2010 IEEE International Conference on Robotics and Automation
Anchorage Convention District
May 3-8, 2010, Anchorage, Alaska, USA

An Analysis of the Operational Space Control of Robots

Ngoc Dung Vuong, Marcelo H Ang Jr, Tao Ming Lim, Ser Yong Lim

***Abstract*— Theoretically, the operational space control framework [1] can be regarded to be the most advanced control framework for redundant robots. However, in practice, the control performance of this framework is significantly degraded in the presence of model uncertainties and discretizing effects. Using the singular perturbation theory, this paper shows that the same model uncertainties can create different effects on the task space and joint space control performance. From the analysis, a multi-rate operational space control was proposed to minimize the effects of model uncertainties on the control performance and while maintaining the advantages of the original operational space framework [2]. In this paper, we present a stability analysis of the multi-rate operational space control framework using the Lyapunov's direct method.**

I. Introduction

Generally, most operational space (or task space) control approaches relate to directly closing the control loop in the operational space. Task space commands from the operational space controller are then translated into joint space commands through kinematic transformations. Operational space control approaches can be divided into three groups based on the way they handle the redundancy [3]. The first group resolves the redundancy at the velocity level [4-5] while the second [6-7] and the third groups [1] are based on the acceleration. The main difference between the second and the third group is whether the task-space and null-space are "kinematically" (second) or dynamically (third) decoupled. Because of the dynamically decoupling property, the third approach sometime has been referred as the force-based operational space control [3].

Theoretically, the force-based operational space control is one of the most advanced control framework for redundant robots. One main reason is because it uses the system's inertial matrix to weight the pseudo-inverse solution. Thus, the solution provided by this framework is an optimal solution since the instantaneous kinetic energy is minimized along the path [8]. In other words, the force-based operational space framework provides a natural choice to decouple the task-space dynamics from its internal (or null-space) dynamics. In order to address the nonlinear effects due to the link inertia, the gravity and so on in task space, Khatib introduced the concept of task-space dynamics [1]. He also suggested a model-based PD controller to achieve asymptotical performance. It is worth noting that this asymptotical performance is only valid when the robot model is accurately known and the controller is continuously implemented. However, the assumption of perfect knowledge is always violated in practice, and therefore the control performance can be significantly degraded due to the imperfect model as has been experimentally shown in [2-3]. In order to overcome this uncertainty issue, a number of approaches such as robust control approaches, adaptive control approaches and so on, have been proposed in the literature.

This paper looks at the model uncertainty issue from a different aspect: a multi-rate operational space control [2]. This approach has also been referred to as the inner-outer loop approach in the literature although the initial motivation is different. One of the main initial motivations for the inner-outer loop approach comes from the fact that most of the industrial robot come with a motion controller at each joint [9]. As a result, the task-space commands can only be realized by creating an outer loop in the operational space. On the other hand, the motivation in our case is the inadequate performance when we implemented the conventional operational space controller [1] on our experimental platform, the Mitsubishi 7-degree-of-freedom (DOF) (from now on, the force-based operational space control introduced by Khaib [1] will be referred to as the conventional operational space control). One explanation for this poor performance is the inaccuracy of the task space inertial matrix [3].

It is also worth noting that the subject of model uncertainties in control has received a considerable attention from the research community. For example, Qu [10] proved that the continuous PD computed-torque is robust with respect to the unknown dynamics. Moreover, the stability of the error system can be made asymptotically stable with a finite-high control gains if the static balancing torque is known. In practice, however, the control gains have been shown to have upper limits due to the presence of model uncertainties and discretizing effects [11]. So far, it is not known whether these upper limits of the control gains can have any adverse effects on the control performance. By analyzing the closed-loop systems of two control schemes (the digital PD joint space computed-torque control and the digital PD task space computed torque control), we provide a detailed explanation why the theoretical more advanced force-based operational space control cannot perform well in the presence of model uncertainties.

Based on the above analysis, a multi-rate control structure

Manuscript received Sep 15, 2009. This work was supported by the CRP project between the National University of Singapore and SIMTech (Singapore Institute of Manufacturing Technology).

Ngoc Dung, Vuong and Marcelo Ang. H, Jr are with the National University of Singapore, 9 Engineering Drive 1, Singapore 117576 (e-mail: <ndvuong, mpeangh>@ nus.edu.sg).

Tao Ming, Lim and Ser Yong, Lim is with SIMTech, 71 Nanyang Drive, Singapore, 638075 (email: <tmlim, sylim>@simtech.a-star.edu.sg)

which was proposed in [2] is reviewed in the next section. Note that this control structure is not new in the sense that it has been mentioned in some previous works such as [9, 12-13]. The contribution here is that the feedback linearization concept has been shifted from the operational space [1] into joint space in order to minimize the effects (if possible) of the model uncertainties. Details of the multi-rate control structure are given in section 2-3. It is also important to point out that in most of the previous works, the robot model is either perfectly known [12] or ignored [9, 13]. As a result, the performance of the inner-loop can be degraded if the nonlinear effects of the robot model are significant.

The main contributions of this paper are as follows:

- An explicit explanation on how the model uncertainties affect the control performance of the force-based operational space control.
- An analysis on the stability of the multi-rate operational space control framework under the presence of model uncertainties.
- The multi-rate operational space controller and its stability are verified experimentally.

The rest of the paper is presented as follows: firstly, the discrete high-gain computed-torque controls are analyzed by the singular perturbation theory. Next, the multi-rate operational space control is reviewed and finally, the stability of the proposed controller is given using Lyapunov's direct method.

II. Computed-torque control for joint space and task space: an analysis

A. High-gain computed-torque control

The following discussion is motivated from [11]. Consider a simple rigid dynamic model of an n-DOF robot (without gravity and joint friction) in joint space and task space:

$$A(q)\ddot{q} + C(\dot{q},q) = \Gamma \tag{1}$$

$$\Lambda(x)\ddot{x} + \mu(x,x) = F \tag{2}$$

where q and x are the generalized coordinate, Γ and F are the generalized force in joint space and task space accordingly. For simplicity, let us only consider non-redundant robots at singular-free configuration in this section. The relationship between the joint space dynamics and task space dynamics can be stated as follows [1]:

$$\Lambda = J^{-T} A J^{-1} \tag{3}$$

$$\mu = \bar{J}^{T} C - \Lambda \dot{J}\dot{q} \tag{4}$$

$$\Gamma = J^{T} F \tag{5}$$

Consider the two set-point PD computed-torque controllers in joint space and task space as follows:

$$\Gamma = \hat{A}u_q + \hat{C} = \hat{A}\left(-k_{vq}\dot{q} + k_{pq}\left(q_d - q\right)\right) + \hat{C} \tag{6}$$

$$F = \hat{\Lambda}u_x + \hat{\mu} = \hat{\Lambda}\left(-k_{vx}\dot{x} + k_{px}\left(x_d - x\right)\right) + \hat{\mu} \tag{7}$$

where q_d and x_d are the desired set point, $k_{vq,x}, k_{pq,x}$ are the control gains and $\hat{A}, \hat{C}, \hat{\Lambda}$ and $\hat{\mu}$ are the estimated/identified dynamic model of the robot. The closed-loop equations in joint space and task space become:

$$\ddot{q} = -\frac{1}{\varepsilon_q} A^{-1}\hat{A}s_q - A^{-1}\tilde{C}, \ddot{x} = -\frac{1}{\varepsilon_x}\Lambda^{-1}\hat{\Lambda}s_x - \Lambda^{-1}\tilde{\mu} \tag{8-9}$$

where:

$$\begin{cases} k_{vq} = \dfrac{1}{\varepsilon_q}, \alpha_q = \dfrac{k_{pq}}{k_{vq}}, k_{vx} = \dfrac{1}{\varepsilon_x}, \alpha_x = \dfrac{k_{px}}{k_{vx}} \\ s_q = \dot{q} - \alpha_q\left(q_d - q\right), s_x = \dot{x} - \alpha_x\left(x_d - x\right) \end{cases} \tag{10-11}$$

After introducing a fast time scale $\tau = t/\varepsilon$, the joint space and task space closed-loop system become:

$$\begin{cases} \dfrac{dq}{dt} = \dot{q} \\ \dfrac{ds_q}{d\tau} = -A^{-1}\hat{A}s_q + \varepsilon_q\left(\alpha_q\dot{q} - A^{-1}\tilde{C}\right) \end{cases}, \begin{cases} \dfrac{dx}{dt} = \dot{x} \\ \dfrac{ds_x}{d\tau} = -\Lambda^{-1}\hat{\Lambda}s_x + \varepsilon_x\left(\alpha_x\dot{x} - \Lambda^{-1}\tilde{\mu}\right) \end{cases} \tag{12-13}$$

At high-gain i.e. $\varepsilon_{q,x} \to 0$, (12-13) reduces to the fast reduced subsystem by the singular perturbation theory [14]:

$$\frac{ds_q}{d\tau} = -A^{-1}\hat{A}s_q = -\Omega_q s_q \text{ and } \frac{ds_x}{d\tau} = -\Lambda^{-1}\hat{\Lambda}s_x = -\Omega_x s_x \tag{14}$$

Note that the inertia matrices A, Λ are always positive definite, thus if the estimated/identified inertia matrices $\hat{A}, \hat{\Lambda}$ is also positive definite (please refer to [15] for detailed discussion on how to obtain an positive definite inertia matrix), the eigenvalues of $\{\Omega_q, \Omega_x\}$ will be all positive [16]. As a result, s_q and s_x tend to 0 exponentially as discussed in [11].

As is seen from above analysis, as long as the control gains can be increased, the effects of the model uncertainties (12-13):

$$\begin{cases} D_q = \alpha_q\dot{q} - A^{-1}\tilde{C} \\ D_x = \alpha_x\dot{x} - \Lambda^{-1}\tilde{\mu} \end{cases} \tag{15}$$

on the close-loop response can be made arbitrary small and the behavior of the closed-loop systems (8-9) can be defined by adjusting α_q and α_x. Practically, because the control laws (6-7) are usually implemented using digital computers, thus, the control gains will have upper limits as discussed in [11] and section IV of [17]. This observation raises a question on how these gain's limits restrict the response of the closed-loop systems (8-9) in practice.

B. Discrete high-gain computed torque control

Before discussing the effects of the discrete high-gain computed-torque control in joint space and task space, let us summarize the question in hand again:

(i) Assume that we have an identified dynamic model of the robot in joint space ($\hat{A}, \hat{C}$). The equivalent task space dynamics can be obtained using (3-4).

(ii) Let the task in joint space and task space be exactly the same i.e. $x_d = F_{KIN}(q_d)$, where F_{KIN} is the forward kinematics of the robot. In addition, let us assume that the kinematics model is accurately known.

(iii) Let the control laws (6-7) be digitally implemented with the same sampling period T, and assume that the control gains are chosen high enough so that the closed-loop systems can be approximated by (14).

The question we are interested in here is how the responses of the closed-loop systems (12-13) will be.

Let us first consider the following Lemma:

Lemma 1: *Under the above assumptions (i, ii, iii), the upper limits of the control gains of the joint space and task space controller (6-7) are the same.*

Proof: note that (14) can be rewritten as follows:

$$\frac{ds_q}{d\tau} = -A^{-1}\hat{A}s_q \Leftrightarrow \frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha_q \end{bmatrix}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ A^{-1}\hat{A}u_q \end{bmatrix} \tag{16}$$

$$\frac{ds_x}{d\tau} = -\Lambda^{-1}\hat{\Lambda}s_x \Leftrightarrow \frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha_x \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \Lambda^{-1}\hat{\Lambda}u_x \end{bmatrix} \tag{17}$$

The discrete forms of the above equations, under the assumption that the computation time of the control law is negligible, are:

$$\begin{bmatrix} q[k+1] \\ \dot{q}[k+1] \end{bmatrix} = \frac{1}{\varepsilon_q}\begin{bmatrix} \frac{1}{\alpha_q}\left(1-e^{-\alpha_q T}-\alpha_q T\right)\Omega_q & \frac{1}{\alpha_q^2}\left(1-e^{-\alpha_q T}-\alpha_q T\right)\Omega_q \\ \left(e^{-\alpha_q T}-1\right)\Omega_q & \frac{1}{\alpha_q}\left(e^{-\alpha_q T}-1\right)\Omega_q \end{bmatrix}\begin{bmatrix} q[k] \\ \dot{q}[k] \end{bmatrix} \tag{18}$$

$$\Leftrightarrow Q[k+1] = \Phi_q Q[k]$$

and

$$\begin{bmatrix} x[k+1] \\ \dot{x}[k+1] \end{bmatrix} = \frac{1}{\varepsilon_x}\begin{bmatrix} \frac{1}{\alpha_x}\left(1-e^{-\alpha_x T}-\alpha_x T\right)\Omega_x & \frac{1}{\alpha_x^2}\left(1-e^{-\alpha_x T}-\alpha_x T\right)\Omega_x \\ \left(e^{-\alpha_x T}-1\right)\Omega_x & \frac{1}{\alpha_x}\left(e^{-\alpha_x T}-1\right)\Omega_x \end{bmatrix}\begin{bmatrix} x[k] \\ \dot{x}[k] \end{bmatrix} \tag{19}$$

$$\Leftrightarrow X[k+1] = \Phi_x X[k]$$

where T is the sampling period. Substitute (3-5) into Ω_x leads to:

$$\Omega_x = \Lambda^{-1}\hat{\Lambda} = JA^{-1}\hat{A}J^{-1} = J\Omega_q J^{-1} \tag{20}$$

Thus, Φ_x can be rewritten as:

$$\Phi_x = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}\left(\frac{1}{\varepsilon_x}\begin{bmatrix} \frac{1}{\alpha_x}\left(1-e^{-\alpha_x T}-\alpha_x T\right)\Omega_q & \frac{1}{\alpha_x^2}\left(1-e^{-\alpha_x T}-\alpha_x T\right)\Omega_q \\ \left(e^{-\alpha_x T}-1\right)\Omega_q & \frac{1}{\alpha_x}\left(e^{-\alpha_x T}-1\right)\Omega_q \end{bmatrix}\right) \tag{21}$$

$$\begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}^{-1} = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}\overline{\Phi}_q\begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}^{-1}$$

Note that by similar matrix property, Φ_x and $\overline{\Phi}_q$ have the same set of eigenvalues [16]. Because the stability of (18) and (19) can only be guaranteed if and only if the eigenvalues of Φ_q and Φ_x are inside the unit circle [18] (this is where the upper limits of the control gains occur), (21) implies that the upper limits of the control gains for both systems are the same. □

From Lemma 1, it is clear that the responses of the closed-loop systems (12-13) will now depend on how significant the disturbances (15) are. The reason is because $\varepsilon_{q,x}$ cannot be arbitrarily reduced to zero to eliminate the effects of the model uncertainties, as in the continuous case.

To see the effects of model uncertainties on the closed-loop systems (12-13), let us further expand the disturbance terms (15):

$$D_q = \alpha_q\dot{q} - A^{-1}\tilde{C} \tag{22}$$

$$D_x = \alpha_x\dot{x} - \Lambda^{-1}\tilde{\mu} = J(\alpha_x\dot{q} - A^{-1}\tilde{C}) + \left(I - JA^{-1}\hat{A}J^{-1}\right)\dot{J}\dot{q} \tag{23}$$

As is seen, the joint space closed-loop system (12) is disturbed by (22) and the responses can be transformed to the operational space using the kinematics relationship between the joint space and task space. However, if the control is done in task space, the closed-loop system (13) has to cope with the disturbance (23) which is the result of the joint-space disturbance (22) multiplied by the Jacobian. Moreover, the uncertainties of the inertial matrix also appear as an extra term in the disturbance equation (23). As a result, if the kinematics model of the robot happens to magnify the modeling errors, the control performance of the task space controller (7) can be much worse than the one in joint space (6). The main reason is because the control gains cannot be further increased to compensate for the model uncertainties.

To verify the above observation, let us consider the simulation of a 1-DOF robot as shown in Figure 1:

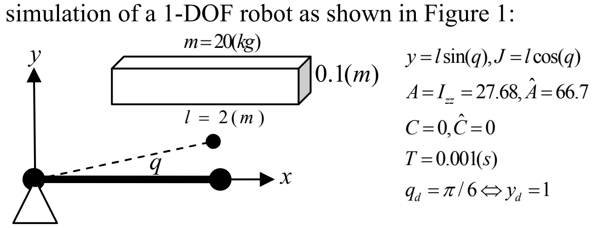
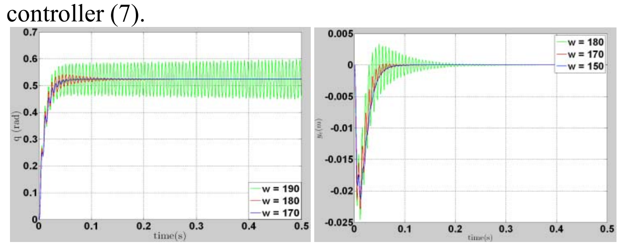


Figure 1: One-DOF arm

For simplicity, let us choose the control gains k_p, k_v as (Hurwitz polynomial):

$$k_v = 2w = \frac{1}{\varepsilon}, k_p = w^2 = \alpha\frac{1}{\varepsilon} \tag{24}$$

The control laws for joint space and task space are (6) and (7) accordingly. Based on the above discussion, the upper limit of the control gains is $w < 200$ for both joint space and task space. Figure 2(a) shows the response of the joint space controller (6) for some w (the responses in task space are similar as the previously proved). This simulation was done using SimMechanics Toolbox® under MatLAB/Simulink environment. Clearly that when w is near to the theoretical unstable value (200), chattering occurred. In order to evaluate the control performances, the difference between the task space responses $y_e = y_q - y_x$ is plotted in Figure 2(b). Here, $y_q = l\sin(q)$ is the response of the controller (6) and $y_x = l\sin(q)$ is the response of the controller (7). As is seen, the difference y_e tends to be negative which implies that the overshoot of y_q is less. In other words, under the same control gains (the maximum gains that the discrete high-gain system can take), the PD joint space controller (6) provides a better response in comparison to the task space controller (7).



(a) (b)

Figure 2: System responses with various gains

In conclusion, the analysis on this section indicates that for an inaccurate identified robot model, it is better to use this dynamic model in joint space rather than task space unless one can guarantee that the disturbance in (22) is much larger than the one in (23).

III. MULTI-RATE OPERATIONAL SPACE CONTROL

As discussed in the previous section, if the "joint space disturbance" (22) is smaller than the "task space disturbance" (23), the computed-torque control should be done in the joint space to minimize the effects of model uncertainties on the control performance. A natural question should be raised is how to get the joint space commands from tasks which are specified in the operational space. As point out in [19], the inverse kinematics approach has several disadvantages such as it ignores the robot dynamics and the complexity is significantly increased for redundant robots. On the other hand, the operational space control framework [1] is supposed to take care of the robot dynamics and dynamically decouple the null space from the task space. To maintain the advantages of the operational space framework and still minimize the model uncertainties on the control performance, the following multi-rate operational space control has been proposed [2]:

- ***Outer loop:*** the operational space command force is computed as in [1] using the identified dynamics model as a reference [2]. This task space command is then applied to the joint space identified model to get the joint acceleration command:

$$\ddot{q}_d = \hat{\bar{J}}\left(u_x - \dot{J}\dot{q}\right) + \left(I - \hat{\bar{J}}J\right)\hat{A}^{-1}\tau_{null} \tag{25}$$

$$u_x = \ddot{x}_d + K_V\left(\dot{x}_d - \dot{x}\right) + K_P\left(x_d - x\right) \tag{26}$$

where K_V, K_P is the control gains in task space, and:

$$\hat{\bar{J}} = \hat{A}^{-1}J^T\left(J\hat{A}^{-1}J^T\right)^{-1} \tag{27}$$

It is worth noting that (27) is actually a inertia-weighted pseudo-inverse at the acceleration level [8]. Thus, equation (27) will give a joint space response with respect to a task space command $\{\ddot{x}_d, \dot{x}_d, x_d\}$ through the identified dynamic model $\hat{A}$. In other words, the purpose of the outer loop is to "transform" the task space command to the joint space command using the identified dynamic model. The output of this outer loop is then realized by the PI computed-torque control at the inner loop as below.

- ***Inner loop:*** a PI computed torque controller is used to control the joint velocity of the robots. The use of the dynamics model here will enhance the performance as in [15]. The input of this controller is the reference joint velocities. The controller can be stated as:

$$\Gamma = \hat{A}u_q + \hat{C} + \hat{g} + \hat{\Gamma}_{Fric} \tag{28}$$

$$u_q = \ddot{q}_d + K(\dot{q}_d - \dot{q}) + K_I\int(\dot{q}_d - \dot{q})dt \tag{29}$$

where K, K_I are the control gains in joint space.

The desired joint velocity $\dot{q}_d$ can be obtained by integrating the desired joint acceleration (25):

$$\dot{q}(t+\Delta t)_d \triangleq \int_t^{t+\Delta t} \ddot{q}(t)_d\, dt \tag{30}$$

with the initial condition as the current $\{\dot{q}, q\}$. Clearly, if the inner velocity control loop is able to bring the manipulator from the current state $\{\dot{q}(t), q(t)\}$ to $\{\dot{q}(t+\Delta t), q(t+\Delta t)\}$ after Δt (sec), the behavior of the robot will be exactly determined by the identified dynamic model as depicted in Figure 3. Because the assumption that the inner velocity control loop can change the system states in Δt (sec) is usually violated in practice, an outer loop, which is the force-based operational space control, is always necessary to ensure the task space performance.

The efficiency of the proposed multi-rate operational space control has been experimentally verified on the Mitsubishi 7-DOF PA10 manipulator as described in the following section.

A. Experiment testbed

The proposed controller has been implemented on the Mitsubishi 7-DOF PA10 manipulator. In order to achieve real-time torque control capability (which is necessary for the inner-loop controller), the original controller of the PA10 has been replaced by our custom controllers. QNX© Neutrino Real-time Operated System (6.3) has been used to implement the above multi-rate control laws as depicted in Figure 3. Note that because the robot is redundant, the following simple null-space controller is used throughout this paper:

$$\tau_{null} = -K_{nD}\dot{q} - \nabla V_0 \tag{31}$$

$$V_0 = \frac{1}{2}(q - q_{rest})K_{nP}(q - q_{rest}) \tag{32}$$

where q_{rest} is some preferred joint configuration.

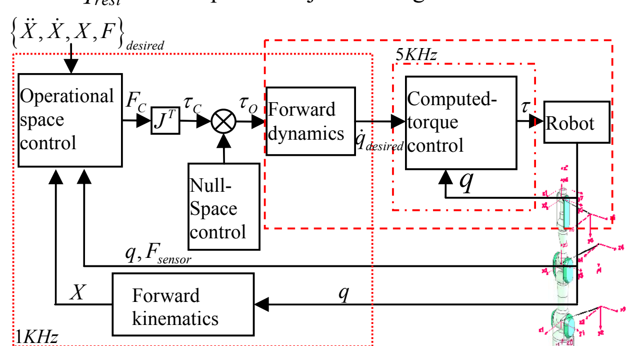


Figure 3: The proposed controller: (dot frame): outer loop, (dot-dash frame): inner loop

B. Experiment results

In order to evaluate the control performance, the end-effector of the manipulator was commanded to move 0.2 meter in the y-direction of the base frame in 2 seconds from the same initial configuration (Figure 4a) using two different controllers: (I): the conventional operational space control introduced by Khatib [1] and (II): the proposed multi-rate operational space control. Both controllers has been tuned the first sign of instability appears i.e. chattering occurs at any control variables.

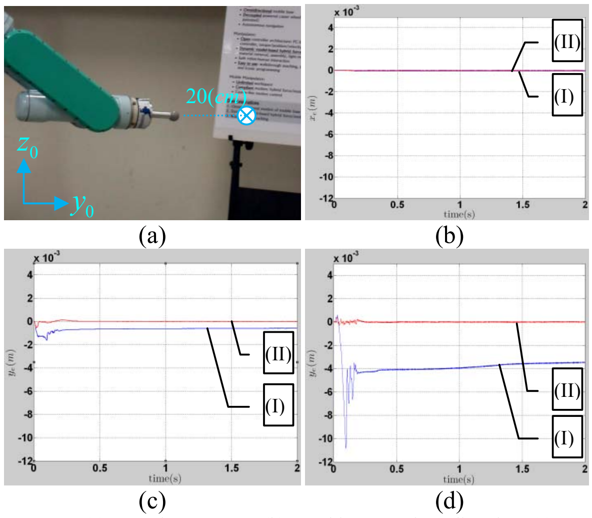


Figure 4: (a) Experiment setup. (b) Tracking error in x-direction. (c) Tracking error in y-direction. (d) Tracking error in z-direction.

As is seen in Figure 4, the tracking performance of the conventional operational space control (I) is poorer than the one using the proposed controller (II). One explanation as discussed in section 2 is because the modeling errors, such as inertial parameters and joint frictions, are magnified through the Jacobian as in (23).

IV. Stability analysis of the Multi-rate OS control

The discussion from section 2 and the experimental results from the previous section is consistent in justifying the usefulness of the proposed multi-rate operational space controller over the conventional one. However, it is still necessary to investigate the stability of the proposed controller, at least for the continuous case. It is important to stress that the stability analysis in this section only serves as a necessary condition for the usefulness of the proposed controller because we do not account for discretizing effects, signal noise and so on. If the control law is digitally implemented, the performance of the closed-loop system will now depend on how high the control gains can take as discussed in section 2. Note that only a sketch of proof is given here due to the limited space.

A. Problem statements

Consider the rigid dynamic model of an n-DOF robot as follows:

$$A(q)\ddot q + C(\dot q, q) + g(q) + \Gamma_{Fric} + D_\Gamma = \Gamma \tag{33}$$

where $\{A, C, g, D_\tau\}$ are inertial matrix, Coriolis-Centrifugal, gravity, friction torque and unknown disturbance in joint space. By applying the control law (25-29), the closed-loop control becomes:

$$\begin{cases} \ddot{\tilde x} + K_V\dot{\tilde x} + K_P\tilde x = J\dot{\tilde w} \\ \dot{\tilde w} + A^{-1}\hat A K\tilde w + A^{-1}\hat A K_I z = A^{-1}\tilde A\ddot q_d + A^{-1}\tilde H \\ \dot z = \tilde w \end{cases} \tag{34}$$

$$\Leftrightarrow \frac{d}{dt}\begin{bmatrix}\tilde x\\ \dot{\tilde x}\\ \tilde w\\ z\end{bmatrix} = \begin{bmatrix} 0 & I & 0 & 0\\ -K_P & -K_V & -JA^{-1}\hat A K & -JA^{-1}\hat A K_I\\ 0 & 0 & -A^{-1}\hat AK - A^{-1}C - \alpha A^{-1}A - A^{-1}K & -A^{-1}\hat A K_I - \alpha A^{-1}C\\ 0 & 0 & I & 0\end{bmatrix}\begin{bmatrix}\tilde x\\ \dot{\tilde x}\\ \tilde w\\ z\end{bmatrix}$$
$$+\begin{bmatrix} 0\\ JA^{-1}\tilde A\ddot q_d + JA^{-1}\tilde H\\ \left(A^{-1}C + \alpha A^{-1}A\right)\tilde w + \left(-\alpha A^{-1}C\right)z + A^{-1}\tilde A\ddot q_d + A^{-1}\tilde H\\ 0\end{bmatrix} \tag{35}$$

$$\Leftrightarrow \dot X = \Omega X + B \tag{36}$$

where:

$$\begin{aligned} &\ddot{\tilde x} = \ddot x_d - \ddot x, \dot{\tilde x} = \dot x_d - \dot x, \tilde x = x_d - x\\ &\tilde w = \dot{\tilde q} = \dot q_d - \dot q, \dot{\tilde w} = \ddot{\tilde q} = \ddot q_d - \ddot q\\ &\tilde H = C + g + \Gamma_{Fric} + D_\Gamma - \hat C - \hat g - \hat\Gamma_{Fric}\end{aligned} \tag{37}$$

Here, we are interested in the stability property of the equilibrium $\begin{bmatrix}\tilde x & \dot{\tilde x} & \tilde w & z\end{bmatrix}_e^T = \begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}$ of the nonlinear system (34). Note that there are extra terms ($-A^{-1}C - \alpha A^{-1}A - A^{-1}K$) in the Ω matrix, and they have been canceled out later in vector B. The purpose of these terms is to simplify the analysis for the nominal system as shown in the next section.

To analyze the stability of (34), we adopt the methodology proposed by Khalil [20], that is:

- Firstly, the asymptotic stability property of the nominal system $\dot X = \Omega X$ is studied.
- Next, the solution of the overall system (34) is shown to be uniformly ultimately bounded.

B. Stability of the nominal system

Consider the following Lyapunov function candidate inspired by [10]:

$$V = X^T P X \tag{38}$$

where:

$$P = \frac{1}{2}\begin{bmatrix} K_v & I & 0 & 0\\ I & I & 0 & 0\\ 0 & 0 & A & \alpha A\\ 0 & 0 & \alpha A & \alpha K + \alpha^2 A\end{bmatrix} \tag{39}$$

is positive definite when $kv > 1, k > 0, \alpha > 0$. The derivative of V is (after making use of the skew-symmetric property of the inertial matrix $X^T\left(\dot A - C - C^T\right)X = 0$):

$$\dot V = -\begin{bmatrix}\tilde x\\ \dot{\tilde x}\\ \tilde w\\ z\end{bmatrix}^T\begin{bmatrix} K_P & 0 & JA^{-1}\hat AK & JA^{-1}\hat AK_I\\ K_P & K_V - I & JA^{-1}\hat AK & JA^{-1}\hat AK_I\\ 0 & 0 & \hat AK + K & \hat AK_I\\ 0 & 0 & \alpha\hat AK & \alpha\hat AK_I\end{bmatrix}\begin{bmatrix}\tilde x\\ \dot{\tilde x}\\ \tilde w\\ z\end{bmatrix} = -X^TQX \tag{40}$$

After some manipulation, it can be shown that $Q > 0$ when:

$$\begin{cases} kp > 0, kv > (1/4)kp + 1, 1 > \alpha > 0\\ (1/2)(kp + kv - 1 - \sqrt{kp^2 + (kp - kv + 1)^2})\\ > (\alpha + 1)k^2k_H^2 / (1 + (1 + \alpha^2)\underline{\lambda} - \sqrt{(1 + (1 + \alpha^2)\underline{\lambda})^2 - 4\alpha^2\underline{\lambda}})\end{cases} \tag{41}$$

where $\underline{\lambda} = \lambda_{\min}(\hat A)$ is the smallest eigenvalues of $\hat A$. Assume that the induced norm of the Jacobian is bounded by $\|J\| < k_J$ then $k_H = k_J\lambda_{\max}(A^{-1})\lambda_{\max}(\hat A)$. As is seen, if the control gain k is fixed (i.e. after the inner-loop control is tuned), (41) can always be satisfied by increasing the task space gains $\{kp, kv\}$. As a result, the nominal system $\dot{X} = \Omega X$ is exponentially stable because:

$$\dot{V} = -X^T QX < -\lambda_{\max}(Q) \| X \|^2, \lambda_{\max}(Q) > 0 \qquad (42)$$

C. Stability of the overall system

Because the disturbance B is a function of joint position and acceleration, using the similar approach as in [21], it can be shown that:

$$\| B \| < \zeta_1 + \zeta_2 \| X \| + \zeta_3 \| X \|^2 \qquad (43)$$

where $\zeta_1, \zeta_2, \zeta_3 > 0$ are the system parameters. Using the same Lyapunov function (38), $\dot{V}$ now becomes:

$$\dot{V} = -X^T QX + X^T B \leq \| X \| (\zeta_1 + (\zeta_2 - \lambda_{\max}(Q)) \| X \| + \zeta_3 \| X \|^2) \qquad (44)$$

Thus, by applying Lemma 3.5 in [22], the overall system is uniformly ultimately bounded. Note that the purpose of this stability analysis is only to show that the proposed control law can be stabilized by a proper choice of control gains. However, in practice, the size of the uniform/uniform ultimate bound cannot be made arbitrarily small because of the upper limits on the control gains. This proposed controller performed better than the conventional operational space control because the model uncertainties has been shifted from task space to joint space as the discussion in section 2-3. Intensive experiments had been done in order to validate the performance of the proposed controller [2].

V. Conclusion

In this paper, we had explicitly shown that the control performance of the conventional operational space control can be significant degraded due to model uncertainties and discretizing effects. As a result, the computed torque technique should be done in joint space to avoid magnifying the modeling errors through the robot kinematics. In order to maintain the advantages of the force-based operational space control, a multi-rate operational space controller was proposed. Experimental results showed significant improvements in comparison to the conventional one. Stability analysis had been carried out to show that the proposed controller is stable in the continuous domain. Since the joint space control of the proposed controller is a simple PI controller, further improvements on the inner loop control can be done in order to enhance the overall performance of the system.

Acknowledgment

The authors would like to thank Prof. Frank Lewis (UTA) for all the invaluable discussions.

References

1. Khatib, O., A unified approach for motion and force control of robot manipulators: The Operational Space Formulation. *IEEE Journal of Robotics and Automation*, 1987. RA-3(1): p. 43-53.
2. Ngoc Dung, V., et al., Multi-rate perational space control of compliant motion in robotic manipulators, in *2009 IEEE SMC 2009*. 2009, San Antonio, Texas, US.
3. Nakanishi, J., et al., Operational space control: A theoretical and empirical comparison. *International Journal of Robotics Research*, 2008. 27(6): p. 737-757.
4. Whitney, D.E., Resolved motion rate control of manipulators and human prostheses. *IEEE Transactions on Man-Machine Systems*, 1969. mms-10(2): p. 47-53.
5. Nakamura, Y. and H. Hanafusa, Optimal Redundancy Control of Robot Manipulators. *The International Journal of Robotics Research*, 1987. 6(1): p. 32-42.
6. Hollerbach, J. and S. Ki, Redundancy resolution of manipulators through torque optimization. *IEEE Journal of Robotics and Automation*, 1987. 3(4): p. 308-316.
7. Hsu, P., J. Hauser, and S. Sastry. Dynamic control of redundant manipulators, *American Automatic Control Council*, 1988. Green Valley, AZ, USA.
8. Featherstone, B. and O. Khatib, Load independence of the dynamically consistent inverse of the Jacobian matrix. *IJRR*, 1997. 16(2): p. 168-70.
9. Freund, E. and J. Pesara, High-bandwidth force and impedance control for industrial robots. *Robotica*, 1998. 16: p. 75-87.
10. Zhihua, Q., et al., Robust control of robots by the computed torque law. *Systems & Control Letters*, 1991. 16(1): p. 25-32.
11. Shang-Teh, W., Digital high-gain PD control of robot manipulators. *J. of Robotic Systems*, 1997. p. 375-87.
12. Kelly, R. and J. Moreno, Manipulator motion control in operational space using joint velocity inner loops. *Automatica*, 2005. 41(8): p. 1423-32.
13. Garrido, R., E. Canul, and A. Soria. Task space robot control using an inner PD loop. *IEEE International Conference on Robotics and Automation (ICRA),* 12-17 May 2009, Piscataway, NJ, USA: IEEE.
14. Kokotovic, P.V., H.K. Khalil, and J. O'Reilly, Singular perturbation methods in control: Analysis and Design. *Classics in applied mathematics,* 1999, Philadelphia.
15. Ngoc Dung, V. and J. Marcelo H. Ang, Dynamic Model Identification For Industrial Manipulator Subject To Advanced Model Based Control, *4th International Conference HNITCCEM*. 2008: Manila, Philippines.
16. Horn, R.A. and C.R. Johnson, Topics in matrix analysis. *New York: Cambridge University Press*. viii, 607p.
17. Yabuta, T., Nonlinear basic stability concept of the hybrid position/force control scheme for robot manipulators. *IEEE Transactions on Robotics and Automation*, 1992. 8(5): p. 663-670.
18. Franklin, G.F., J.D. Powell, and M.L. Workman, Digital control of dynamic systems. 3rd, *Addison-Wesley*, 742p.
19. Bruyninckx, H. and O. Khatib. Gauss' principle and the dynamics of redundant and constrained manipulators, *Robotics and Automation, 2000. Proceedings*.
20. Khalil, H.K., Nonlinear systems. 3rd, *Upper Saddle River*, N.J. Prentice Hall, 750p.
21. Camarillo, K., et al., Stability analysis of the operational space control for industrial robots using their own joint velocity PI controllers. *Robotica*, 2008. 26(6): p.729-738.
22. Qu, Z. and D.M. Dawson, Robust tracking control of robot manipulators. *New York: IEEE Press*, 233p.