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Recursive convex replanning for the trajectory tracking of wheeled mobile robots

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***Abstract*— The article consider the Cartesian trajectory tracking of wheeled mobile robots to be performed by a hybrid control scheme with feedforward inverse control and a state feedaback that is only updated periodically and relies on a recursive convex replanning of the reference trajectory. This approach applied to the standard unicycle model is shown to maintain its efficacy also in presence of noise or unmodeled robot dynamics. Explicit, sufficient conditions are provided to ensure global boundedness of the tracking error. Experimental results are presented using Lego Mindstorm mobile robots.**

I. Introduction

The motion control of wheeled mobile robots (WMRs) is a research subject that still attracts researchers providing new contributions an improvements. This is probably due a variety of reasons e.g. the growing importance of autonomous navigation, the emergence of new application fields (e.g. service robotics, etc.), the integration of new sensor devices (laserscanners, vision cameras, etc.). A recent survey on this subject with an extensive bibliography has appeared in [1].

Focusing on the specific problem of trajectory tracking of WMRs, relevant global control methods were proposed based on nonlinear feedback laws [2], feedback linearization [3], [4], and integrator backstepping [5]. These methods and others as reported in [1] rely on continuous-time feedback from the WMR's state and on the absence of noise affecting the robot model in the analysis of the asymptotic tracking properties.

However, there are cases where continuous-time or high-frequency revelation of the robot state is not available or not economical and therefore only low-frequency state feedback may be practicable. With this scenario, the WMR motion is commanded by feedforward (i.e. open-loop) control inputs and periodically the feedforward signals are adjusted to compensate the inevitable motion errors. This kind of hybrid feedforward/feedback strategy has been chosen in [6] for the asymptotic stabilization of fixed robot (or system) postures and also in [7], [8] where it was denominated *iterative state steering*. The usefulness of feedforward/feedback strategies was also shown in [9] where an application to lane following of a vision-based autonomous car was developed using iterative steering and feedforward inverse control.

In this paper within the framework of hybrid feedforward/feedback control schemes we propose a trajectory tracking problem of a WMR modeled by a unicycle model affected by norm-bound noise. Given a desired, feasible Cartesian trajectory to be tracked by the WMR, the proposed control scheme uses a recursive convex replanning method to compute a new reference trajectory whenever the WMR's state is real-time available at a frequency assigned by the replanning time period T (cf. Section II). Then this new reference trajectory that is still feasible is used to generate the feedforward inverse command velocities that help in reducing the tracking errors (see Figure 5). If the replanning period T is sufficiently small relative to the noise magnitude, explicit closed-form bounds on the global tracking error are provided (cf. Corollary 1). In such a way a "practical" tracking convergence to the desired trajectory is achieved.

The paper is organized as follows. Section II exposes the recursive convex replanning method, poses the problem of finding conditions that guarantee global boundedness of the tracking errors in spite of noise and provides these conditions in Corollary 1. Section III introduces the recursive replanning method for general nonlinear systems and presents a main result (Proposition 1) over which Corollary 1 is deduced by means of the deductions and calculations reported in Section IV. Simulation results with a comparison with a classic control scheme are exposed in Section V. Experimental results using Lego Mindstorm mobile robots are presented in Section VI. Concluding remarks end the article in Section VII.

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II. Trajectory tracking for the unicycle

This section presents the recursive tracking approach discussed in this paper in the case of the kinematic unicycle.

Consider the following model for the unicycle (see Figure 1)

$$\left\{ \begin{array}{rcl} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} & = & v(t) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \eta(t) \\ \dot{\theta} & = & \omega(t) + \eta_\theta(t) \end{array} \right. , \qquad (1)$$

where $(x, y) \in \mathbb{R}^2$ is the position of the center of the unicycle, θ is the orientation angle and v, ω are the velocity control inputs, and set $z = (x, y, \theta)$. Functions η and η_θ are noise terms that satisfy the bounds $\forall t \in \mathbb{R}$

$$\left\{ \begin{array}{rcl} \|\eta(t)\| & \leq & N \, , \\ |\eta_\theta(t)| & \leq & N_\theta \, . \end{array} \right. \qquad (2)$$

When the noise terms are not present, (i.e. $N = 0$ and $N_\theta = 0$) system (1) is called the *nominal unicycle*.

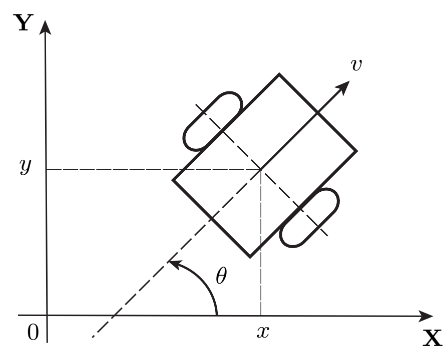


Fig. 1. Schematic of a unicycle mobile robot

Consider a reference trajectory γ_0 defined as follows.

Assumption 1: Let $\gamma_0 : \mathbb{R}^+ \to \mathbb{R}^2$ be a reference trajectory with C^3 continuity such that:

1) $0 < V_m \leq \|\dot{\gamma}_0(t)\| \leq V_M$,
2) $\|\ddot{\gamma}_0(t)\| \leq A_M$.

Exact tracking of γ_0 is achieved when $\forall t \geq 0$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \gamma_0(t) \, .$$

The following straightforward result characterizes completely the exact tracking problem for the nominal unicycle.

Property 1: Exact tracking is achieved for the nominal unicycle (1), i.e. $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \gamma_0(t)$, $\forall t \geq 0$, if and only if the following conditions hold:

a) $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \gamma_0(0)$,

b) $v(0) \begin{pmatrix} \cos\theta(0) \\ \sin\theta(0) \end{pmatrix} = \dot\gamma_0(0)$,

c) $v(t) = \|\dot\gamma_0(t)\|, \forall t \geq 0$,

d) $\omega(t) = \frac{d}{dt} \arg(\dot\gamma_0(t)), \forall t \geq 0$.

Conditions a), b) imply that the initial conditions must be such that at the initial time the unicycle lies at the beginning of the curve with orientation angle parallel to the tangent vector to the curve γ_0. Conditions c), d) actually define the controls that must be used to exactly track the given reference. These controls are *feed-forward* velocity input signals because depend only on the reference γ_0.

Remark: Having chosen a C^3-trajectory reference, i.e. a trajectory function that is continuous with its derivatives till to the third order, we obtain by means of c) and d) smooth velocities $v(t)$, $\omega(t)$ with continuous accelerations, i.e. $v, \omega \in C^1(\mathbb{R}^+)$. A weaker condition to still ensure continuous accelerations is to assume $\gamma_0 \in C^2(\mathbb{R}^+)$ and γ_0 is a G^3-curve, i.e. a curve with third order geometric continuity (continuity along the curve of the tangent vector, curvature, and derivative of the curvature with respect to the arc length) [10].

Obviously, using feedforward control only, defined by c) and d), the tracking error may grow unbounded if $N > 0$, $N_\theta > 0$. In order to keep the error bounded one may use continuous-time feedback control. In this paper another approach is considered, based on an idea similar to *iterative steering* (see [8]). The method consists in using at all times the feedforward controls given by c), d) but the reference trajectory is periodically replanned. When $t \in [0, T]$, γ_0 is used as reference trajectory, for $t \in [T, 2T]$ a different curve γ_1 is used and, in general the reference trajectory γ_i is used for $t \in [iT, (i+1)T]$. Each references γ_i is defined *recursively* with respect to γ_{i-1} in such a way to keep the tracking error limited. Before explaining in detail the overall feedforward/feedback strategy, the replanning operator to be used to construct each reference γ_i from γ_{i-1} is defined as follows:

Definition 1 (Replanning Operator): Let be given a (current) reference trajectory $\gamma : [t_0, +\infty) \to \mathbb{R}^2$ and a robot's state $z_0 = (x_0, y_0, \theta_0)$. Define a new reference trajectory $\gamma_{z_0,t_0,\gamma} : [t_0, +\infty) \to \mathbb{R}^2$ according to the convex replanning:

$$\begin{aligned} \gamma_{z_0,t_0,\gamma}(t) = \lambda(t - t_0) \left[(x_0, y_0) + R(e_\theta(t_0))(\gamma(t) - \gamma(t_0)) \right] \\ + (1 - \lambda(t - t_0))\, \gamma(t) \ , \end{aligned} \tag{3}$$

where

- $\lambda : \mathbb{R}^+ \to [0, 1]$ is a monotone decreasing C^3-function with $\lambda(0) = 1$, $D^i\lambda(0) = 0$, $i = 1, 2, 3$ and $\lim_{t\to+\infty} \lambda(t) = 0$;
- $R(x) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$ is the rotation matrix;
- $e_\theta(t_0) = \theta_0 - \arg\dot\gamma(t_0)$ is the heading angle error at time t_0.

The curve $\gamma_1 = \gamma_{z_0,t_0,\gamma_0}$ is a C^3-function and enjoys the following properties

$$\gamma_1(t_0) = (x_0, y_0) \ ,$$

$$\arg\dot\gamma_1(t_0) = \theta_0 \ ,$$

$$\lim_{t\to\infty} \gamma_1(t) - \gamma_0(t) = 0 \ .$$

In other words, trajectory γ_1 at t_0 is equal to $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ and its derivative has the direction given by θ_0. Asymptotically γ_1 converges to γ_0 and the rate of convergence is controlled by the monotone decreasing function λ. Remark that the replanned curve γ_1 is determined trough a linear convex combination, weighted by $\lambda(t)$, of function γ_0 and another trajectory obtained by rotating and translating γ_0 itself, as depicted in Figure 2. For instance, one may choose λ using C^3-transition polynomials [11] and setting the transition time equals to $2T$:

$$\begin{aligned} \lambda(t) &= 1 - 35\left(\frac{t}{2T}\right)^4 + 84\left(\frac{t}{2T}\right)^5 \\ &- 70\left(\frac{t}{2T}\right)^6 + 20\left(\frac{t}{2T}\right)^7, \ t \in [0, 2T] \\ \lambda(t) &= 0, \ t > 2T; \end{aligned} \tag{4}$$

the graph of this function is reported in Figure 3

The motion control method can be summarized as follows (it is assumed that a), b) of Property 1 hold).

I. For $t \in [0, T]$, where $T > 0$ is the *replanning time*, the control functions are given by c), d) (in Property 1)

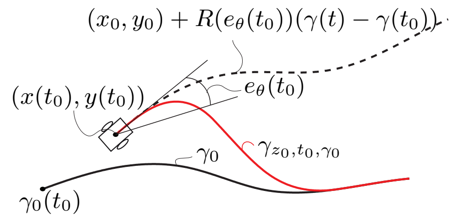


Fig. 2. Convex replanning

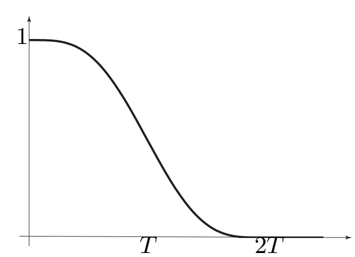


Fig. 3. The C^3-transition polynomial $\lambda(t)$

II. For $t \in [iT, (i+1)T]$, with $i = 1, 2, \ldots$, the control velocities are defined by

$$u(t) = \|\dot\gamma_i(t)\| \ , \tag{5}$$

$$\omega(t) = \frac{d}{dt} \arg(\dot\gamma_i(t)) \ , \tag{6}$$

where $\gamma_i(t)$ is the trajectory determined via the convex replanning operator (3):

$$\gamma_i = \gamma_{z(iT), iT, \gamma_{i-1}} \ . \tag{7}$$

That is, for $t \in [iT, (i+1)T]$, an open loop control is applied, that would drive the nominal system from state $\begin{pmatrix} x(iT) \\ y(iT) \end{pmatrix}$ with orientation $\theta(iT)$, to reference trajectory γ_{i-1}. Therefore the reference trajectory γ_i is defined recursively with respect to trajectory γ_{i-1}, as shown in Figure 4.

The overall control scheme is depicted in Figure 5 where the Recursive Convex Replanning Operator block takes care of the iterative trajectory generation and the Inverse Control Operator block computes the actual control by means of differential relations (5,6).

The control method just outlined draws on the idea of iterative state steering (see [8]), the main difference lies in the fact that each replanned trajectory is defined recursively with respect to the previous one. With respect to the iterative state steering, this method has the following significant differences:

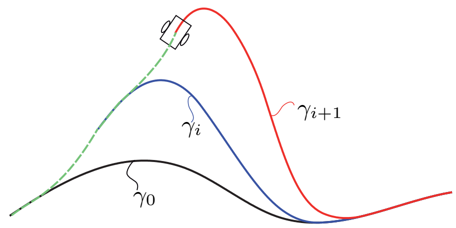


Fig. 4. Recursive generation of reference trajectories

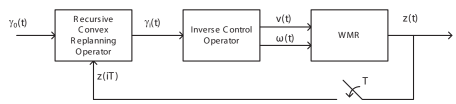


Fig. 5. The hybrid feedforward/feedback scheme for the trajectory tracking of wheeled mobile robots.

- If the noise functions η and η_θ affecting system (1) are zero during time interval $[iT,(i+1)T]$ the replanned trajectory coincides with the previous one, i.e. $\gamma_{i+1}=\gamma_1$. No replanning is actually performed in absence of noise.
- The replanning does not affect the control smoothness as ω and v remain C^1-functions, linear and angular accelerations remain continuous. Actually, these control functions could be made arbitrarily regular by choosing sufficiently regular reference γ_0 and function λ.
- Even if a direct comparison is difficult, the sufficient conditions for applying this method are somehow weaker than the one appearing in [8] since it is not required that the tracking error decreases in norm after the replanning time T (see (c) of Assumption 1 in [8]).

In this paper, this method will be analyzed, conditions will be found that allow keeping the tracking error limited and bounds will be provided. The problem that will be solved is therefore the following.

The Problem: Find conditions on trajectory γ_0, replanning time T and noise magnitude that guarantee that the tracking error is bounded, and find an estimate on the error norm.

In Section III, this problem will be considered for a general class of systems that includes the unicycle model (1). The main result of this work (Proposition 1), when applied to the case of the unicycle vehicle with function λ defined as in (5), brings to the following result.

Corollary 1: Consider control laws given by (5) and (6) and let $\lambda(t)$ be given by (4). If $T<\frac{32}{83N_\theta}$ then the following bounds hold

$$\|\dot\gamma_i(t)-\dot\gamma_0(t)\|\le\bar V_2:=\frac{\frac{83}{32}\,T\,N_\theta\,V_M+\left(\frac{T}{2}^2\,N_\theta+T\,N\right)}{1-\frac{83}{32}\,T\,N_\theta}\,, \tag{8}$$

$$\|\gamma_i(t)-\gamma_0(t)\|\le\left(1+\frac{T}{4}\right)\,T\,N_\theta(\bar V_2+V_M)+\frac{T}{2}\,N\,. \tag{9}$$

This results means that if the product of the replanning time T and the noise bound N_θ is sufficiently small, then the difference between the replanned curves γ_i and the reference curve γ_0 is bounded (the tracking error has similar bounds). Obviously, the provided bounds grow as the replanning time T increases and decrease with the noise bounds N, N_θ. Exact tracking is guaranteed only when $N=0$ and $N_\theta=0$.

III. Recursive tracking in a general setting

In this section we introduce the recursive tracking problem in a more general way and present a technical result (Proposition 1) which will permit to find tracking bounds for the case of the unicycle vehicle discussed in Section II.

Consider system

$$\left\{\begin{array}{rcl}\dot z(t)&=&f(z(t),u(t))+\eta(t)\\ z(t_0)&=&z_0\end{array}\right. \tag{10}$$

where $z(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$ and η is a noise term that satisfies the following constrain

$$\|\eta(t)\|\le N\quad\forall t\in\mathbb{R}\,, \tag{11}$$

with $N\in\mathbb{R}^+$. As in the case of the unicycle, when $N=0$, the system above is called the *nominal system* (10).

Define as *feasible trajectory* a reference function which can be exactly tracked by the nominal system (10):

Definition 2: A continuous function $\gamma_0:\mathbb{R}\to\mathbb{R}^n$ is a feasible trajectory for (10) with control u_0, if the following differential equation is satisfied

$$\dot\gamma_0(t)=f(\gamma_0(t),u_0(t)),t\ge 0\,. \tag{12}$$

The following is the fundamental assumption for defining a recursive iterative tracking. For every feasible system trajectory γ, every initial state $\bar z$ and time $\bar t$, it is assumed that there exists a feasible replanned trajectory that brings asymptotically the state from $\bar z$ to the reference γ.

Assumption 2: If γ_0 is a feasible trajectory for (10) then $\forall\bar z\in\mathbb{R}^n$ and $\bar t\in\mathbb{R}$ there exist continuous functions $u_{\bar z,\bar t,\gamma_0}:[\bar t,+\infty)\to\mathbb{R}^m$ and $\gamma_{\bar z,\bar t,\gamma_0}:[\bar t,+\infty)\to\mathbb{R}^n$, such that

$$\left\{\begin{array}{rcl}\dot\gamma_{\bar z,\bar t,\gamma_0}(t)&=&f(\gamma_{\bar z,\bar t,\gamma_0}(t),u_{\bar z,\bar t,\gamma_0}(t))\\ \gamma_{\bar z,\bar t,\gamma_0}(\bar t)&=&\bar z\,,\end{array}\right. \tag{13}$$

and

$$\lim_{t\to+\infty}\gamma_{\bar z,\bar t,\gamma_0}(t)-\gamma_0(t)=0\,. \tag{14}$$

Assumption 2 allows defining a recursive iterative control (as has been done in the case of the unicycle vehicle in Section II) in the following way.

Control law: Given a reference trajectory γ_0, the control function $\bar u$ for system (1) is defined as follows

$$\begin{array}{l}\bar u(t)=u_0(t),\ \text{if } t\in[0,T]\\ \bar u(t)=u_{z(iT),iT,\gamma_{i-1}}(t)\ \text{if } t\in[iT,(i+1)T]\,,\end{array} \tag{15}$$

where

$$\begin{array}{rcl}\dot z(t)&=&f(z(t),\bar u(t))\\ \gamma_i(t)&=&\gamma_{z(iT),iT,\gamma_{i-1}}(t),\ i>0\,.\end{array} \tag{16}$$

The following defines a particular class of positive definite operators, similar to Lyapunov functions.

Definition 3: Let n be a positive integer, then $U:\mathbb{R}^n\to\mathbb{R}$, is a seminorm if the following conditions hold

1) $V(0)=0$;
2) $V(z)\ge 0,\forall z\in\mathbb{R}^n$;
3) $V(z_1+z_2)\le V(z_1)+V(z_2),\ \forall z_1,z_2\in\mathbb{R}^n$.

Moreover $V=(V_1,V_2,\ldots,V_l):\mathbb{R}^n\to\mathbb{R}^l$ is a vector of seminorms if each component V_i is a seminorm.
(Notation: for any relational operator $<_R$ and $x,y\in\mathbb{R}^n$, $x<_R y$ means $x_i<_R y_i$, $i=1,\ldots,n$).

Definition 4: Given a function $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ and a seminorm U, we say that system (10) is (U,φ)-bounded, if, when $\bar\gamma$ is a feasible trajectory with control $\bar u$ and z is the solution of the following system

$$\left\{\begin{array}{rcl}\dot z(t)&=&f(z(t),\bar u(t))+\eta(t)\\ z(t_0)&=&\bar\gamma(t_0)\end{array}\right.$$

then, $\forall t\ge t_0$

$$U(z(t)-\bar\gamma(t))\le\varphi(t-t_0)\,. \tag{17}$$

The following proposition is the main result of this section.

Proposition 1: Let V be a vector of seminorms and U a seminorm, γ_0 a feasible trajectory for (10), with control function

u_0. Let $z(t)$ and γ_i be defined according to (15), (16). Let function $\Phi : \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ be such that

$$V(\bar{\gamma}_{z_0,t_0,\gamma}(t)-\gamma(t)) \leq \Phi(U(z(t_0)-\gamma(t_0)), t-t_0, W(\gamma-\gamma_0)) \ , \quad (18)$$

and Φ is monotone increasing with respect to each component of the argument W, defined as $W(\gamma) = \sup_{t\in\mathbb{R}} V(\gamma(t))$. Moreover, assume that there exists a function $\varphi(t)$, such that (10) is (U,φ)-bounded.

If there exists $\bar{V} \in \mathbb{R}^l$ such that

$$\bar{V} \geq \sum_{k=1}^{+\infty} \Phi(\varphi(T), t-kT, \bar{V}) \quad (19)$$

then, $\forall t \in \mathbb{R}$ and $\forall i \in \mathbb{N}$,

$$V(\gamma_i(t) - \gamma_0(t)) \leq \bar{V} \ . \quad (20)$$

Proof: Proposition 1 can be proved by induction as follows. Consider first $i = 0$, in this case inequality (20) holds since, by 1) of Definition 3,

$$V(\gamma_0(t) - \gamma_0(t)) = V(0) = 0 \leq \bar{V} \ .$$

Moreover assume that (20) is verified for $i = 0, 1, \ldots, l-1$, then from (18) and 2) of Definition 3 the following relation is obtained

$$\begin{aligned} V(\gamma_l(t) - \gamma_0(t)) &= V\left(\sum_{k=1}^{l}(\gamma_k(t) - \gamma_{k-1}(t))\right) \\ &\leq \sum_{k=1}^{l} V(\gamma_k(t) - \gamma_{k-1}(t)) \\ &\leq \sum_{k=1}^{l} \Phi(U(z(kT) - \gamma_{k-1}(kT)), \\ &\quad t-kT, W(\gamma_{k-1}(t) - \gamma_0(t))) \ . \end{aligned} \quad (21)$$

From (17), with $\bar{\gamma} = \gamma_{k-1}$ and $t_0 = (k-1)T$, $\forall k = 1, \ldots, l$, the following inequality holds

$$U(z(kT) - \gamma(kT)) \leq \varphi(T) \ ,$$

Since by the inductive hypothesis relation (20) is true for $i = 0, 1, \ldots, l-1$, $\forall t \in \mathbb{R}$:

$$W(\gamma_{k-1}(t) - \gamma_0(t)) \leq \bar{V} \ ,$$

therefore, the following inequality is obtained

$$V(\gamma_l(t) - \gamma_0(t)) \leq \sum_{k=1}^{l} \Phi(\varphi(T), t-kT, \bar{V}) \ . \quad (22)$$

and finally, combining (22) and (19), it follows that

$$V(\gamma_i(t) - \gamma_0(t)) \leq \bar{V} \ .$$

■

Remark that instead of finding separately a function Φ and φ which satisfy (18) and (17), one can find directly the composite function $\Phi(\varphi(T), t-t_0, W(\gamma-\gamma_0))$ which appears in (19), as will be done for the unicycle vehicle.

The idea behind proposition (1) is the following. The key element for finding bounds for trajectories γ_i defined in (16) consists in finding the function $\Phi(\varphi(T), t-t_0, W(\gamma-\gamma_0))$, which provides bounds on the norm at time t of the difference of a curve replanned at t_0 with the previous one (γ), as a function of the replanning time T, the time elapsed since the parameterization $(t-t_0)$ and the maximum value of the norms of the difference between γ and the reference curve γ_0.

IV. Application to the tracking problem for the unicycle

In this section, we apply Proposition 1 to the tracking problem for the unicycle vehicle, introduced in Section II.

The following lemma estimates the error on the feed-forward control of system (1) caused by the noise terms.

Lemma 1: Consider system (1), assume that a) and b) in Property 1 holds and that the controls u and ω are given by c) and d). Then the following inequalities hold

$$|\theta(t) - \arg(\gamma_0(t))| \leq N_\theta t \ , \quad (23)$$

$$\left\| \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} - \gamma_0(t) \right\| \leq \frac{t^2}{2} N_\theta V_M + Nt \ . \quad (24)$$

Proof: Define $e_\theta(t) = \theta(t) - \arg(\dot{\gamma}_0(t))$ and $e(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} - \gamma_0(t)$, then $\dot{e}_\theta(t) = \eta_\theta(t)$ and $|\dot{e}_\theta(t)| \leq N_\theta$, from which (23) is obtained. Moreover $\dot{e} = v\begin{pmatrix} \cos\theta(t) - \cos(\arg\dot{\gamma}(t)) \\ \sin\theta(t) - \sin(\arg\dot{\gamma}(t)) \end{pmatrix} + \eta$ and $\|\dot{e}(t)\| \leq V_M\sqrt{2}\sqrt{1-\cos e_\theta} + N$. Since $\cos x \geq 1 - \frac{x^2}{2}$, then $\|\dot{e}(t)\| \leq V_M \frac{t^2 N_\theta}{2} + Nt$, from which (24) follows. ■

The following result represents the direct application of Proposition 1 to the case of the unicycle.

Proposition 2: Consider system (1), where the control u is defined by (5)-(6) and the reference function γ_0 satisfies Assumption 1. Moreover suppose that

$$\chi = TN_\theta\left[\sum_{i=0}^{+\infty}\lambda(iT) + \sum_{i=0}^{+\infty}|\dot{\lambda}(iT)|iT\right] < 1 \ .$$

Define

$$\begin{aligned} \bar{V}_2 = (1-\chi)^{-1}\Bigg(&TN_\theta V_M TN_\theta(\sum_{i=1}^{+\infty}\lambda(Ti) \\ &+\dot{\lambda}(Ti)T_i) + (\frac{TN_\theta}{2} + NT)\sum_{i=0}^{+\infty}\lambda(T_i)\Bigg) \ . \end{aligned} \quad (25)$$

$$\begin{aligned} \bar{V}_1 = &\left(\frac{T}{2}^2 N_\theta(\bar{V}_2 + V_M) + NT\right)\sum_{i=1}^{+\infty}\lambda(Ti) \\ &+TN_\theta(\bar{V}_2 + V_M)\sum_{i=1}^{+\infty}\lambda(Ti) \end{aligned} \quad (26)$$

$$\begin{aligned} \bar{V}_3 = &\left(1 - TN_\theta\sum_{i=0}^{+\infty}\lambda(iT)\right)^{-1}\{T\,N_\theta\,(\bar{V}_2 + V_M) \\ &\left(\sum_{i=1}^{+\infty}\ddot{\lambda}(iT) + \dot{\lambda}(iT)\right) + A_M\,T\,N_\theta\sum_{i=1}^{+\infty}\lambda(iT) \\ &+\left(\frac{T}{2}^2 N_\theta + T\,N\right)\sum_{i=1}^{+\infty}\ddot{\lambda}(iT)\} \end{aligned} \quad (27)$$

and suppose that the following condition is verified

$$V_m < \bar{V}_2 \ ,$$

then the following inequalities hold, $\forall i \in \mathbb{N}$ and $\forall t \geq iT$,

$$\|\gamma_i(t) - \gamma_0(t)\| \leq \bar{V}_1 \ , \quad (28)$$

$$\|\dot{\gamma}_i(t) - \dot{\gamma}_0(t)\| \leq \bar{V}_2 \ , \quad (29)$$

$$\|\ddot{\gamma}_i(t) - \ddot{\gamma}_0(t)\| \leq \bar{V}_3 \ . \quad (30)$$

Moreover the controls defined by (5) and (6) satisfy the following bounds, $\forall i \in \mathbb{N}$, $\forall t \geq iT$

$$u(t) \in [V_m - \bar{V}_2, V_M + \bar{V}_2] \ , \quad (31)$$

$$\|\omega(t)\| \leq \frac{A_M + \bar{V}_3}{V_m - \bar{V}_2} \ . \tag{32}$$

Proof: Since, input functions $u(t)$, $\omega(t)$ defined in (5) and (6) are C^1 and respectively C^0, then the extended state $z = \{x, y, \theta, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}\}$ is well defined. Set $V = (V_1, V_2, V_3)$ with $V_1 = \|(x, y)\|$, $V_2 = \|(\dot{x}, \dot{y})\|$ and $V_3 = \|(\ddot{x}, \ddot{y})\|$. Remark that V satisfies Definition 3. In order to use Proposition 1, we now define the function $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ such that (18) holds. To define Φ_1, consider the following bound

$$\begin{aligned} V_1(\gamma_{z(t_0),t_0,\gamma} - \gamma) = \|\lambda(t - t_0)\,\{R(e_\theta(t_0))\,[\gamma(t) - \gamma(t_0)] \\ +\gamma(t_0) + e_\gamma(t)\} + [1 - \lambda(t - t_0)]\,\gamma(t) - \gamma(t)\| \\ = \|\lambda(t - t_0)\,\{[R(e_\theta(t_0)) - I]\,[\gamma(t) - \gamma(t_0)] + e_\gamma(t_0)\}\| \ , \end{aligned}$$

together with $\|R(_\theta(t_0)) - I\| \leq |e_\theta(t_0)|$ and

$$\begin{aligned} \|\gamma(t) - \gamma(t_0)\| &\leq (t - t_0) \sup_{t \geq t_0} \|\dot{\gamma}(t)\| \\ &\leq (t - t_0)\left[V_M + \sup_{t \geq t_0} \|\dot{\gamma}(t) - \dot{\gamma}_0(t)\|\right] \\ &\leq (t - t_0)\,[V_M + W_2(\gamma - \gamma_0)] \ . \end{aligned}$$

Therefore, by Lemma 1, we find the bound

$$\begin{aligned} V_1(\gamma_{z(t_0),t_0,\gamma} - \gamma) \leq \lambda(t - t_0)\,\{|e_\theta(t_0)|\,(t - t_0) \\ [W_2(\gamma - \gamma_0) + V_M] + |e_\gamma(t_0)|\} \\ \leq \Phi_1(T, t - t_0, W(\gamma - \gamma_0)) \ . \end{aligned}$$

Analogously

$$\begin{aligned} V_2(\gamma_{z(t_0),t_0,\gamma} - \gamma) = \|\dot{\lambda}(t - t_0)\,\{[R(e_\theta(t_0)) - I] \\ [\gamma(t) - \gamma(t_0)] + e_\gamma(t_0)\} + \dot{\gamma}(t) \\ \{\lambda(t - t_0)\,[R(e_\theta(t_0)) - I]\}\| \\ \leq \Phi_2(T, t - t_0, W(\gamma - \gamma_0)) \ . \end{aligned}$$

Finally

$$\begin{aligned} V_3(\gamma_{z(t_0),t_0,\gamma} - \gamma) = \|\ddot{\lambda}(t - t_0)\,\{[R(e_\theta(t_0) - I] \\ [\gamma(t) - \gamma(t_0) + e_\gamma(t_0)]\} + \dot{\lambda}(t - t_0)\,[R(e_\theta(t_0)) - I] \\ \dot{\gamma}(t) + \ddot{\gamma}(t)\,\{1 + \lambda(t - t_0)\,[R(e_\theta(t_0) - I]\} \\ +\dot{\gamma}(t)\,\ddot{\lambda}(t - t_0)\,[R(e_\theta(t_0) - I] - \ddot{\gamma}(t)\| \\ \leq \Phi_3(T, t - t_0, W(\gamma - \gamma_0)) \ . \end{aligned}$$

From (25), (26) and (27) it follows that, for $k = 1, 2, 3$

$$\bar{V}_k \geq \sum_{i=1}^{+\infty} \Phi_k(T, iT, \bar{V}_k) \ ,$$

and, by (20) of Proposition 1, relations (28), (29) and (30) hold. Moreover, $\forall t \in [iT, (i+1)T]$

$$\begin{aligned} u(t) = \|\dot{\gamma}_i(t)\| = \|\dot{\gamma}_0(t) + \dot{\gamma}_i(t) - \dot{\gamma}_0(t)\| \\ \in \left[V_m - \sup_{t \geq iT}\{\|\dot{\gamma}_i(t) - \dot{\gamma}_0(t)\|\}, V_M + \sup_{t \geq iT}\{\|\dot{\gamma}_i(t) - \dot{\gamma}_0(t)\|\}\right] \\ \subset \left[V_m - \bar{V}_2, V_M + \bar{V}_2\right] \ , \end{aligned}$$

hence (31) holds. Furthermore,

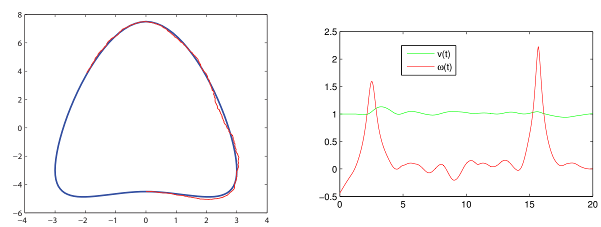
$$\begin{aligned} |\omega(t)| = |\frac{d}{dt}\arg(\dot{\gamma}_i(t))| \leq \frac{|\det[\ddot{\gamma}_i(t), \dot{\gamma}_i(t)]|}{\|\dot{\gamma}_i(t)\|^2} \\ \leq \frac{\|\ddot{\gamma}_i(t)\|}{\|\dot{\gamma}_i(t)\|} \leq \frac{\|\ddot{\gamma}_0(t)\| + \sup_{t \geq iT}\{\|\ddot{\gamma}_i(t) - \ddot{\gamma}_0(t)\|\}}{\|\dot{\gamma}_0(t)\| - \sup_{t \geq iT}\{\|\dot{\gamma}_i(t) - \dot{\gamma}_0(t)\|\}} \\ \leq \frac{A_M + \bar{V}_2}{V_m - \bar{V}_2} \ , \end{aligned}$$

therefore (32) holds and the proof of Proposition 2 is complete. ∎

Corollary 1 follows from Proposition 2 when λ is given by (4) .

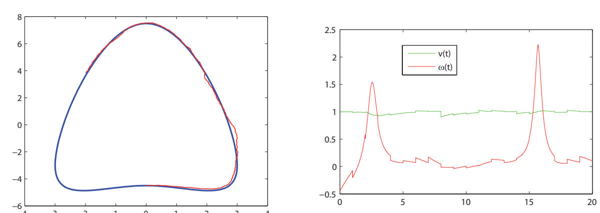
V. Simulations

We have compared in simulation the method presented in Section II, with the controller for the unicycle presented in [1, p. 809]. We have assumed that the state is measured only at regular intervals $T = 1$ s, which represents also the replanning time for our algorithm. The state appearing in the feedback control law presented in [1] is obtained through a discontinuous open loop observer which is updated at each observation time. The gain in this controller have been set to have controls signals of magnitude similar to the method presented in Section II. As reference trajectory we have considered a periodic spline followed with constant speed 1 m/s. The noise bounds appearing in (2) are given by $N = 0.5\sqrt{2}$ and $N_\theta = 0.5$. The obtained results are presented in Figures 6 and 7. The two methods showed a similar performance in terms of tracking error. However, the control method presented in this paper has the advantage of providing overall continuous input signals whereas the control signals of the classic controller are discontinuous (even if this is a consequence of having used a discontinuous observer). We believe that our method has the advantage of guaranteeing an arbitrary class of continuity in the input signals. Moreover it is not a ad-hoc solution for the unicycle, since it can be applied in principle to any system satisfying the conditions presented in Section III.



a) b)

Fig. 6. a) The robot trajectory and b) The control inputs for the recursive method



a) b)

Fig. 7. a) The robot trajectory and b) The control inputs for the method presented in [1, p. 809]

VI. Experimental Results

We have implemented an experimental setting for the method presented in Section II. We have used a mobile robot built with Lego Mindstorm NXT pieces, depicted in Figure 8. The traction is provided by 2 front wheels, a passive rear castor wheel is used to prevent the robot from falling over. The inputs variable are ω_l and ω_r, the angular velocity of left and right wheels. Set $v = r\frac{\omega_l + \omega_r}{2}$ and $\omega = \frac{r}{L}\frac{\omega_l - \omega_r}{2}$, where r is the driving wheels radius and L is the distance between the two wheels. After this substitution this differential drive robots can be described with the unicycle model (1).

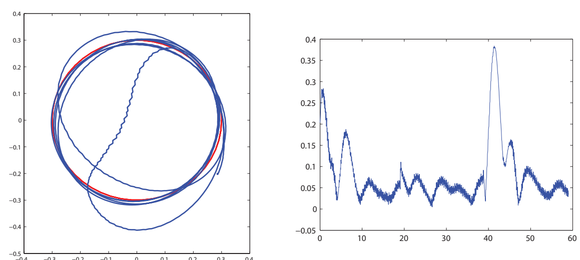
Two red markers of different sizes have been placed on the robot and the system state (x, y, θ) is measured 10 times per second through a Unibrain firewire camera, using standard computer vision techniques. A personal computer running Matlab contains

Fig. 8. The LEGO NXT robot

a systems observer for finding the robot state and implements the recursive controller presented in (5), (6) and (7). The control signals are computed and sent to the wheeled robot via Bluetooth.

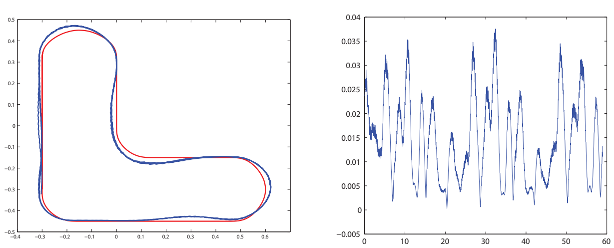
In these experiments the replanning time has been set to $T = 0.8$ s.

This experimental setting is characterized by some difficulties, in particular the Bluetooth transmission introduces in the control loop a delay of 80 ms , and the wheels occasionally experiment slippage.



a) b)

Fig. 9. a) Reference and actual trajectory for a circle b) The norm of the (x, y) component of the tracking error with respect to time



a) b)

Fig. 10. a) Reference and actual trajectory for a composite $\boldsymbol{\eta}^3$-spline b) The norm of the (x, y) component of the tracking error with respect to time

Figure 9-a) shows the experimental results obtained when the reference trajectory is a circle of radius equal to 30 cm, followed with a constant speed of 0.2 m/s. The red line represents the reference trajectory γ_0 and the blue line the robot observed position. In the middle of the test the robot has been moved with a rod to test the robustness of the controller, this explain the large transient error present in the figure. The reader should watch the attached video (which shows the same experiment) to have a better idea of the controller behavior in this case. In Figure 9-b), the norm of the (x, y) component of the tracking error is showed; the spike on time $t = 40$ it is due to the test of the robustness of the controller. Figure 10-a) shows another experiment where the desired trajectory γ_0 is obtained using $\boldsymbol{\eta}^3$-splines [12], reparameterized with constant speed 0.15 m/s. The associated tracking error is shown in Figure 10-b). Remark that the evaluation of functions γ_i in (7) require the use of a recursive function. If function λ reaches 0 in finite time τ, then the maximum order of recursion is given by the ratio $\frac{\tau}{T}$ (recall that T is the replanning time). Since the order of recursion is deterministic, the proposed control law can be implemented in a real time controller. Parameter T must be carefully chosen. In fact, on one hand, by (8), (9), reducing T improves the tracking performances. On the other hand, it increases the ratio $\frac{\tau}{T}$, the number of recursions needed to implement the controller and the computational effort.

VII. Conclusion

This article has exposed a new controller scheme for the hybrid feedforward/feedback trajectory tracking of WMRs. A key idea of the approach is the replanning operator (cf. (3)) that is applied recursively and computes straightforwardly a new reference trajectory by means of a convex combination of the earlier trajectory with the same trajectory rotated and translated according to the current robot state. The authors retain that this replanning operator can be generalized and be used in broader contexts such as e.g. in aerial robotics and in certain classes of n-dimensional nonlinear systems.

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