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Immobilizing 2D Serial Chains in Form Closure Grasps

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Abstract *The immobilization of non-rigid objects is a relatively unexplored area in grasp mechanics. This paper considers the immobilization of 2D serial chains of n hinged polygons at a given placement using frictionless point fingers. The paper sets the problem in the context of classical grasping theory by showing that chain immobilization can only be achieved with equilibrium grasps. In earlier work we described an immobilization procedure for serial chains of $n \neq 3$ polygons using $n+2$ frictionless point fingers. However, the immobilization of three-link chains remained an open problem. This paper establishes that three-link chains can usually be immobilized with five point fingers, but certain three-link chains can only be immobilized with six point fingers. The paper then considers the robust immobilization of serial chains under small contact placement errors. We describe a procedure for robust immobilization of n-link chains that requires only one extra contact for the entire chain, using a total of $n+3$ frictionless point fingers. The immobilization procedures and the exceptional three-link chains are illustrated with examples.*

I. INTRODUCTION

The concept of *form closure* ensures grasp and fixture safety by immobilizing the grasped object with respect to the fingertips or fixels [2, 3, 18]. The immobilization of *non-rigid* objects is a relatively unexplored area in grasp mechanics. Serial chains of hinged bodies can model a large set of non-rigid objects. Such chains arise as sub-assemblies during manufacturing operations, they appear during humanoid robot interaction with devices such as doors and windows, and are useful in the modeling of skeletal structures during robotic surgery and rehabilitation [7]. This paper focuses on the planar version of the chain immobilization problem. The paper seeks to establish upper bounds on the number of frictionless point fingers required to immobilize serial chains of hinged polygons at a given placement, and to provide procedures for synthesizing such immobilizing arrangements.

The notion of *form closure,* formulated by Reuleaux [12], assumes that a rigid object is held by rigid and stationary fingertips or fixels via frictionless contacts. It requires that all local motions of the object be prevented by the rigidity of the object and the surrounding bodies. (Force closure, which is based on constraining forces rather than constraining contacts, is dual to form closure in the absence of friction [10].) *First-order* form closure achieves object immobilization based on the contact normal directions [12]. *Second-order* form closure achieves object immobilization by additionally exploiting curvature effects at the contacts [14]. Both types of form closure can be analyzed in the object's configuration space, where the fingertips are regarded as c-space obstacles constraining the object's motions. The configuration space of a 2D rigid object has dimension $m = 3$. Based on first-order geometric effects, Markenscoff et al. [8] and Mishra et al. [9] showed that every 2D piecewise smooth object (except for a circular object) can be immobilized with $m + 1 = 4$ frictionless point fingers. Based on curvature effects, Czyzowicz et al. [6] and Rimon et. al. [13] showed that polygonal objects with non-parallel edges can be immobilized with $m = 3$ frictionless point fingers. (The latter bound applies to general piecewise smooth objects provided that the fingers have a sufficiently flat curvature at the contacts [13]).

All of these results deal with *rigid* objects. Likewise, all papers on the synthesis of immobilizing grasps and fixtures focus on rigid objects (e.g. [4, 10, 17]). The literature on non-rigid objects is mainly concerned with manipulation of large deformable objects such as flexible metal sheets (e.g. [16]), or the motion planning of non-rigid mechanisms such as serpentine robots (e.g. [11]). Closer to the chain immobilization problem is the recent work of Balkcom et. al [1] on grasping cloth polygons with "pinching" point fingers. The 2D chains considered here can eventually serve as models for deformable objects.

The paper first establishes that chain immobilization can only be achieved with equilibrium grasps. Then it describes our earlier result on second-order immobilization of 2D serial chains of $n \neq 3$ polygons by $n+2$ frictionless point fingers. The paper next describes a procedure for immobilizing 2D serial chains based on first-order geometric effects. Using only one extra contact for the entire chain, serial chains of n polygons can be immobilized by $n+3$ frictionless point fingers. Moreover, these immobilizing grasps are robust with respect to small contact placement errors. Finally the paper considers the special case of three-link chains. Focusing on three-link chains of convex polygons, the paper establishes that curvature effects can usually immobilize these chains with five point fingers. However, certain three-link chains can only be immobilized with six point fingers.

The paper's structure is as follows. Section II introduces our setup and discusses the necessity of equilibrium grasps for chain immobilization. Section III summarizes our curvature-based immobilization procedure for serial chains of $n \neq 3$ polygons. Section IV describes a robust immobilization procedure for serial chains of n polygons based on first-order effects. Section V discusses the special curvature based immobilization of three-link chains. The concluding section discusses extension of the results to the immobilization of 3D chains.

II. Preliminaries

The paper considers 2D serial chains of n rigid polygons, denoted $(\mathcal{B}_1, \ldots, \mathcal{B}_n)$. Each polygon $\mathcal{B}_{i+1}$ is attached to its predecessor $\mathcal{B}_i$ by a rotational joint θ_i, which is located at a common vertex of the two polygons. The joint θ_i allows the adjacent polygons to rotate relative to each other, but the two polygons may not overlap. A serial chain of n polygons has $n-1$ joints and therefore $n+2$ configuration parameters (the joint angles and the position and orientation of one of the chain's links). The chain's configuration space, or *c-space*, is parametrized by $q \in I\!R^{n+2}$. Given a non-overlapping placement of the chain at a configuration q_0, we wish to determine how many frictionless point fingers suffice to immobilize the chain. For practical reasons it is undesirable to place the fingers at convex vertices of the polygons. Therefore we require that the fingers contact the polygons only along edge interiors or at concave vertices. In particular, the fingers may not contact the chain's joints, since every joint is a convex vertex of at least one of the adjacent polygons.

We now introduce two types of immobility. The first type is best described in the chain's c-space. Let the chain be held at a configuration q_0 by k point fingers $p_1, \ldots, p_k$. Each finger p_i induces a c-space obstacle, or *c-obstacle,* consisting of all configurations at which the chain overlaps p_i. The c-obstacle's boundary consists of all configurations at which the chain's boundary touches p_i. When the chain is held by k fingers at q_0, the point q_0 lies at the intersection of the k c-obstacle boundaries. Chain immobility is defined as follows.

Definition 1: A chain $(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ held at a configuration q_0 by stationary point fingers $p_1, \ldots, p_k$ is **immobilized** if any local c-space trajectory of the chain which starts at q_0 penetrates the interior of one of the finger c-obstacles.

Chain immobility is based on the rigidity of its links and the constraints imposed by the point fingers. For practical reasons we also introduce the following notion of robust immobility.

Definition 2: Let a chain $(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ be immobilized by stationary point fingers $p_1 \ldots p_k$. The chain is **robustly immobilized** if it remains immobile for any local perturbation of the finger contacts along the edges of $\mathcal{B}_1, \ldots, \mathcal{B}_n$.

We now describe two properties of k-finger grasps that will serve to construct immobilizing grasps. The first property, which is a key result of this paper, asserts that equilibrium grasps are necessary for immobilization.

Theorem 1: A necessary condition for immobilization of a chain $(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ by $k \geq 2$ fingers is that the fingers hold the chain in a feasible **equilibrium grasp.**

Proof: Let the chain be held by point fingers $p_1, \ldots, p_k$ at a configuration $q_0 \in I\!R^{n+2}$. Let $\eta_i \in I\!R^{n+2}$ be the unit outward normal to the i^{th} finger c-obstacle at q_0. When a

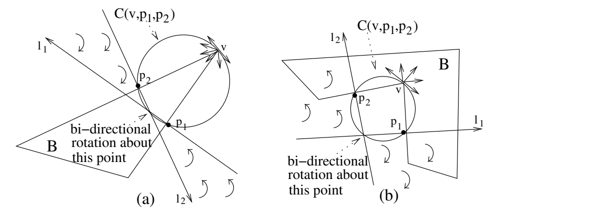

Fig. 1. The free motions of $\mathcal{B}$ with respect to two point fingers contacting edges meeting at (a) a convex, and (b) a concave vertex of $\mathcal{B}$.

finger p_i applies a force f_i on the chain, it induces a *wrench,* $\boldsymbol{w}_i \in I\!R^{n+2}$, consisting of moments about the chain's joints as well as force and torque on the link containing the chain's reference frame. Based on the virtual work principle, it can be verified that when f_i acts along the edge's inward normal, $\boldsymbol{w}_i$ is a positive multiple of η_i, $\boldsymbol{w}_i = \lambda_i \eta_i$ for $\lambda_i \geq 0$. At a k-finger equilibrium grasp the net wrench on the chain (i.e. the net moment about each joint and the net force and torque on the link containing the chain's reference frame) must be zero:

$$\lambda_1 \eta_1 + \cdots + \lambda_k \eta_k = \vec{0} \qquad \lambda_1 \ldots \lambda_k \geq 0. \qquad (1)$$

We now show that when the chain is *not* held at an equilibrium grasp, it can simultaneously escape the k fingers and is therefore *not* immobilized. Let $\mathcal{W}$ be the collection of net wrenches that can affect the chain at q_0:

$$\mathcal{W} = \left\{ \boldsymbol{w} \in I\!R^{n+2} : \boldsymbol{w} = \lambda_1 \eta_1 + \cdots + \lambda_k \eta_k \;\; \lambda_1 \ldots \lambda_k \geq 0 \right\}.$$

The set $\mathcal{W}$ forms a closed convex cone based at q_0. When the fingers do not form a feasible equilibrium grasp, $\mathcal{W}$ does not contain any full one-dimensional line passing through q_0 (such a line can only be generated by two opposing rays and is associated with an equilibrium grasp involving at least two fingers). The cone *dual* to $\mathcal{W}$ is given by $\mathcal{W}^* = \{h \in I\!R^{n+2} : h \cdot \boldsymbol{w} \leq 0 \text{ for all } \boldsymbol{w} \in \mathcal{W}\}$. The dual cone must have a non-empty interior, otherwise $(\mathcal{W}^*)^* = \mathcal{W}$ contains a full one-dimensional line. Every vector h from the interior of $\mathcal{W}^*$ satisfies $h \cdot \boldsymbol{w} < 0$ for all $\boldsymbol{w} \in \mathcal{W}$. Let h be a vector from the interior of $\mathcal{W}^*$. Let $\dot{q} = -h$ be a tangent vector representing a particular instantaneous motion of the chain at q_0. Since $\eta_i \in \mathcal{W}$ for $i = 1 \ldots k$, $\dot{q}$ satisfies the inequality $\eta_i \cdot \dot{q} > 0$ for $i = 1 \ldots k$. Since the c-space trajectory $q(t)$ such that $q(0) = q_0$ and $\dot{q}(0) = \dot{q}$ moves away from the k finger c-obstacles, the chain is *not* immobilized by the k fingers. Equilibrium grasp is thus necessary for immobilization. □

The following lemma bounds the area where a vertex of a polygon can lie when it is contacted by two point fingers. Given a vertex v and stationary point fingers at p_1 and p_2, let $C(v, p_1, p_2)$ be the unique circle through v, p_1, and p_2. Let $\mathsf{int}\big(C(v, p_1, p_2)\big)$ denote the *interior* of $C(v, p_1, p_2)$ including its bounding circle; let $\mathsf{ext}\big(C(v, p_1, p_2)\big)$ denote the *exterior* of $C(v, p_1, p_2)$ including its bounding circle.

Lemma 2.1 (Circle Lemma): Let two point fingers fixed at p_1 and p_2 contact a polygon $\mathcal{B}$ along two edges incident to its vertex v. The free motions of $\mathcal{B}$ cause v to locally move into $\mathsf{int}\big(C(v, p_1, p_2)\big)$ when v is a convex vertex, and into $\mathsf{ext}\big(C(v, p_1, p_2)\big)$ when v is a concave vertex (Figure 1).

A proof of the lemma appears in [5, 15]. An intuitive feel for the lemma can be gained with Reuleaux's graphical technique [12]. Let l_i be the line through p_i directed along $\mathcal{B}$'s inward normal at p_i ($i = 1, 2$). The instantaneous free motions of $\mathcal{B}$ with respect to p_i are counterclockwise rotations about points on the left of l_i, clockwise rotations about points on the right of l_i, and bi-directional rotations about points on l_i. The instantaneous free motions of $\mathcal{B}$ with respect to *both* p_1 and p_2 are bi-directional rotation about the intersection point of l_1 and l_2, and two regions of uni-directional rotations. The bi-directional rotation moves v along the tangent to $C(v, p_1, p_2)$. The uni-directional rotations move v into the halfplane tangent to $C(v, p_1, p_2)$ from its inside when v is a convex vertex (Figure 1(a)), and into the halfplane tangent to $C(v, p_1, p_2)$ from its outside when v is a concave vertex (Figure 1(b)).

III. Second-Order Immobilization of 2D Serial Chains

This section summarizes our curvature-based procedure for immobilizing 2D serial chains of $n \neq 3$ polygons [5]. The procedure starts with a single polygon.

Lemma 3.1 (Single-Link Chain): Every polygon with non-parallel edges can be immobilized by **three** frictionless point fingers.

The immobilization of a polygon $\mathcal{B}$ is based on its largest inscribed disc [6, 13]. Such a disc generically touches the boundary of $\mathcal{B}$ at two or three points. When the disc touches the boundary at three points, these points form an immobilizing three-finger equilibrium grasp. When the disc touches the boundary at two points, one point is necessarily a concave vertex. A local splitting of the point opposing the vertex immobilizes $\mathcal{B}$ in a three-finger equilibrium grasp. The immobilization procedure next considers two-link chains.

Lemma 3.2 (Two-Link Chain): Every chain of two polygons with non-parallel edges can be immobilized by **four** frictionless point fingers.

Proof sketch: Let the common vertex of $\mathcal{B}_1$ and $\mathcal{B}_2$ at the joint, v, be a convex vertex of both polygons (Figure 2(a)). In this case there exists a line l through v which separates the two links. Let $l^{\perp}$ be the line perpendicular to l at v. Construct a sufficiently small circle centered on $l^{\perp}$, which passes through v and intersects two interior points on the edges of $\mathcal{B}_i$ incident to v ($i = 1, 2$). Place four fingers p_1, p_2, p_3, and p_4 at the points where the two circles intersect the edges of $\mathcal{B}_1$ and $\mathcal{B}_2$. According to the circle lemma, v can move only inside the circles $C(v, p_1, p_2)$ and $C(v, p_3, p_4)$. Since the two circles touch only at v, this vertex is immobilized by the four point fingers. Since each pair of fingers resists opposite rotations of $\mathcal{B}_i$ about v, each $\mathcal{B}_i$ is immobilized with respect to v. The case where v is a concave vertex of $\mathcal{B}_1$ or $\mathcal{B}_2$ can be treated as depicted in Figure 2(b). $\square$

The radii of the circles constraining the two-link chain depend on the finger c-obstacles curvature. Hence this is a *curvature based* immobilization. The three-link chains form a striking exception and are discussed in Section V. Let us therefore proceed with chains of $n \geq 4$ polygons. The immobilization procedure treats these chains as consisting of

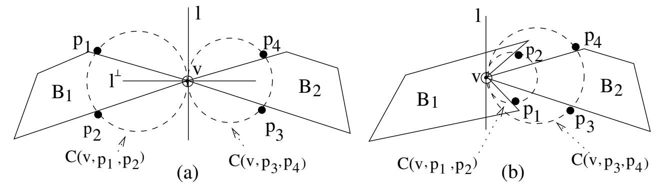

Fig. 2. Immobilization of a two-link chain by four point fingers when $\mathcal{B}_1$ and $\mathcal{B}_2$ are (a) both convex, and (b) concave and convex at the joint.

short subchains separated by two-link gaps. The following lemma considers chains containing a two-link gap.

Lemma 3.3: Let a chain $(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ of $n \geq 4$ polygons consist of two subchains separated by two links, $(\mathcal{B}_1, \ldots, \mathcal{B}_i)$ and $(\mathcal{B}_{i+3}, \ldots, \mathcal{B}_n)$. If the subchains $(\mathcal{B}_1, \ldots, \mathcal{B}_i)$ and $(\mathcal{B}_{i+3}, \ldots, \mathcal{B}_n)$ are immobilized, the entire chain $(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ is immobilized.

To illustrate the lemma, consider the four-link chain $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4)$. Let v be the common vertex of $\mathcal{B}_2$ and $\mathcal{B}_3$ at the θ_2 joint. When $\mathcal{B}_1$ and $\mathcal{B}_4$ are immobilized, v can only trace the circles centered at the θ_1 and θ_3 joints. As these circles intersect at two discrete points, one of which is the nominal position of the θ_2 joint, the entire four-link chain is immobilized when $\mathcal{B}_1$ and $\mathcal{B}_4$ are immobilized.

The number of polygons of an n-link chain can be written as $n = (n \bmod 4) + 4N$, where $n \bmod 4 = 0, 1, 2, 3$ and N is a positive integer. Hence one can break any n-link chain into an initial subchain having $n \bmod 4$ links, and N subchains each having four links. However, an initial three-link subchain is exceptional in terms of the number of point fingers required for its immobilization. Hence we will express the number n as $n = n_0 + 4N$, where $n_0 = 0, 1, 2, 7$ is the length of the initial subchain. As a preparation for the immobilization procedure, consider the immobilization of seven-link chains.

Corollary 3.4: Every chain of seven polygons can be immobilized by **nine** frictionless point fingers.

Proof: Partition the chain $(\mathcal{B}_1, \ldots, \mathcal{B}_7)$ into the single-link chains $\mathcal{B}_1$, $\mathcal{B}_4$, and $\mathcal{B}_7$, separated by the two-link chains $(\mathcal{B}_2, \mathcal{B}_3)$ and $(\mathcal{B}_5, \mathcal{B}_6)$. Each of the single links can be immobilized by three point fingers. As these links are separated by two-link gaps, the entire chain is immobilized by nine point fingers according to Lemma 3.3. $\square$

Second-order immobilization procedure:

1. Partition the n-link chain into an initial subchain of $n_0 = 0, 1, 2, 7$ links, and N four-link subchains.
2. Immobilize the initial subchain by n_0+2 frictionless point fingers using Lemmas 3.1, Lemma 3.2, and Corollary 3.4.
3. In each four-link subchain, $(\mathcal{B}_i, \mathcal{B}_{i+1}, \mathcal{B}_{i+2}, \mathcal{B}_{i+3})$, immobilize its distal two-link chain, $(\mathcal{B}_{i+2}, \mathcal{B}_{i+3})$, by four frictionless point fingers using Lemma 3.2.

The following theorem asserts that the procedure immobilizes the n-link chain with $n + 2$ frictionless point fingers.

Theorem 2 (2'nd Order Form Closure): The above procedure **immobilizes** every 2D serial chain of $n \neq 3$ polygons with non-parallel edges using $\boldsymbol{n+2}$ frictionless point fingers.

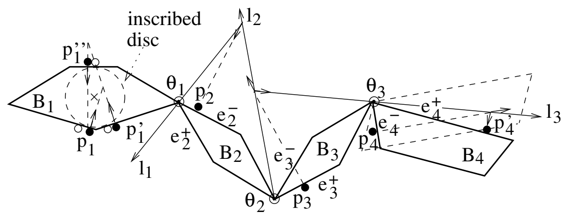

Fig. 3. Robust immobilization of a four-link chain by seven point fingers.

Proof: The procedure immobilizes the initial subchain by n_0+2 point fingers. In each subchain $(\mathcal{B}_i, \mathcal{B}_{i+1}, \mathcal{B}_{i+2}, \mathcal{B}_{i+3})$, $(\mathcal{B}_{i+2}, \mathcal{B}_{i+3})$ is immobilized by four point fingers. Since $(\mathcal{B}_i, \mathcal{B}_{i+1})$ forms a two-link gap between $(\mathcal{B}_{i+2}, \mathcal{B}_{i+3})$ and the previous subchain, the entire n-link chain is immobilized according to Lemma 3.3. Substituting $N = \frac{1}{4}(n-n_0)$, the total number of point fingers is $(n_0+2) + 4 \cdot \frac{1}{4}(n-n_0) = n+2$. $\square$

IV. Robust Immobilization of 2D Serial Chains

This section describes a robust immobilization procedure for serial chains of n polygons using first-order geometric effects. We will use the following notation. The normal line to a polygon edge at p_i is denoted $n(p_i)$. The *force line* of a finger p_i is the line $n(p_i)$ directed along the edge's inward normal. The edges of a polygon $\mathcal{B}_i$ incident to the joint θ_{i-1} are denoted e_i^- and e_i^+, according to the sign of the moment generated about the joint by the edge's inward normal. The line passing through the joints θ_{i-1} and θ_i is denoted h_i for $i > 1$. For $i = 1$, h_1 is the line through the joint θ_1 and the center of $\mathcal{B}_1$'s maximal inscribed disc.

The procedure accepts as input a serial chain $(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ with joints $\theta_1, \ldots, \theta_{n-1}$ at a given placement. The polygons $\mathcal{B}_2, \ldots, \mathcal{B}_n$ can have any shape, but $\mathcal{B}_1$ must have non-parallel edges (this assumption is relaxed in [15]). The procedure first places two point fingers on $\mathcal{B}_n$, then one point finger on each $\mathcal{B}_i$ for $1 < i < n$, and finally three point fingers on $\mathcal{B}_1$.

Robust immobilization procedure:

1. Starting with $\mathcal{B}_n$, consider the parallelogram spanned by the normals to the edges e_n^- and e_n^+ incident to θ_{n-1}. The joint line h_{n-1} partitions the parallelogram into at most two regions. Select any point x_n in the interior of these regions. Place the fingers p_n and p_n' at the points on e_n^- and e_n^+ such that the normal lines $n(p_n)$ and $n(p_n')$ intersect at x_n. Denote by l_{n-1} the net force line of p_n and p_n', which is based at x_n and acts on $\mathcal{B}_{n-1}$ through the θ_{n-1} joint (Figure 3).
2. Consider each $\mathcal{B}_i$ for $1<i<n$ in order of decreasing index. The net force line l_i does not coincide with the joint line h_i. Hence l_i generates a non-zero moment about θ_{i-1}. Set $e_i = e_i^-$ when l_i generates a *positive* moment about θ_{i-1}, and $e_i = e_i^+$ when l_i generates a *negative* moment about θ_{i-1}. Consider the line segment $\mathcal{I}_i = \{n(p) \cap l_i : p \in e_i\}$. The next joint line, h_{i-1}, partitions $\mathcal{I}_i$ into at most two subsegments. Select any point x_i from the interior of one of these subsegments. Place the finger p_i at the point on e_i such that $n(p_i) \cap l_i = \{x_i\}$. Denote by l_{i-1} the net force line of p_i and l_i, which is based at x_i and acts on $\mathcal{B}_{i-1}$ through the θ_{i-1} joint.
3. Immobilize $\mathcal{B}_1$ as follows. The maximal inscribed disc of $\mathcal{B}_1$, D_1, generically touches its boundary at two or three points. When D_1 touches the boundary at three points, place the fingers p_1, p_1', and p_1'' at these points. When D_1 touches the boundary at two points, one point is necessarily a concave vertex of $\mathcal{B}_1$ (since $\mathcal{B}_1$ has non-parallel edges), while the opposing point lies on an edge of $\mathcal{B}_1$. Place the finger p_1 at the concave vertex. Place the fingers p_1' and p_1'' at the opposing point, then perturb their position to both sides of this point.
4. Obtain a robust immobilization of $\mathcal{B}_1$ as follows. When D_1 touches the boundary of $\mathcal{B}_1$ at two points, the grasp obtained in Stage 3 is already robust. When D_1 touches the boundary of $\mathcal{B}_1$ at three points, $n(p_1)$, $n(p_1')$, and $n(p_1'')$ meet at the inscribed disc center x_1. Consider the net force line l_1 acting on $\mathcal{B}_1$ through the θ_1 joint. When l_1 generates a positive moment about x_1, perturb the three point fingers (in the same direction) such that $n(p_1)$, $n(p_1')$, and $n(p_1'')$ would generate a *negative* net moment about x_1. When l_1 generates a negative moment about x_1, perturb the three point fingers such that $n(p_1)$, $n(p_1')$, and $n(p_1'')$ would generate a *positive* net moment about x_1. In either case, the perturbations should be sufficiently small to prevent the triangle bounded by $n(p_1)$, $n(p_1')$, and $n(p_1'')$ from intersecting the force line l_1.

Example: Consider the four-link chain of Figure 3. The first step is to place the point fingers p_4 and p_4' in a way that resists rotations of $\mathcal{B}_4$ about θ_3. The net force line of p_4 and p_4', l_3, acts on $\mathcal{B}_3$ through the θ_3 joint and generates a negative moment about θ_2. Hence the point finger p_3 is placed on the edge e_3^+ of $\mathcal{B}_3$. This is followed by a placement of p_2 on the edge e_2^- of $\mathcal{B}_2$. Finally, the inscribed disc of $\mathcal{B}_1$ touches its boundary at three points. The point fingers p_1, p_1', and p_1'' are first placed at these points. Since the net force line l_1 generates a negative moment about the disc's center, x_1, the three fingers are perturbed in counterclockwise direction as to generate a positive net moment about x_1. The resulting seven-finger grasp immobilizes the chain as stated in the following theorem.

Theorem 3 (1'st Order Form Closure): The above procedure **immobilizes** the serial chain $(\mathcal{B}_1, \ldots, \mathcal{B}_n)$ using $\boldsymbol{n+3}$ frictionless point fingers.

Proof sketch: Consider the chain's polygons from $\mathcal{B}_n$ to $\mathcal{B}_1$. The fingers p_n and p_n' jointly prevent rotations of $\mathcal{B}_n$ about θ_{n-1}. So when θ_{n-1} cannot move the polygon $\mathcal{B}_n$ is immobilized. Similarly, the net force line l_i and the point finger p_i prevent rotations of $\mathcal{B}_i$ about the joint θ_{i-1}. So when θ_{i-1} cannot move the polygon $\mathcal{B}_i$ is immobilized for $i > 1$. Finally, when $\mathcal{B}_1$ has non-parallel edges its maximal inscribed disc induces an immobilizing three-finger grasp [13, 17]. This result also extends to the case where $\mathcal{B}_1$ has parallel edges [15]. Once $\mathcal{B}_1$ is immobilized its joint θ_1 is immobilized. It now follows by induction that the $n+3$ point fingers $p_1, p_1', p_1'', p_2, \ldots, p_{n-1}, p_n, p_n'$ immobilize the entire chain. $\square$

The following proposition asserts that the immobilizing grasps are robust with respect to small finger placement errors.

Proposition 4.1 (Robust Immobilization): The above pro-

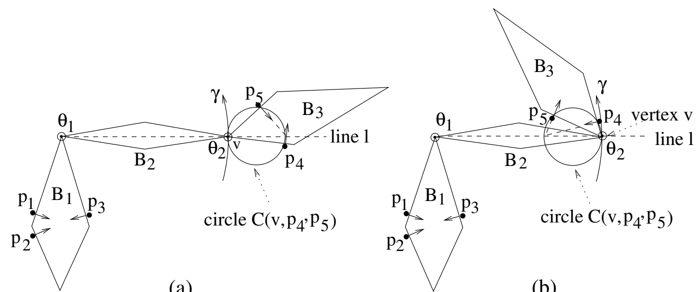

Fig. 4. A three-link chain held in a $(3, 0, 2^*)$ equilibrium grasp such that $C(v, p_4, p_5)$ is tangent to γ from its (a) outside, and (b) inside.

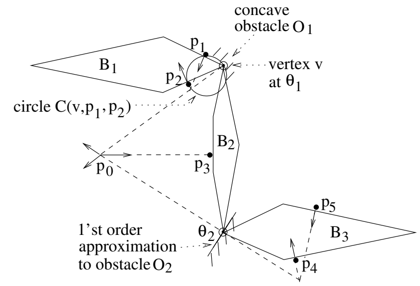

Fig. 5. A three-link chain held in a $(2^*, 1, 2)$ equilibrium grasp.

cedure **robustly** immobilizes the serial chain $(\mathcal{B}_1, \ldots, \mathcal{B}_n)$.

Proof sketch: Any perturbation of p_n and p'_n along the edges e_n^- and e_n^+ continues to prevent rotations of $\mathcal{B}_n$ about θ_{n-1}. The distance of the intersection point of $n(p_n)$ and $n(p'_n)$, x_n, from the line h_{n-1} is $\epsilon_n > 0$. Hence a sufficiently small perturbation (depending on ϵ_n and the orientations of e_n^- and e_n^+) of p_n and p'_n will prevent the perturbed intersection point, $\tilde{x}_n$, from crossing the line h_{n-1}. Hence the perturbed net force line, $\tilde{l}_{n-1}$, still generates the *same moment sign* about θ_{n-2}. Similarly, any small perturbation of p_i along the edge e_i induces a perturbed force line $\tilde{l}_i$ which still generates the *same moment sign* about about θ_{i-2} (or x_1 if $i = 2$) for all $i > 2$. Finally, suppose D_1 touches three edges of $\mathcal{B}_1$. All sufficiently small perturbations of p_1, p'_1, and p''_1 will keep the center of D_1 within the triangle bounded by $n(p_1)$, $n(p'_1)$, and $n(p''_1)$, as well as keep the triangle from crossing the perturbed force line $\tilde{l}_1$. It follows that the n-link chain is robustly immobilized by the $n+3$ point fingers. □

V. The Special Case of Three-Link Chains

Let us focus on three-link chains of convex polygons and show that a large class of these chains can be immobilized by five point fingers. Consider a three-link chain, $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$, with joints θ_1 and θ_2. Let (i, j, k) denote the number of fingers assigned to the three links. Since the fingers must form an equilibrium grasp, the fingers contacting $\mathcal{B}_1$ must generate zero net moment about θ_1, while the fingers contacting $\mathcal{B}_3$ must generate zero net moment about θ_2. Assuming that the edge normals of $\mathcal{B}_1$ and $\mathcal{B}_3$ do not pass through the θ_1 and θ_2 joints, equilibrium requires that $\mathcal{B}_1$ and $\mathcal{B}_3$ each be held by at least *two* fingers. The possible five-finger equilibrium grasps are thus $(2, 1, 2)$, $(2, 0, 3)$, and $(3, 0, 2)$. Let $(2^*, 1, 2)$ and $(2^*, 0, 3)$ denote grasps in which two fingers contact the edges of $\mathcal{B}_1$ incident to θ_1. Let $(2, 1, 2^*)$ and $(3, 0, 2^*)$ denote grasps in which two fingers contact the edges of $\mathcal{B}_3$ incident to θ_2. The following lemma describes how to immobilize three-link chains using $(2^*, 0, 3)$ or $(3, 0, 2^*)$ equilibrium grasps.

Lemma 5.1: When a chain of three convex polygons with non-parallel edges can be held in a $(2^*, 0, 3)$ or a $(3, 0, 2^*)$ equilibrium grasp, there exists an immobilizing five-finger grasp of the chain (Figure 4).

Proof sketch: Let the chain be held in a $(3, 0, 2^*)$ equilibrium grasp. Let the fingers p_4 and p_5 contact the edges of $\mathcal{B}_3$ incident to θ_2. Equilibrium grasp requires that the net force line of p_4 and p_5 pass through the θ_2 joint. Since the middle polygon $\mathcal{B}_2$ is held in equilibrium grasp, the net force line of p_4 and p_5 must also pass through the θ_1 joint and hence coincide with the line passing through the two joints, denoted l. The edge normals at p_4 and p_5 thus intersect on l, and in this case the center of the circle $C(v, p_4, p_5)$ also lies on l [15].

Now re-locate the three fingers contacting $\mathcal{B}_1$ such that this polygon would become immobilized (Lemma 3.1). The vertex v of $\mathcal{B}_3$ at the θ_3 joint can only trace a circular arc, γ, centered on the θ_1 joint. Since the center of $C(v, p_4, p_5)$ lies on l, γ is tangent to $C(v, p_4, p_5)$ at v. Since v is a convex vertex of $\mathcal{B}_3$, it can locally move only inside the circle $C(v, p_4, p_5)$ (Lemma 2.1). Let $l^\perp$ be the line perpendicular to l at v. When γ and $C(v, p_4, p_5)$ lie on opposite sides of $l^\perp$, v is clearly immobilized (Figure 4(a)). When γ and $C(v, p_4, p_5)$ lie on the same side of $l^\perp$, one can slide the fingers p_4 and p_5 toward the θ_2 joint until the radius of γ is strictly larger than the radius of $C(v, p_4, p_5)$. In this case, too, the vertex v is immobilized (Figure 4(b)). Once $\mathcal{B}_2$ is immobilized, $\mathcal{B}_3$ is also immobilized since p_4 and p_5 resist opposite rotations of $\mathcal{B}_3$ about θ_2. □

The remaining five-finger equilibrium grasps are $(2, 1, 2)$ arrangements. The next lemma describes how to immobilize 3-link chains using $(2^*, 1, 2)$ or $(2, 1, 2^*)$ equilibrium grasps.

Lemma 5.2: When a chain of three convex polygons with non-parallel edges can be held in a $(2^*, 1, 2)$ or a $(2, 1, 2^*)$ equilibrium grasp, there exists an immobilizing five-finger grasp of the chain (Figure 5).

Proof sketch: Let the chain be held in a $(2^*, 1, 2)$ equilibrium grasp. Let p_1 and p_2 contact $\mathcal{B}_1$, p_3 contact $\mathcal{B}_2$, and p_4 and p_5 contact $\mathcal{B}_3$. Let v be the vertex of $\mathcal{B}_1$ at the θ_1 joint. Since v is a convex vertex, it can locally move only inside the circle $C(v, p_1, p_2)$. Let u be the vertex of $\mathcal{B}_2$ attached to v at the θ_1 joint. From u's perspective $C(v, p_1, p_2)$ forms a fixed "obstacle," $\mathcal{O}_1$, constraining its motions. As illustrated in Figure 5, the obstacle $\mathcal{O}_1$ lies on the exterior of $C(v, p_1, p_2)$ and is therefore *concave* at its contact with u. Let u' be the vertex of $\mathcal{B}_2$ at the θ_2 joint. The fingers p_4 and p_5 contacting $\mathcal{B}_3$ also induce a fixed "obstacle," $\mathcal{O}_2$, constraining the motions of u'. However, the boundary of $\mathcal{O}_2$ can have any shape at u'.

Let us analyze the free motions of $\mathcal{B}_2$ as a free body held by the "fingers" $\mathcal{O}_1$, $\mathcal{O}_2$, and the point finger p_3. The forces generated by $\mathcal{O}_1$ and $\mathcal{O}_2$ act on $\mathcal{B}_2$ at u and u'. Since $\mathcal{B}_2$ is held in a three-finger equilibrium grasp, the finger force lines must intersect at a common point, p_0, such that their directions positively span the origin of $\mathbb{R}^2$ (Figure 5). The only instantaneous free motion of $\mathcal{B}_2$ with respect to

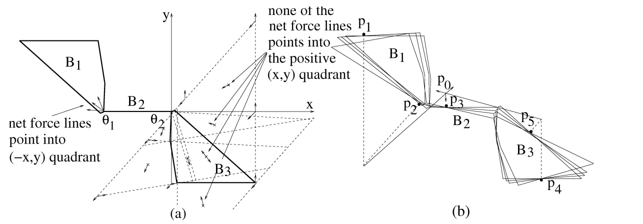

Fig. 6. A likely counter example of a three-link chain that cannot be immobilized by five point fingers.

the three fingers consists of pure rotation about p_0. When curvature effects prevent this instantaneous rotation, $\mathcal{B}_2$ is immobilized by the three fingers. One needs to verify that the *c-space relative curvature form* [15] is negative along this instantaneous rotation. The obstacle $\mathcal{O}_1$ is bounded by the circle $C(v, p_1, p_2)$. This circle can be made arbitrarily small by sliding the fingers p_1 and p_2 toward the θ_1 joint. Since the curvature of $\mathcal{O}_1$ attains an arbitrary large negative value during this sliding, the c-space relative curvature becomes negative (hence immobilizing) for all placements of p_1 and p_2 sufficiently close to θ_1 [15]. $\square$

A likely counter example: Three-link chains of convex polygons can usually be immobilized by five point fingers based on Lemmas 5.1-5.2. However, Figure 6(a) depicts a symmetric three-link chain which does not satisfy the requirements of these lemmas. The chain cannot be held in a $(2^*, 0, 3)$ equilibrium grasp, since the edges of $\mathcal{B}_1$ adjacent to θ_1 do not support a net force line passing through both joints. Let us verify that the chain cannot be held in a $(2^*, 1, 2)$ equilibrium grasp. All the net force lines generated by the edges of $\mathcal{B}_1$ adjacent to θ_1 point into the $(-x, y)$ quadrant at the θ_1 joint (Figure 6(a)). The single finger contacting $\mathcal{B}_2$ can only apply a vertical force. Hence $\mathcal{B}_2$ is held in equilibrium grasp when the net force line of the two fingers contacting $\mathcal{B}_3$ points into the positive (x, y) quadrant at the θ_2 joint. However, none of the net force lines associated with the edge pairs of $\mathcal{B}_3$ points into this quadrant at the θ_2 joint. It follows that the chain cannot be held in a $(2^*, 1, 2)$ equilibrium grasp.

A full investigation of the chain's free motions is under progress. One has to determine all possible $(3, 0, 2)$ and $(2, 1, 2)$ equilibrium grasps of the chain, then verify the existence of free motions in each of these grasps. Figure 6(b) shows a feasible $(2, 1, 2)$ equilibrium grasp, as well as a motion which allows the chain to escape the contacting fingers.

VI. Conclusion

This paper considered the immobilization of 2D serial chains of n polygons by frictionless point fingers. The paper established that chain immobilization can only be achieved with equilibrium grasps. Then it described two immobilization approaches based on first and second-order geometric effects. Based on curvature effects, all serial chains of $n \neq 3$ polygons with non-parallel edges can be immobilized by $n+2$ frictionless point fingers. Based on first-order geometric effects, all serial chains of n polygons can be immobilized by $n+3$ frictionless point fingers, such that the grasps are robust with respect to small contact placement errors. Note that a linear joint can be modeled as a rotational joint with axis at infinity. The immobilization techniques of this paper thus extend to serial chains having any mix of rotational and linear joints.

Our current work focuses on the immobilization of 3D serial chains of n polyhedral bodies. In contrast with 2D chains, there is a rich variety of joint types as well as inter-link attachments. A single 3D rigid polyhedron can be immobilized by *four* frictionless point fingers based on curvature effects, and by *seven* frictionless point fingers based on first-order geometric effects. What would be the corresponding bounds for 3D serial chains? A more practical objective concerns with the immobilization of the human spine for robotic surgery and rehabilitation. Which 3D serial chain is most suitable for modeling the spine? Which immobilization objectives are most useful for such medical applications? These are challenging problems that we plan to pursue in future research.

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