2010 IEEE International Conference on Robotics and Automation
Anchorage Convention District
May 3-8, 2010, Anchorage, Alaska, USA

Gait synthesis for a three-link planar biped walker with one actuator

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***Abstract*— We consider a 3-link planar walker with two legs and an upper body. An actuator is introduced between the legs, and the torso is kept upright by torsional springs. The model is a 3-DOF impulsive mechanical system, and the aim is to induce stable limit-cycle walking in level ground. To solve the problem, the ideas of the virtual holonomic constraints approach are explored, used and extended. The contribution is a novel systematic motion planning procedure for solving the problem of gait synthesis, which is challenging for non-feedback linearizable mechanical systems with two or more passive degrees of freedom. For a preplanned gait we compute an impulsive linear system that approximates dynamics transversal to the periodic solution. This linear system is used for the design of a stabilizing feedback controller. Results of numerical simulations are presented to illustrate the performance of the closed loop system.**

***Index Terms*— Biped robots, virtual holonomic constraints, motion planning, orbital stabilization of periodic trajectories.**

I. INTRODUCTION

In recent years, dynamically stable walking machines have begun having an impact in the research of legged locomotion. The main motivation is the introduction of more anthropomorphic and energy efficient legged robots, by considering the design of underactuated mechanical systems [3]. This solution would in principle allow to exploit the robot natural dynamics. However, achieving a dynamically stable gait on an underactuated legged robot has proved to be difficult in both motion planning and control system design

Some studies have been dedicated to the class of planar biped robots with one passive degree of freedom. These low-dimensional walkers constitute the basic models for the development of gait synthesis methods and control system design strategies. Recently, some approaches were introduced to solve both problems [7], [17], [13], [1], [8], [6]. In particular, the virtual holonomic constraints (VHC) approach has become of interest due to its successful analytical and practical applications in robots of this class [17], [10]. The main idea of this approach is to define a number of geometrical relations (synchronization) among the robot's degrees of freedom, which are imposed into the robot dynamics by feedback control. For gait synthesis, different methods were explored to define the constraint functions, e.g. from human recorded data [11], to heuristic polynomials that specify the walking posture constraints along a walking step [17], [2].

From the motion planning perspective, different questions are still opened regarding the mentioned approach. It is unclear, for example, how to plan a gait in the case where the robot is subject to two or more passive links. The main difficulty is to specify geometric relations that result in a convenient representation of a gait, and also satisfy the dynamic constraints of the robot model, and different performance criteria. In opposite, the controller design is more tractable, since it can be done by following the generic procedure presented in [5], [14].

The main contribution of this article is providing a non-trivial extension of motion planning by the VHC approach, considering a modified benchmark example of the compass biped with upper torso [18], i.e. three degrees of freedom and only one actuator. For this example of underactuation two, we develop a systematic method for searching virtual constraint functions. The resulting gait is used for the design of an orbitally stabilizing controller, exemplifying the framework of [5].

The remainder of this paper is organized as follows. In Section II, we describe the model of the walker. The motion planning procedure for finding forced hybrid cycles is formulated in Section III. In Section IV, we present the design of a feedback controller to achieve orbital stability of the hybrid periodic gait. Discussion of the results supported by numerical simulations and final concluding remarks are provided in Sections V and VI.

II. MODEL DESCRIPTION

We consider the three link planar model depicted in Fig. 1, consisting of a torso, a hip and two symmetric legs. The torque is applied only between the support and swing legs. The torso is kept upright by torsional springs between the legs and the torso. The hybrid dynamics of the biped robot, describing the continuous phase and discrete events, is presented below.

This work has been partially supported by the Swedish Research Council (grant 2008-5243) and Karriärbidrag from Umeå University.

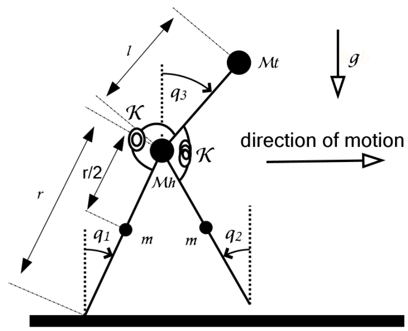

Fig. 1. Schematics of the biped in the sagittal plane and level ground. A walking motion is described by the time-evolution of the support leg angle q_1, the swing leg angle q_2, and the torso angle q_3. The length of the legs is denoted by r and the torso's by l. The masses of the legs, denoted by m, are lumped at $r/2$. The hip mass is denoted by M_h and the torso's by M_t, and it is lumped at a distance l from the hip. The torsional springs between the legs and the torso have stiffness coefficient K.

A. Swing phase dynamic model

In a walking gait, the stance leg acts as a pivot joint. Under the assumption of non-sliding ground foot contact, the Euler-Lagrange equations yield the dynamic model for the continuous-in-time part of the motion in the single support phase as

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Bu, \tag{1}$$

where $M(q)$ is a positive definite matrix of inertias, $G(q)$ is the vector of gravity, and $C(q,\dot{q})$ is the matrix of Coriolis and centrifugal forces. Entries of these matrices are given in appendix A, while the physical parameters of the model are listed in table I[1].

Table I: Physical parameters

Parameters	***Legs***	***Hip***	***Torso***
Mass [kg]	m = 5	M_h = 15	M_t = 15
CoM [m]	$r/2$ = 0.5	r = 1	l = 1
Length [m]	r = 1		
Gravity	g = 9.81 m/s^2		
Spring constant	K, to be found		

B. Impact Model

Following the collision model of [9], and the derivations in [12], [2], [7], the instantaneous jump in the values of the states due to impact can be computed as follows[2]

$$\Gamma^+ \ni q^+ = \Delta \cdot q^-, \quad q^- \in \Gamma^-, \quad \dot{q}^+ = F(q^-)\cdot \dot{q}^-, \tag{2}$$

where Δ is a relabeling operator and $F(q)$ is a matrix of algebraic terms. The notation $(\cdot)^-$, $(\cdot)^+$ is for the state of the system just prior and immediately after the impact event.

[1]These values are taken from a similar model analyzed in [7], where the authors consider the case of underactuation degree one.

[2]The authors of [7] give a description of this computation, and also provide MATLAB code in their website.

The walking plane or switching surfaces of the impact event are determined by

$$\Gamma^+ = \Gamma^- = \left\{ q \in \mathbb{R}^2 : \cos(q_1+\psi) - \cos(q_2+\psi) = 0 \right\}, \tag{3}$$

where ψ denotes the angle of the slope.

C. Problem Formulation

Given the hybrid model (1)-(2), the search for a symmetric gait consists of finding

- a spring constant $K > 0$,
- a vector of initial conditions

$$[q_1^*(0),\, q_2^*(0),\, q_3^*(0),\, \dot{q}_1^*(0),\, \dot{q}_2^*(0),\, \dot{q}_3^*(0)] \in \Gamma^+, \tag{4}$$

- a C^1-smooth scalar control input function $u = u^*(t)$, $t \in [0, T_e]$, for some $T_e > 0$,

such that the forced solution $q^*(t)$ of the hybrid model (1)-(2), with $u = u^*(t)$ and initial conditions (4) at time moment $t = T_e$, reaches the switching surface Γ^-, and after the jump returns to the starting point (4).

The second task is to design a feedback controller such that this newly planned forced hybrid periodic solution becomes orbitally asymptotically stable for the closed-loop system.

III. Planning a hybrid cycle for the hybrid model (1)-(2)

The difficulty in finding such a solution $q^*(t)$ is that neither a function $u^*(t)$, nor a period T_e of a gait, nor the initial conditions (4) for a cycle are known a priori[3].

Below we propose a procedure for a convenient representation of forced hybrid periodic motions of the walker (1)-(2) with two passive degrees of freedom. In turn, these arguments lay out the background for solving this infinite dimensional problem and reducing it to the search of appropriate initial conditions for an auxiliary set of differential equations, i.e. making the original problem finite dimensional. By doing so, we do not loose the generality and do not simplify the task, but organize the simultaneous search for such hybrid motion of (1)-(2) and the corresponding control input $u^*(t)$.

A. Re-parametrization for a continuous-in-time part of a hybrid cycle of (1)-(2) with one jump

The continuous time evolution of the generalized coordinates of a hybrid forced periodic solution can be represented as

$$q_1 = q_1^*(t), \quad q_2 = q_2^*(t), \quad q_3 = q_3^*(t), \quad t \in [0, T_e], \tag{5}$$

where the interval $[0, T_e]$ represents the duration of one step up to the time of the impact event. If the evolution of the stance leg angle $q_1(\cdot)$ along the cycle is monotonic, then one can rewrite the relations (5) as

$$q_1 = \theta_*(t),\ q_2 = \phi_2(\theta_*(t)),\ q_3 = \phi_3(\theta_*(t)),\ \theta_* \in [0, \Theta_e], \tag{6}$$

[3]For the case of purely passive walking this control input is trivial, i.e. $u_*(t) \equiv 0\ \forall\, t \in (0, T_e)$. However initial conditions and period of a cycle, if exists, are unknown and so finding the gait is still a challenging task [4].

and introduce two functions ϕ_2 and ϕ_3 that describe synchronizations of coordinates along the cycle. These functions in (6)—the so-called virtual holonomic constraints—have been previously used for motion planning and controller design for systems of underactuation degree one, see e.g. [15], [17], [13], [4].

Following [16] (and it can be readily checked for this case), the Euler-Lagrange equations (1) restricted by the invariance of the geometrical relations

$$q_1 = \theta, \quad q_2 = \phi_2(\theta), \quad q_3 = \phi_3(\theta), \tag{7}$$

can be written as the set of three differential equations:

$$\begin{array}{rcl} \alpha_1(\theta)\ddot{\theta} + \beta_1(\theta)\dot{\theta}^2 + \gamma_1(\theta) & = & 0, \\ \alpha_2(\theta)\ddot{\theta} + \beta_2(\theta)\dot{\theta}^2 + \gamma_2(\theta) & = & 0, \\ \alpha_3(\theta)\ddot{\theta} + \beta_3(\theta)\dot{\theta}^2 + \gamma_3(\theta) & = & -u \end{array} \tag{8}$$

To obtain (8), we substitute the following relations into (1):

$$\begin{array}{rcl} \dot{q}_1 & = & \dot{\theta}, \quad \ddot{q}_1 = \ddot{\theta}, \\ \dot{q}_i & = & \phi_i'(\theta)\dot{\theta}, \quad \ddot{q}_i = \phi_i''(\theta)\dot{\theta}^2 + \phi_i'(\theta)\ddot{\theta}, \; i = \{2,3\}. \end{array} \tag{9}$$

The coefficients $\alpha_j(\cdot)$, $\beta_j(\cdot)$, and $\gamma_j(\cdot)$ are written with the following abuse of notation

$$\begin{array}{rcl} \alpha_j(\theta) & = & \alpha_j\left(\theta, \phi_2(\theta), \phi_3(\theta), \phi_2'(\theta), \phi_3'(\theta)\right), \\ \beta_j(\theta) & = & \beta_j\left(\theta, \phi_2(\theta), \phi_3(\theta), \phi_2'(\theta), \phi_3'(\theta), \phi_2''(\theta), \phi_3''(\theta)\right), \\ \gamma_j(\theta) & = & \gamma_j\left(\theta, \phi_2(\theta), \phi_3(\theta)\right), \qquad j = \{1,2,3\}. \end{array}$$

For the gait (6) the variable $\theta = \theta_*(t)$ must be a simultaneous solution for each of the differential equations in (8). It is worth to note that the first two differential equations (8) are integrable; their general integral of motion is given in Appendix B.

B. *Equations for Constraint Functions $\phi_i(\cdot)$*

The equations (8), which by construction are valid along the solution (5), can be used for defining constraint functions ϕ_i. Indeed, selecting different pairs of equations (8) allows to rewrite the evolutions of $\ddot{\theta} = \ddot{\theta}_*(t)$ and $\dot{\theta}^2 = \dot{\theta}_*^2(t)$ along the gait as follows:

$$\begin{array}{rcl} \begin{bmatrix} \ddot{\theta} \\ \dot{\theta}^2 \end{bmatrix}_1 & = & -\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_3 & \beta_3 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_1 \\ \gamma_3 + u \end{bmatrix}, \\ \begin{bmatrix} \ddot{\theta} \\ \dot{\theta}^2 \end{bmatrix}_2 & = & -\begin{bmatrix} \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_2 \\ \gamma_3 + u \end{bmatrix}, \\ \begin{bmatrix} \ddot{\theta} \\ \dot{\theta}^2 \end{bmatrix}_3 & = & -\begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}. \end{array} \tag{10}$$

Furthermore, along the solution (5), the functions $\ddot{\theta}_*$ and $\dot{\theta}_*^2$ are related as [16]

$$\ddot{\theta}_*(t) = \frac{1}{2}\frac{d}{d\theta}\left(\dot{\theta}_*^2(t)\right). \tag{11}$$

Denoting the right hand sides of (10) as

$$\begin{bmatrix} \ddot{\theta} \\ \dot{\theta}^2 \end{bmatrix}_j = \begin{bmatrix} D_{1j} \\ D_{2j} \end{bmatrix}, \qquad j = \{1,2,3\} \tag{12}$$

we can rewrite (11) for each $j = \{1,2,3\}$ in the form

$$2D_{1j} = \frac{d}{d\theta}D_{2j} = \chi_j + \frac{\partial D_{2j}}{\partial \phi_2''}\phi_2''' + \frac{\partial D_{2j}}{\partial \phi_3''}\phi_3''' + \frac{\partial D_{2j}}{\partial u}u' \tag{13}$$

with

$$\chi_j = \frac{\partial D_{2j}}{\partial \theta} + \frac{\partial D_{2j}}{\partial \phi_2}\phi_2' + \frac{\partial D_{2j}}{\partial \phi_3}\phi_3' + \frac{\partial D_{2j}}{\partial \phi_2'}\phi_2'' + \frac{\partial D_{2j}}{\partial \phi_3'}\phi_3''.$$

Here we assumed that the control input $u = u_*(t)$ along the gait can be rewritten as a C^1-smooth function of $\theta_*(t)$. Resolving the equations (13) with respect to the highest derivatives of the constraint functions ϕ_2''', ϕ_3''' and the input torque u', we obtain the following set of differential equations:

$$\begin{bmatrix} \phi_2''' \\ \phi_3''' \\ u' \end{bmatrix} = \begin{bmatrix} \frac{\partial D_{21}}{\partial \phi_2''} & \frac{\partial D_{21}}{\partial \phi_3''} & \frac{\partial D_{21}}{\partial u} \\ \frac{\partial D_{22}}{\partial \phi_2''} & \frac{\partial D_{22}}{\partial \phi_3''} & \frac{\partial D_{22}}{\partial u} \\ \frac{\partial D_{23}}{\partial \phi_2''} & \frac{\partial D_{23}}{\partial \phi_3''} & \frac{\partial D_{23}}{\partial u} \end{bmatrix}^{-1} \begin{bmatrix} 2D_{11} - \chi_1 \\ 2D_{12} - \chi_2 \\ 2D_{13} - \chi_3 \end{bmatrix}, \tag{14}$$

with some initial conditions

$$\xi = \left[\phi_2^+, \phi_3^+, \phi_2^{'+}, \phi_3^{'+}, \phi_2^{''+}, \phi_3^{''+}, u^+\right]^T \in \mathbb{R}^7, \tag{15}$$

where $(\cdot)^+$ denotes the values at $\theta_*(0)$. Solutions of these differential equations define the walking gait of the robot. Besides, it is worth to observe that the infinite dimensional difficult task of computing

- an unknown in advance time interval (a gait period), and
- an unknown external generalized force—a control signal $u = u_*(t)$—that shapes a gait

is converted into a simpler finite dimensional problem: finding initial conditions (15) for the differential equations (14). The last task is similar to those that may appear in searching for a gait in passive walkers [4], [12].

C. *Procedure for Searching a Symmetric Gait of the Walker (1)-(2)*

The iterative search of a symmetric gait for the walker (1)-(2) is explicitly based on the differential equations (14), which requires an appropriate initialization (15), the step interval $[\theta^+, \theta^-]$ and the spring coefficient K. We can define the initialization vector for the search as

$$\mathcal{X}_* = \left[K,\ \theta^+,\ \phi_3^+,\ \dot{\theta}^+,\ \phi_2^{'+},\ \phi_3^{'+},\ u^+\right]^T \in \mathbb{R}^7. \tag{16}$$

since the reminder of the initial conditions, i.e. $\phi_2^{''+}$ and $\phi_3^{''+}$, can be computed by solving (8) with respect to $\ddot{\theta}$, $\phi_2^{''}$, and $\phi_3^{''}$ and substituting the values from (16). In addition, (36) can be used to verify the solutions obtained.

As an example, we present one result for level ground walking, i.e. $\psi = 0$. The initial conditions for the found gait, written in the form (15) for the dynamical system (14), are

$$\begin{array}{r} \xi = [0.3375,\ 0.5816,\ -0.0883,\ 0.1620, \\ -4.4508,\ 1.7286,\ 2.7132]^T, \end{array} \tag{17}$$

and the found spring stiffness is $K = 20.3429$. The evolution of θ is within the step interval $[-0.3375, 0.3375]$. The constraint functions $\phi_2(\theta)$, $\phi_3(\theta)$ and the control signal $u = u_*(t)$ (expressed as a function of θ_*) that generate the gait are shown in Fig. 2. The corresponding vector of initial

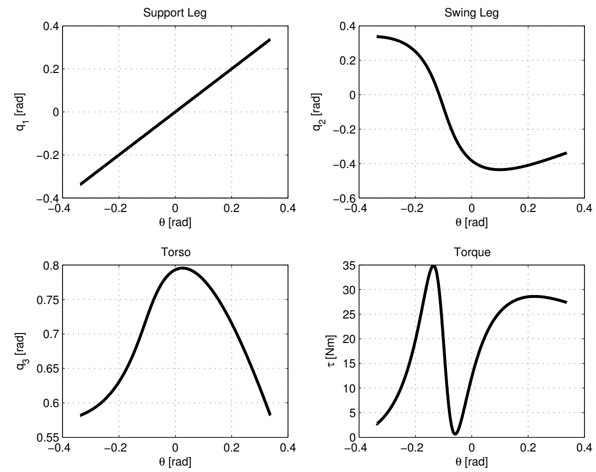

Fig. 2. Solutions of (14) as functions of θ with initial conditions (17). Top Left: Support leg. Top right: Swing Leg. Bottom left: Torso. Bottom right: Nominal input torque.

conditions of the gait written for the dynamics of (1) are

$$q_{1*}(0) \approx -0.3375,\ q_{2*}(0) \approx 0.3375,\ q_{3*}(0) \approx 0.5816$$
$$\dot q_{1*}(0) \approx 1.3470,\ \dot q_{2*}(0) \approx -0.1189,\ \dot q_{3*}(0) \approx 0.2182 \tag{18}$$

However, it can be verified that the found forced hybrid periodic motion of (1)-(2) is not stable. A possible design of a state feedback control law to make it orbitally exponentially stable is explained next.

IV. Design of an Orbital Stabilizing Feedback Control Law

For the synthesis of a controller, and analysis of the closed-loop system dynamics, the technique recently proposed in [13], [14] is applied. Such arguments rely on concepts of a moving Poincaré section and a linearization of hybrid transverse dynamics, which are of importance for analysis of dynamics in a vicinity of the gait, and can be efficiently constructed for forced motions of mechanical systems and further used for controller design.

Below we briefly present the general steps in constructing a transverse linearization, and then show how it is used for controller design.

A. Computing a Hybrid Transverse Linearization for a Gait of (1)-(2)

A hybrid transverse linearization of the 3-DOF walker dynamics (1)-(2) along its gait of period T_e is a hybrid linear control system defined as:

$$\tfrac{d}{d\tau}\,\hat x_\perp(\tau) = A(\tau)\hat x_\perp(\tau) + B(\tau)\hat u_\perp(\tau) \quad \tau \in [0, T_e], \tag{19}$$

$$\hat x_\perp(T_e^+) := L\,\hat x_\perp(T_e^-) \qquad \tau \in T_e, \tag{20}$$

where the linear system approximates dynamics transverse to the trajectory (5) and $L \in \mathbb{R}^{5\times 5}$ corresponds to a linearization of the impact map. This system is initialized by a vector $\hat x_\perp(0) = \hat x_\perp^0$, which is re-defined after each impact event as $\hat x_\perp^0 := \hat x_\perp(T_e^-)$.

Having known the nominal model for dynamics of the walker (1)-(2), its gait (5), and the associated constraint functions (6), the matrices $A(\tau)$, $B(\tau)$, and L of this linear system can be found analytically as described in [13]. As soon as a stabilizing feedback control law for (19), (20) is found, it can be transfered into a nonlinear orbitally exponentially stabilizing state feedback control law for the nonlinear system (1)-(2), see [13], [14]. To proceed with the necessary computations, we need to define a few specific coordinate and feedback transformations.

First, introduce the following new generalized coordinates for (1) in a vicinity of the motion:

$$\theta, \quad y_1 = q_2 - \phi_2(\theta), \quad y_2 = q_3 - \phi_3(\theta), \tag{21}$$

with the derivatives

$$\dot\theta \quad , \quad \dot y_i = \dot q_{i+1} - \phi'_{i+1}(\theta)\dot\theta, \tag{22}$$
$$\ddot\theta \quad , \quad \ddot y_i = \ddot q_{i+1} - \left(\phi''_{i+1}(\theta)\dot\theta^2 + \phi'_{i+1}(\theta)\ddot\theta\right), \tag{23}$$

where $i = \{1, 2\}$. Observe that as long as a control action makes the desired cycle invariant and the initial conditions are on the cycle, we have

$$\theta = \theta_*(t), \quad \dot\theta = \dot\theta_*(t), \quad y_i = 0, \quad \dot y_i = 0. \tag{24}$$

Dynamics (1) in the new coordinates can be found by introducing the expression from (21), (22) and (23) into dynamics of the robot (1). This substitution yields the dynamics of y in the following form:

$$\ddot y_i = R_i(y_i, \theta, \dot y_i, \dot\theta) + N_i(y_i, \theta)u, \quad i = \{1, 2\}, \tag{25}$$

The θ-dynamics is also found by the same substitution and it can be written as the following differential equation

$$\alpha_i(\theta)\ddot\theta + \beta_i(\theta)\theta^2 + \gamma_i(\theta) = g_i(\theta, \dot\theta, y, \dot y, \ddot y), \tag{26}$$

where $g_i(\cdot)$ is a smooth function that is equal to zero on the desired orbit.

It can be verified that $N_1(0, \theta_*(t)) \neq 0$ for all $t \in [0, T_e]$; hence, we define the following feedback transformation:

$$u = \frac{1}{N_1}u_\perp - \frac{R_1}{N_1}, \tag{27}$$

with $u_\perp \equiv 0$ along the target motion. Then, the nominal input $u_*(t)$ on the orbit is[4]

$$u_*(t) = U(\theta_*(t), \dot\theta_*(t), 0, 0) = -\frac{R_1(0, \theta_*(t), 0, \dot\theta_*(t))}{N_1(0, \theta_*(t), 0, \dot\theta_*(t))}, \tag{28}$$

and this brings the y_1-dynamics into the form: $\ddot y_1 = u_\perp$. Substituting (27) into (25) yields the y-dynamics as:

$$\begin{bmatrix} \ddot y_1 \\ \ddot y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ R_2 - \frac{N_2}{N_1}R_1 \end{bmatrix}}_{R(\theta,\dot\theta,y,\dot y)} + \underbrace{\begin{bmatrix} 1 \\ \frac{N_2}{N_1} \end{bmatrix}}_{N(\theta,y)} u_\perp, \tag{29}$$

[4]Note that it can also be expressed as a function of θ_* only.

The dynamical system (26), (29) possesses a natural choice of $(2n-1)$ - transverse coordinates

$$x_\perp = [I^{(i)}(\theta, \dot{\theta}, \theta_*(0), \dot{\theta}_*(0)), y_1, y_2, \dot{y}_1, \dot{y}_2]^T, \qquad (30)$$

where $I^{(i)}$ is a conserved quantity that measures the Euclidean distance to the desired cycle projected onto the $[\theta, \dot{\theta}]$ plane, see Appendix B. Consider the nonlinear dynamical system (29), (26) and its solution defined for $t \in [0, T_e]$

$$y_1 \equiv 0, \quad y_2 \equiv 0, \quad \theta \equiv \theta_*(t), \quad u_{\perp *} \equiv 0, \qquad (31)$$

the linearization of dynamics of the transverse coordinates (30) along (31) can be represented by the time-varying system (19).

Now, we proceed with designing a stabilizing feedback controller for the system (19)–(20).

B. Feedback controller design

Suppose there exist a C^1-smooth vector of gains $K(\cdot)$, such that the feedback control law

$$\hat{u}_\perp(\tau) = K(\tau)\, \hat{x}_\perp(\tau), \qquad (32)$$

stabilizes the trivial equilibrium of the hybrid linear system (19)–(20). Then, an orbitally stabilizing controller for the nonlinear system (29), (26) can be constructed as follows [13], [14]:

$$u_\perp(t) = K(s) x_\perp(t), \qquad s = s(\theta(t)), \qquad (33)$$

where $x_\perp(t)$ is given by (30), and $s(\cdot)$ is an index parameterizing the particular leaf of the moving Poincaré section, to which the vector $x_\perp$ belongs at time moments t, see [13].

Finally, the gain $K(\tau)$ can be chosen via a few steps of the numerical minimization procedure for the maximum absolute value of the eigenvalues of the state transition matrix $\Phi(T_e+)$ computed as follows

$$\begin{array}{ll} \frac{d}{d\tau}\Phi(\tau) = \big(A(\tau) + B(\tau)K(\tau)\big)\Phi(\tau), & 0 < \tau < T_e \\ \Phi(0^+) = I, \qquad \Phi(T_e^+) = L\,\Phi(T_e^-). & \end{array} \qquad (34)$$

In general, $K(\tau)$ can be computed via solving the associated differential Riccati equation on the period of the gait with additional constraints at the beginning and the end of the interval. i.e. $\dot{R} + A^T R + RA + G = RB\Gamma^{-1}B^T R$, $\forall \tau \in [0, T]$, with $G(\tau) > 0$ being a n-by-n positive definite matrix, and $\Gamma > 0$, such that $K(\tau) = -\Gamma^{-1}B(\tau)^T R(\tau)$. The boundary conditions should satisfy $R(0) < W_c$ and $R(T) = L^T W_c L$, with $W_c > 0$. Despite the challenge of this numerical problem, we have followed this general method and have computed $K(\tau)$ for the found gait, and its components are shown in Fig. 3.

V. Simulation Results

To verify that the transverse-linearization-based controller (33) is orbitally stabilizing the gait of the biped with a reasonable region of attraction, simulations with various initial conditions were performed. The results for one of such simulations of the nonlinear closed-loop system are shown on Fig. 4, depicting the exponential convergence of three

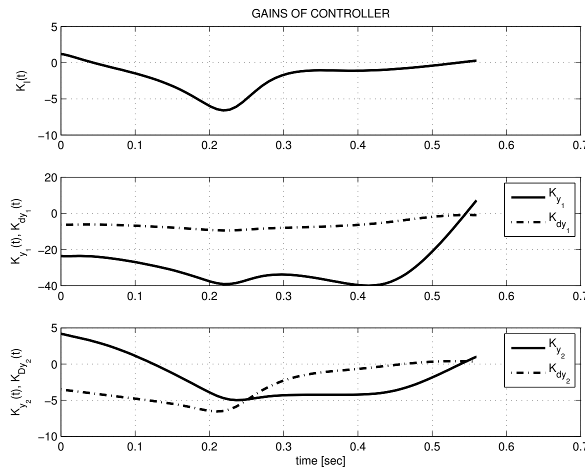

Fig. 3. Evolution of $K_I(t)$, $K_{y_1}(t)$, $K_{\dot{y}_1}(t)$, $K_{y_2}(t)$ and $K_{\dot{y}_2}(t)$ for one period of the walking gait.

transverse coordinates to the equilibrium, i.e. I, y_1, y_2. The other transverse coordinates – $\dot{y}_1(\cdot)$ and $\dot{y}_2(\cdot)$ – converge to zero with a similar rate. The limit cycle is depicted in Fig. 5.

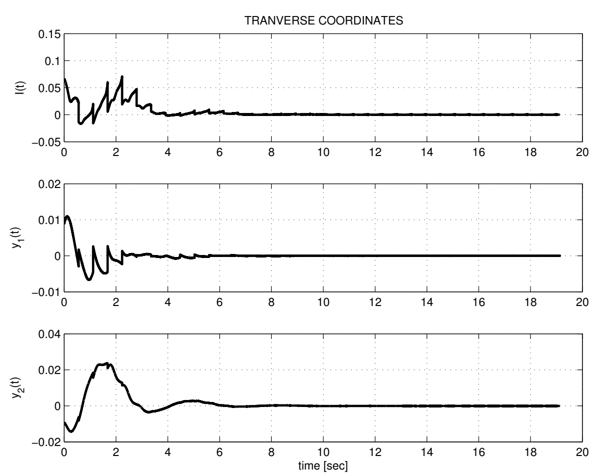

Fig. 4. Evolution of the transverse coordinates along a solution of the closed loop system with the perturbed initial conditions. The values of the transverse coordinates are about the order of 10^{-4} after 12 seconds.

VI. Conclusions

We have considered the model of a planar 3-DOF biped robot with two passive links. The main result is a new systematic procedure for gait synthesis based on the concept of virtual holonomic constraints. The procedure allows to represent forced hybrid periodic motions of the walker in a more convenient form. In fact, we have transformed the original infinite dimensional task of searching a gait into a finite dimensional one, consisting on finding appropriate initial conditions for an auxiliary set of differential equations.

For a found periodic gait, a hybrid transverse linearization has been computed. This auxiliary system, which represents dynamic transverse to the desired motion, has been used for the design of an exponentially orbitally stabilizing controller.

Simulation results have been shown to visualize the performance in closed loop.

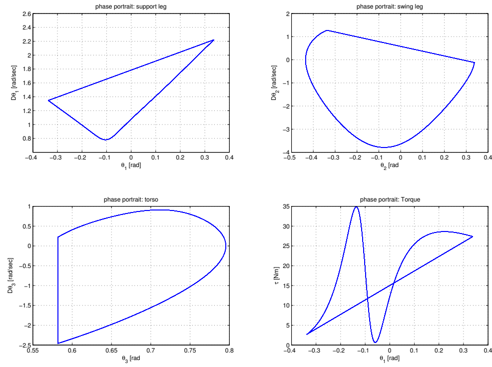

Fig. 5. Phase portrait for each degree of freedom. Top left: support leg, $\dot{\theta}$ vs. θ . Top right: swing leg, $\dot{q}_2(\theta)$ vs. $q_2(\theta)$. Bottom left: torso, $\dot{q}_3(\theta)$ vs. $q_3(\theta)$. Bottom right: torque, $u(t)$ vs. θ.

VII. Acknowledgement

The authors would like to thank Sergei V. Gusev for his help in resolving the numerical difficulties with solving the periodic matrix differential Riccati equation.

Appendix

A. Matrices for the dynamic model (1):

$$M = \begin{bmatrix} \left(M_h + \frac{5}{4}m + M_t\right) r^2 & -\frac{1}{2}mr^2 c_{12} & M_t r l c_{13} \\ -\frac{1}{2}mr^2 c_{12} & \frac{1}{4}mr^2 & 0 \\ M_t r l c_{13} & 0 & M_t l^2 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & -\frac{1}{2}mr^2 s_{12}\dot{q}_2 & M_t r l s_{13}\dot{q}_3 \\ \frac{1}{2}mr^2 s_{12}\dot{q}_1 & 0 & 0 \\ -M_t r l s_{13}\dot{q}_1 & 0 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} (q_1 - q_3)K - \left(M_h + \frac{3}{2}m + M_t\right) r s_1 g \\ (q_2 - q_3)K + \frac{1}{2}mgr^2 s_2 \\ (2q_3 - q_2 - q_1)K - M_t g l s_3 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

where

$$s_{1j} = \sin(q_1 - q_j), \quad c_{1j} = \cos(q_1 - q_j),$$

with $j = \{2, 3\}$.

B. Integral of motion

The first two second order differential equations of (8) are integrable.

Lemma 1: [16] Along the solutions of the differential equation

$$\alpha_i(\theta)\ddot{\theta} + \beta_i(\theta)\dot{\theta}^2 + \gamma_i(\theta) = 0, \qquad i = \{1, 2\} \tag{35}$$

initiated at $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0$, the integral function

$$I^{(i)}(\theta, \dot{\theta}, \theta_0, \dot{\theta}_0) = \dot{\theta}^2 - \Psi(\theta_0, \theta)\left[\dot{\theta}_0^2 - \int_{\theta_0}^{\theta} \Psi(s, \theta_0)\frac{2\gamma_i(s)}{\alpha_i(s)}ds\right], \tag{36}$$

with

$$\Psi(\theta_1, \theta_2) = \exp\left\{-2\int_{\theta_1}^{\theta_2} \frac{\beta_i(\tau)}{\alpha_i(\tau)} d\tau\right\} \tag{37}$$

preserves its zero value.

References

[1] Fumihiko Asano and Zhi Wei Luo. Asymptotic stability of dynamic bipedal gait with constraint on impact posture. In *Proceedings of the International Conference on Robotics and Automation*, pages 1246–1251, Pasadena, USA, May, 19-23 2008.

[2] C. Chevallereau, G. Bessonnet, G. Abba, and Y. Aoustin. *Bipedal Robots: Modeling, Design and Walking Synthesis*. John Wiley and Sons, New Jersey, 2009.

[3] S. Collins, A. Ruina, R. Tedrake, and M. Wisse. Efficient bipedal robots based on passive dynamic walkers. *Science Magazine*, 307:1082–1085, 2005.

[4] L.B. Freidovich, U. Mettin, A.S. Shiriaev, and M.W. Spong. A passive 2-DOF walker: Hunting for gaits using virtual holonomic constraints. *IEEE Trans. on Robotics*, 25(5):1202–1208, 2009.

[5] L.B. Freidovich and A.S. Shiriaev. Transverse linearization for mechanical systems with passive links, impulse effects, and friction forces. In *Proceedings of Joint 48th IEEE Conference on Decision and Control, and 28th Chinese Control Conference*, pages 6490–6495, Shanghai, P.R. China, 2009.

[6] A. Goswami, B. Espiau, and A. Keramane. Limit cycles in a passive compass gait biped and passivity-mimicking control laws. *Journal of Autonomous Robots*, 4(3):273–286, 1997.

[7] J.W. Grizzle, G. Abba, and F. Plestan. Asymptotically stable walking for biped robots: Analysis via systems with impulse effects. *IEEE Transactions on Automatic Control*, 46(1):51–64, 2001.

[8] Y. Hurmuzlu, F. Genot, and B. Brogliato. Modeling, stability and control of biped robots - a general framework. *Automatica*, 40(10):1647–1664, 2004.

[9] Y. Hurmuzlu and D.B. Marghitu. Rigid body collisions of planar kinematic chains with multiple contact points. *The International Journal of Robotics Research*, 13(1):82–92, 1994.

[10] F. Iida I. R. Manchester, U. Mettin and R. Tedrake. Stable dynamic walking over rough terrain: Theory and experiment. In *Proceedings of the 14th International Symposium on Robotics Research*, Lucerne, Switzerland, Aug. 31 - Sept. 3 2009.

[11] U. Mettin, P.X. La Hera, L.B. Freidovich, A.S. Shiriaev, and J. Helbo. Motion planning for humanoid robots based on virtual constraints extracted from recorded human movements. *Intelligent Service Robotics*, 1(4):289–301, 2008.

[12] A. L. Schwab and M.Wisse. Basin of attraction of the simplest walking model. In *Proceedings International Conference on Noise and Vibration*, Pittsburgh, Pennsylvania, Sept. 9-12 2001.

[13] A.S. Shiriaev and L.B. Freidovich. Transverse linearization for impulsive mechanical systems with one passive link. *IEEE Trans. on Automatic Control*, 54(12):2882–2888, 2009.

[14] A.S. Shiriaev, L.B. Freidovich, and S.V. Gusev. Transverse linearization for controlled mechanical systems with several passive degrees of freedom. *IEEE Transactions on Automatic Control*, 55(4), 2010.

[15] A.S. Shiriaev, J.W. Perram, and C. Canudas-de-Wit. Constructive tool for orbital stabilization of underactuated nonlinear systems: Virtual constraints approach. *IEEE Transactions on Automatic Control*, 50(8):1164–1176, 2005.

[16] A.S. Shiriaev, J.W. Perram, A. Robertsson, and A. Sandberg. Periodic motion planning for virtually constrained Euler–Lagrange systems. *Systems and Control Letters*, 55:900–907, 2006.

[17] E.R. Westervelt, J.W. Grizzle, C. Chevallereau, J.H. Choi, and B. Morris. *Feedback Control of Dynamic Bipedal Robot Locomotion*. CRC Press, Taylor and Francis Group, 2007.

[18] M. Wisse, A. L. Schwab, and F. C. T. van der Helm. Passive dynamic walking model with upper body. *Robotica*, 22(6):681–688, 2004.