.2, the local control strategy $G = HH^T J^T \left(JHH^T J^T\right)^{-1}$ has an underlying involutive distribution, where:

$$G = \begin{bmatrix} c_3^2 & 0 \\ s_3 c_3 & 1 \\ -s_3 & 0 \end{bmatrix} \tag{22}$$

In [11] a cyclic method has been proposed for the same manipulator. The local control strategy is the following one:

$$\hat{G} = \begin{bmatrix} 1+s_3 & s_3 \\ -c_3 & 1-c_3 \\ 1 & 1 \end{bmatrix} \tag{23}$$

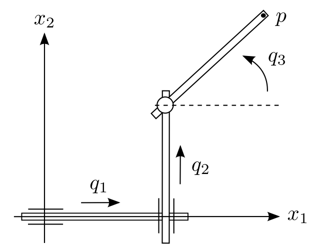


Fig. 3. A planar PPR manipulator

Letting $\hat{g}(q) = q_1 + q_2 - q_3 + s_3 + c_3$, it can be shown that $\hat{J}_G = \partial \hat{g} / \partial q$ is such that $\hat{J}_G \hat{G} \equiv 0$ and the extended Jacobian $\hat{J}_A$ is non singular, where:

$$\hat{J}_A = \begin{bmatrix} 1 & 0 & -s_3 \\ 0 & 1 & c_3 \\ 1 & 1 & -1+c_3-s_3 \end{bmatrix} \tag{24}$$

Moreover a basis of the null space of $\hat{J}_G$ is the following matrix:

$$\hat{N}_G = \begin{bmatrix} 1 & 0 \\ -1 & 1+s_3-c_3 \\ 0 & 1 \end{bmatrix} \tag{25}$$

If we then let $\hat{H} = \begin{bmatrix} \hat{N}_G & N \end{bmatrix}$ it is straightforward to verify that $\hat{G} = \hat{H}\hat{H}^T J^T \left(J\hat{H}\hat{H}^T J^T\right)^{-1}$. In other words, the method (23) like any other holonomic method is represented by the general expression (7), as claimed in Theorem III.2.
In order to confirm the effectiveness of the proposed method some simulations have been run. The proposed holonomic method G is compared to the Moore-Penrose pseudo-inverse $\bar{G} = J^T \left(JJ^T\right)^{-1}$ which is non-holonomic for the given manipulator.
Starting from the same joint configuration, a half of a circumference trajectory centered in $x_c = 1.2070$ m, $y_c = 0.4036$ m with radius $r = 0.4024$ m is used as input for the kinematic inversion local control strategies. The path is covered clockwise and counter-clockwise. Figs. 4 and 5 show the time histories of the joint variables obtained with the two local control strategies. As one can see in Fig. 4, the final configuration of the robot depends on the covered path when a non-holonomic local control strategy was used. On the other hand, Fig. 5 shows the motion computed with a holonomic local control strategy: the final configuration does

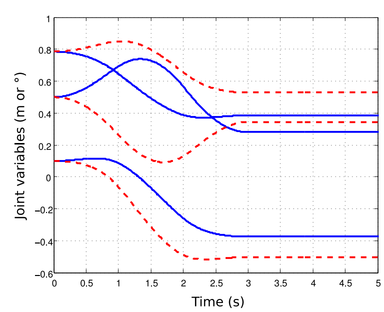


Fig. 4. Moore-Penrose local control strategy: clockwise (solid blue) and counter-clockwise (dashed red)

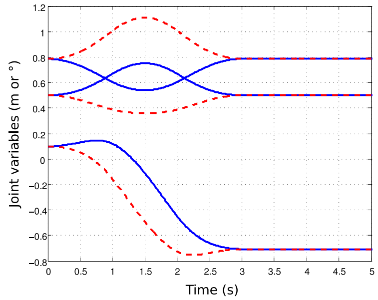
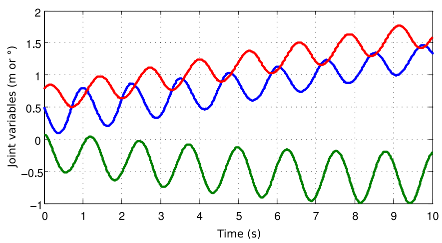
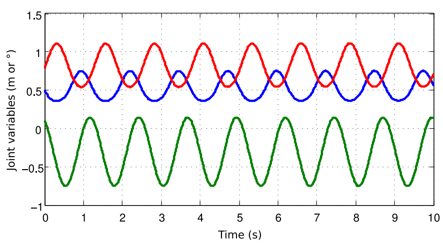


Fig. 5. Holonomic local control strategy: clockwise (solid blue) and counter-clockwise (dashed red)

not depend on the end-effector path, but only on the end-effector final configuration, as expected. Moreover, in Fig. 6



(a) Moore-Penrose local control strategy



(b) Holonomic local control strategy

Fig. 6. Behaviour on a cyclic end-effector trajectory

the planned joint variables (computed with a holonomic and a non-holonomic local control strategy) during a cyclic end-effector motion are shown. As one can see, the holonomic strategy, Fig. 6(b), ensures the repeatability of the motion, while the non-holonomic strategy, Fig. 6(a), results in a significant trend which may lead, in general, to a singular configuration or to a joint limit.

V. Experimental verification

In order to further verify the proposed method some experimental tests have been carried out on a real manipulator. The 6 axes ABB IRB-140 industrial manipulator was used for this purpose. The manipulator is redundant with respect to the task of positioning the TCP with unspecified orientation ($n = 6 > m = 3$). The Denavit-Hartenberg parameters of the manipulator are listed in Tab. I. Let J be the Jacobian of the manipulator partitioned into the position Jacobian J_p and

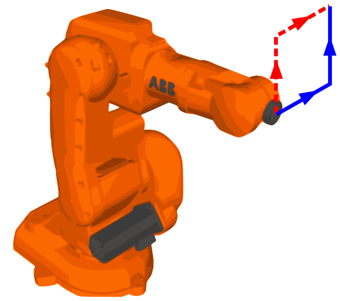


Fig. 7. ABB IRB-140, 6 axes manipulator

TABLE I
DENAVIT-HARTENBERG PARAMETERS

axis, i	$a_i[m]$	$d_i[m]$	α_i	θ_i
1	0.07	0	90	q_1
2	0.36	0	0	q_2
3	0	0	90	q_3
4	0	0.38	-90	q_4
5	0	0	90	q_5
6	0	0.065	0	q_6

the orientation Jacobian J_o (according to ZYZ Euler angles representation):

$$J = \begin{bmatrix} J_p \\ J_o \end{bmatrix} \tag{26}$$

The following local control strategy is taken into account:

$$G = HH^T J_p^T \left(J_p HH^T J_p^T \right)^{-1} \tag{27}$$

where $H = \begin{bmatrix} N_G & N \end{bmatrix}$, $N = null\left(J_p\right)$ and $N_G = null\left(J_o + J_p\right)$. This way, the following holonomic constraint between Euler angles and the Cartesian coordinates of the end-effector is enforced:

$$J_p \dot{q} + J_o \dot{q} = 0 \tag{28}$$

Since N_G is a null-space basis of a Jacobian matrix, the distribution associated to N_G is involutive (see Lemma III.1). Moreover, since the augmented Jacobian

$$J_A = \begin{bmatrix} J_p \\ J_p + J_o \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ I_3 & I_3 \end{bmatrix} J \tag{29}$$

is non-singular away from singularities of J, then H is non-singular, as well. Therefore the proposed method is holonomic.
Starting from the same joint configuration, a half of a square trajectory laying on the vertical plane with side length $L = 0.1\ m$ is used as input for the proposed local control strategy. The scenario is sketched in Fig. 7. As one can see from Fig. 8, even if the final TCP position is reached with different paths, the final posture of the manipulator is the same and does not depend on the followed task trajectory, as expected. The accompanying video shows the experiments described here.

VI. Conclusions and future work

In this paper, a general parameterization of all holonomic local control strategies is proposed. It has been shown that only the knowledge of a null-space basis of the Jacobian

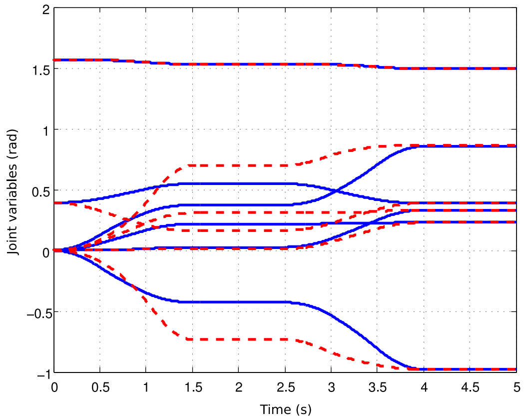


Fig. 8. Holonomic local control strategy: clockwise (blue) and counter-clockwise (dashed red)

matrix is needed in order to design all the possible holonomic control strategies. Moreover, the parameterization does not depend on the choice of the minimal representation of the orientation of the end-effector.

Thanks to the proposed parameterization it is possible to design a holonomic kinematic control strategy for a given redundant manipulator. Further research is required in order to parameterize the space of all possible matrices N_G that generate all the holonomic strategies.

VII. Appendix

Proof: [Proof of Lemma III.1] Since N is a basis of the null space of J it follows that[2]

$$\forall(k,i) : \sum_s J_{ks} N_{si} = 0 \tag{30}$$

Computing the partial derivative with respect to q_u of equation (30), we obtain

$$\forall(k,i,u) : \sum_s \frac{\partial J_{ks}}{\partial q_u} N_{si} + J_{ks} \frac{\partial N_{sj}}{\partial q_u} = 0 \tag{31}$$

similarly, changing index i with j:

$$\forall(k,j,u) : \sum_s \frac{\partial J_{ks}}{\partial q_u} N_{sj} + J_{ks} \frac{\partial N_{sj}}{\partial q_u} = 0 \tag{32}$$

Multiplying equation (31) by N_{uj} and (32) by N_{ui} and subtracting

$$\begin{aligned} \forall(k,i,j,u) : &\sum_s N_{uj} \frac{\partial J_{ks}}{\partial q_u} N_{si} + N_{uj} J_{ks} \frac{\partial N_{si}}{\partial q_u} - \\ &\sum_s N_{ui} \frac{\partial J_{ks}}{\partial q_u} N_{sj} + N_{ui} J_{ks} \frac{\partial N_{sj}}{\partial q_u} = 0 \end{aligned} \tag{33}$$

Summing over all possible values of u

$$\begin{aligned} \forall(k,i,j) : &\sum_{u,s} N_{uj} \frac{\partial J_{ks}}{\partial q_u} N_{si} + N_{uj} J_{ks} \frac{\partial N_{si}}{\partial q_u} - \\ &\sum_{u,s} N_{ui} \frac{\partial J_{ks}}{\partial q_u} N_{sj} + N_{ui} J_{ks} \frac{\partial N_{sj}}{\partial q_u} = 0 \end{aligned} \tag{34}$$

[2] A_{ij} is the element of A on the i-th row and the j-th column

Since the formula (34) is closed with respect to u and s and the Hessian matrix is symmetrical, the first and the third term vanish once the sum is expanded. Then

$$\forall(k,i,j) : \sum_s J_{ks} \sum_u N_{uj} \frac{\partial N_{si}}{\partial q_u} - N_{ui} \frac{\partial N_{sj}}{\partial q_u} = 0 \tag{35}$$

Using the notion of Lie Brackets, see [9], equation (35) is equivalent to the following one:

$$\forall(i,j) : J\,[N_i, N_j] = 0 \tag{36}$$

Then, since $[N_i, N_j] \in Null\,(J) \equiv Range\,(N)$, the distribution associated to N is involutive. ■

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