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Non-rigid Formations of Nonholonomic Robots

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Abstract— **The paper deals with a general class of leader-follower formations of unicycle robots induced by a *constraint function* that depends on the position and the orientation of the vehicles. We study the *flexibility* of such formations by introducing the notion of *formation internal dynamics*, characterize its equilibria and give sufficient geometric conditions for their existence. In particular, we show that the displacement and the relative orientation of each follower with respect to the leader's reference frame are fixed if and only if the robots either move along circular paths or parallel straight lines. These equilibrium configurations always exist if the trajectory of the leader is a circle of sufficiently small curvature or a straight line.**

I. Introduction

Multiagent systems and cooperative control are nowadays research topics of increasing popularity, as witnessed by several monographs and books on the subject [1]–[4]. The research in this area has been stimulated by the recent technological advances in wireless communications and processing units, and by the observation that a team of agents offers numerous advantages over a single unit, such as, e.g., increased fault tolerance, greater area coverage, lower costs, greater accuracy, faster goal achievement.

One of the fundamental tasks of any multiagent system is the ability to achieve and maintain a desired formation. There is a large body of work on *formation control* in the literature, where a wide range of issues have been addressed: a list of key references, yet far from being complete is [5]–[8]. In the past few years, the interest in formation control has been awakened by the introduction of the original notion of *rigidity*, that essentially measures how much a formation can be deformed by a smooth motion without affecting the distance between neighboring agents. Moving from [9], Anderson and coworkers have started to systematically apply the *rigid graph theory* to the analysis of formations of autonomous robots and shown the relevance of the notion of rigidity in several branches of engineering (see [10] and the references therein). In [11], [12], graph rigidity ideas have been used to design decentralized gradient control laws for the stabilization of a group of kinematic points to a target formation. Recently, in [13], a distributed algorithm that stabilizes the shape of a relative sensing network to a desired formation has been proposed: the algorithm overcomes the main drawback in [11], [12], and relies on the global minimization of the *stress majorization function* (a tool from multidimensional scaling theory) associated to the network.

As it is apparent from the previous literature review, most of the research in the rigidity framework has focused, up to today, on graph theoretical issues and little attention has been devoted to the *physical constraints* of the agents. In this respect, two challenging open issues include the characterization of rigidity for formations of *nonholonomic robots* and the definition of general classes of such formations sharing the same rigidity properties. This last problem appears of special practical interest, since real-world applications generally require a specific degree of flexibility of the formation.

This paper builds upon previous contributions of the authors on leader-follower formation control [14] and addresses the previous open issues from a nonlinear control perspective. A general definition of formation for unicycles with a single leader robot, is introduced: the formation is induced through a constraint function F, that depends on the position and the orientation of the vehicles. We state conditions on F that guarantee that the followers' controls (i.e., their linear and angular velocities) are uniquely defined. In this setting, the relative position and orientation of the followers with respect to the leader's reference frame is not fixed (i.e., the formation is *not rigid*). The original notion of *formation internal dynamics* is introduced to study the flexibility of the robotic network. The equilibrium configuration of the internal dynamics is characterized by two main geometric results. A first theorem states that the displacement and the relative orientation of each follower with respect to the leader's reference frame are fixed if and only if the robots either move along circular paths or parallel straight lines. A second theorem shows that such equilibrium configurations always exist if the trajectory of the leader is a circle of sufficiently small curvature or a straight line.

The work is organized as follows: Sect. II presents the main theoretical results of the paper. In Sect. III the theory is applied to two specific constraint functions, and in Sect. IV conclusions are drawn.

The following notation will be used through the paper: S^1 denotes the quotient space $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, where $\mathbb{Z}$ is the set of the integer numbers; $\forall\, x = (x_1, x_2, \ldots, x_n)^T$, $y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$ $(n \geq 1)$, $\langle x,\, y\rangle = \sum_{i=1}^n x_i\, y_i$, $\|x\| = \sqrt{\langle x,\, x\rangle}$; $\forall\, \theta \in S^1$, $\tau(\theta) = (\cos\theta,\ \sin\theta)^T$, $\eta(\theta) = (-\sin\theta,\ \cos\theta)^T$; Given a differential manifold $\mathcal{M}$, $T_x\,\mathcal{M}$ denotes the tangent plane of $\mathcal{M}$ at $x \in \mathcal{M}$; Given two functions $f(t)$ and $g(t)$, $f(t) \circ g(t) = f(g(t))$ indicates their composition.

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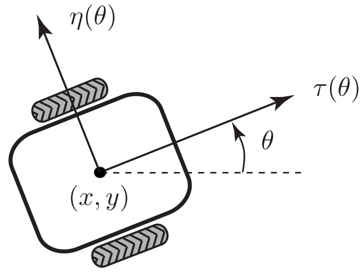

Fig. 1. Variables of the unicycle robot.

II. FORMATIONS OF UNICYCLE ROBOTS

Consider the following definition of robot as a velocity controlled unicycle model [14]:

***Definition** 1 (Unicycle robot):* A function $r = (x, y, \theta) \in \mathcal{C}^1([0, +\infty), \mathbb{R}^2 \times S^1)$ is called a *unicycle robot* (or a *robot*, for short) if there exists a control function $(v, \omega) \in \mathcal{C}^0([0, +\infty), \mathbb{R}^2)$ such that,

$$\begin{cases} \dot{x} = v\cos\theta, \\ \dot{y} = v\sin\theta, \\ \dot{\theta} = \omega. \end{cases}$$

For any $t \in [0, +\infty)$, denote by $(x(t),\, y(t))$ the position of the robot at time t, $\theta(t)$ its heading, $\tau(\theta(t))$ the normalized velocity vector and $\eta(\theta(t))$ the normalized vector orthogonal to $\tau(\theta(t))$. Hence, $\{\tau(\theta(t)),\, \eta(\theta(t))\}$ represents the robot reference frame at time t (see Fig. 1).

The next definition introduces an ordered set of $n + 1$ robots, with $n \geq 1$.

***Definition** 2 ((n + 1)-tuple of robots):* Let

$$\mathcal{X} = (\mathbb{R}^2 \times S^1)^{n+1} = \{\xi = (\xi_0, \ldots, \xi_n) \mid \xi_i = (x_i, y_i, \theta_i) \in \mathbb{R}^2 \times S^1, \forall\, i = 0, \ldots, n\}.$$

A $\mathcal{C}^1$ map $\mathcal{R} : [0, +\infty) \longrightarrow \mathcal{X}$, defined by $\mathcal{R}(t) = (r_0(t), \ldots, r_n(t))$, where $r_i = (x_i, y_i, \theta_i)$, $i = 0, 1, \ldots, n$, is a robot, is called an $(n + 1)$*-tuple of robots*. The set $\mathcal{X}$ is called the *configuration space*, r_0 the *leader* and $r_1, r_2, \ldots, r_n$ the *followers*.

***Definition** 3 (Constraint function):* Let F be the $\mathcal{C}^1$ map defined on $\mathcal{X}$ given by $F = (F_1, F_2, \ldots, F_n)$, where $F_i : \mathcal{X} \longrightarrow \mathbb{R}^2$, $\forall\, i = 1, \ldots, n$. The map F is called a *constraint function* if,

$$\partial_{\xi_j} F_i(\xi) = 0,\ \forall\, \xi \in \mathcal{X},\ i = 1, \ldots, n - 1,\ j = i + 1, \ldots, n, \tag{1}$$

i.e., every F_i depends only on $(\xi_0, \ldots, \xi_i)$.

***Definition** 4 (Constraint set):*
The set $\mathcal{F} = \{\xi \in \mathcal{X} \mid F(\xi) = 0\}$ is called the *constraint set*: it is the set of the configurations ξ compatible with the constraints $F_1, F_2, \ldots, F_n$.

The constraint function F on the ordered set of $(n + 1)$-robots, or $(n+1)$-tuple of robots, imposes a structure on the set of constraints: in fact, constraint F_i depends only on the position and the orientation of robots that have index less or equal than i.

The following definition introduces the notion of F-formation used through the paper.

***Definition** 5 (F-formation):* Let F be a constraint function. An $(n+1)$-tuple of robots $\mathcal{R}$ is said in F*-formation* if,

$$F(\mathcal{R}(t)) = 0,\ \forall\, t \geq 0\,,$$

i.e., if $\mathcal{R}(t) \in \mathcal{F}$, $\forall\, t \geq 0$.

Set $v_f = (v_1, v_2, \ldots, v_n)$, $\omega_f = (\omega_1, \omega_2, \ldots, \omega_n)$, where $(v_i,\, \omega_i)$ is the control function of i-th follower. In this way the kinematic equations of the $n + 1$ robots can be written in a compact form as,

$$\dot{\xi} = g(\xi, v_0, \omega_0, v_f, \omega_f), \tag{2}$$

where $(v_0,\, \omega_0)$ is the control of the leader and $g = (g_0, g_1, \ldots, g_n)$ is such that $\forall\, i = 0, \ldots, n$: $g_i(\xi, v_0, \omega_0, v_f, \omega_f) = (v_i\cos\theta_i,\ v_i\sin\theta_i,\ \omega_i)$.

***Proposition** 1:* Let F be a constraint function and suppose that $\forall\, i = 1, \ldots, n,\ \forall\, \xi \in \mathcal{F}$:

$$\det(\cos\theta_i\, \partial_{x_i} F_i(\xi) + \sin\theta_i\, \partial_{y_i} F_i(\xi),\ \partial_{\theta_i} F_i(\xi)) \neq 0. \tag{3}$$

Then the following facts hold:

i) $\mathcal{F}$ is a differential manifold of dimension $n + 3$.

ii) For each $\xi \in \mathcal{F}$ and for each $(v_0,\, \omega_0) \in \mathbb{R}^2$, there exist unique controls $v_f(\xi, v_0, \omega_0)$, $\omega_f(\xi, v_0, \omega_0)$ such that,

$$g(\xi, v_0, \omega_0, v_f(\xi, v_0, \omega_0), \omega_f(\xi, v_0, \omega_0)) \in T_\xi\, \mathcal{F}.$$

Proof: Denote by $F'(\xi)$ the Jacobian matrix of F. Since $\mathrm{rank}(F'(\xi)) = 2n$ by (3), we get that i) holds by the implicit function theorem. To prove ii), set $\forall\, i = 1, \ldots, n$, $\forall\, j = 0, \ldots, n$, $\partial_{\xi_j} F(\xi) = (\partial_{x_j} F_i(\xi), \partial_{y_j} F_i(\xi), \partial_{\theta_j} F_i(\xi))$, which is a 2×3 matrix, and $\forall\, i = 1, \ldots, n$,

$$\begin{aligned} A_{i,j}(\xi) &= \partial_{\xi_j} F_i(\xi) \begin{pmatrix} \cos\theta_j & 0 \\ \sin\theta_j & 0 \\ 0 & 1 \end{pmatrix} \\ &= (\cos\theta_j\, \partial_{x_j} F_i(\xi) + \sin\theta_j\, \partial_{y_j} F_i(\xi), \partial_{\theta_j} F_i(\xi)). \end{aligned}$$

Then, by hypothesis (3), $\forall\, (v_0, \omega_0) \in \mathbb{R}^2$, $v_f = (v_1, v_2, \ldots, v_n)$, $\omega_f = (\omega_1, \omega_2, \ldots, \omega_n)$ are given by the unique solution of the following triangular system:

$$A_{i,i}(\xi) \begin{pmatrix} v_i \\ \omega_i \end{pmatrix} = -\sum_{j=0}^{i-1} A_{i,j}(\xi) \begin{pmatrix} v_j \\ \omega_j \end{pmatrix},\ i = 1, \ldots, n. \qquad \blacksquare$$

***Remark** 1:* Condition (3) in Proposition 1 guarantees that if the vehicles are in F-formation at time $t = 0$, then there exist unique controls v_f, ω_f for the followers such that they remain in F-formation for all $t \geq 0$, independently on the leader's controls v_0, ω_0, i.e., independently on the trajectory of the leader.

***Definition** 6 (Regular constraint function):* Let F be a constraint function. We say that F is *regular* if condition (3) is satisfied.

***Remark** 2:* In Definition 3 we introduced $2n$ scalar constraints for the followers on the $(n+1)$-tuple of robots. This is a necessary condition for the uniqueness of the controls v_f, ω_f, and it is not a loss of generality since with less constraints it is always possible to restate the problem by adding a suitable number of virtual ones.

To properly study the relative position and orientation between the robots, we introduce the following definition.

***Definition** 7 (Rototranslation invariance):* For any parameter $p = (\bar{x}, \bar{y}, \bar{\theta}) \in \mathbb{R}^2 \times S^1$, define the map $H_p : \mathbb{R}^2 \times S^1 \longrightarrow \mathbb{R}^2 \times S^1$ as,

$$H_p(x, y, \theta) = \big(R(\bar{\theta}) \begin{pmatrix} x + \bar{x} \\ y + \bar{y} \end{pmatrix}, \theta + \bar{\theta}\big),$$

where matrix $R(\theta) = (\tau(\theta), \eta(\theta))$. For any $\xi = (\xi_0, \dots, \xi_n) \in \mathcal{X}$, set $H_p(\xi) = (H_p(\xi_0), \dots, H_p(\xi_n))$. A map $G \in \mathcal{C}^1(\mathcal{X}, \mathbb{R}^m)$, is called *rototranslation invariant* if

$$G(\xi) = G(H_p(\xi)), \forall\, \xi \in \mathcal{X},\ \forall\, p \in \mathbb{R}^2 \times S^1.$$

In other words, a function of the configuration space $\mathcal{X}$ is rototranslation invariant when its value does not change when any rototranslation H_p is applied to its argument.

***Remark** 3:* Let F be a rototranslation invariant constraint function and suppose that the $(n+1)$-tuple of robots $\mathcal{R} = (r_0, \dots, r_n)$ is in F-formation. Therefore for any $\lambda \in \mathcal{C}^1([0, +\infty), \mathbb{R})$, $\widetilde{\mathcal{R}}(t) = \mathcal{R} \circ \lambda$ is in F-formation. Hence if $\|v_0(t)\| \neq 0$, $\forall\, t \geq 0$, taking $\lambda(t) = \int_0^t v_0^{-1}(\tau)\, d\tau$, it follows that the forward velocity of the leader of $\widetilde{\mathcal{R}}$ is always 1.

This justifies the following assumption:

***Assumption** 1:* We will henceforth suppose that the forward velocity v_0 of the leader r_0 is equal to 1.

As a consequence, $\omega_0(t)$ represents the *curvature* of the path $(x_0(t), y_0(t))$ followed by the leader r_0 at time t.

Let us now introduce the following *equivalence relation* "$\sim$" on $\mathcal{X}$.

***Definition** 8:* Given two vectors $\xi, \vartheta \in \mathcal{X}$, $\vartheta \sim \xi$ if $\exists\, p \in \mathbb{R}^2 \times S^1$ such that $\vartheta = H_p(\xi)$.

We denote by $[\![\xi]\!] = \{\vartheta \in \mathcal{X} \mid \vartheta \sim \xi\}$ and by $\mathcal{X}/\!\sim\, = \{[\![\xi]\!] \mid \xi \in \mathcal{X}\}$ the quotient set.

Note that G is a rototranslation invariant function if and only if for any $\xi \in \mathcal{X}$, G is constant on the set $[\![\xi]\!]$.

In the case of rototranslation invariant constraint functions, it is natural to define a *reduced constraint set* as follows:

***Definition** 9 (Reduced constraints set):* The *reduced constraints set* $\mathcal{F}_r$ is the set: $\mathcal{F}_r = \mathcal{F}/\!\sim$.

In this way, each element of $\mathcal{F}_r$ represents a set of configurations for the formation that differ only by a rototranslation.

***Proposition** 2:* Let F be a regular rototranslation constraint function and $\mathcal{R}^1$, $\mathcal{R}^2$ be two $(n+1)$-tuples of robots in F-formation. Suppose that,

$$\mathcal{R}^1(0) \sim \mathcal{R}^2(0), \tag{4}$$

then

$$\mathcal{R}^1(t) \sim \mathcal{R}^2(t),\ \forall\, t \geq 0\,. \tag{5}$$

Proof: By (4) there exists $p \in \mathbb{R}^2 \times S^1$, such that $H_p(\mathcal{R}^1(0)) = \mathcal{R}^2(0)$. Proving (5) is equivalent to show that,

$$H_p(\mathcal{R}^1(t)) = \mathcal{R}^2(t)\ , \forall\, t \geq 0. \tag{6}$$

In fact $F(H_p(\mathcal{R}^1(t))) = 0$ since F is rototranslation invariant: moreover $H_p(\mathcal{R}^1(t))$ is still an $(n+1)$-tuple of robots. Then (6) holds by part $ii)$ of Proposition 1. ■

Let $\omega_0 \in \mathcal{C}([0, +\infty), \mathbb{R})$ be the control for the leader. As a consequence of Proposition 2 the following map is well defined: $\Phi_{\omega_0} : [0, +\infty) \times \mathcal{F}_r \longrightarrow \mathcal{F}_r$, $(t, [\![\zeta]\!]) \rightsquigarrow \Phi_{\omega_0}(t, [\![\zeta]\!])$, where,

$$\Phi_{\omega_0}(t, [\![\zeta]\!]) = [\![\xi(t)]\!]\,, \tag{7}$$

being ξ the only solution of the following system,

$$\dot{\xi} = g(\xi, 1, \omega_0, v_f(\xi, 1, \omega_0), \omega_f(\xi, 1, \omega_0)),\ \ \xi(0) = \zeta,$$

where g is given by (2).

***Remark** 4:* Let F be a rototranslation invariant constraint function. Set,

$$\Gamma = \{\xi = (\xi_0, \dots, \xi_n) \in \mathcal{X} \mid \xi_0 = 0,\ F(\xi) = 0\}. \tag{8}$$

Then the map,

$$\begin{array}{rcl} \mathcal{F}_r & \longrightarrow & \Gamma \\ [\![\xi]\!] & \rightsquigarrow & H_{-\xi_0}(\xi)\ , \end{array} \tag{9}$$

is a bijection. Moreover an $(n+1)$-tuple of robots $\mathcal{R}$ is in F-formation if and only if $H_{-r_0(t)}(\mathcal{R}(t)) \in \Gamma$, $\forall\, t \geq 0$.

As suggested by (9), $\mathcal{F}_r$ is the set of all configurations of followers in the leader's reference frame that are compatible with the constraint function.

***Definition** 10 (Reduced motion):* Given an $(n+1)$-tuple of robots $\mathcal{R}$, the *reduced motion* of $\mathcal{R}$ is the map $[\![\mathcal{R}]\!] : [0, +\infty) \longrightarrow \mathcal{X}/\!\sim$ defined by,

$$[\![\mathcal{R}]\!](t) = [\![\mathcal{R}(t)]\!],\ \forall\, t \geq 0.$$

Note that $[\![\mathcal{R}]\!]$ describes the motion of the followers in the leader's reference frame.

***Remark** 5:* Let $\mathcal{R} = (r_0, \dots, r_n)$ be an $(n+1)$-tuple of robots. $[\![\mathcal{R}]\!]$ is constant if and only if there exists $\bar{\xi} \in \Gamma$ such that,

$$H_{-r_0(t)}(\mathcal{R}(t)) = \bar{\xi},\ \ \forall\, t \geq 0\,.$$

Since $\mathcal{F}_r$ is the quotient set of $\mathcal{F}$ by the equivalence relation $\sim$, it has a lower dimension than $\mathcal{F}$ as specified by the following proposition.

***Proposition** 3:* If F is a regular rototranslation invariant constraint function, $\mathcal{F}_r$ is a differential manifold of dimension n, diffeomorphic to Γ.

Proof: By (3) and the implicit function theorem, the subset of $\mathcal{F}$ given by Γ (see equation (8)) is a submanifold of $\mathcal{X}$ of dimension n. Since the map (9) is a bijection, this induces a natural structure of differential manifold on $\mathcal{F}_r$. ■

***Remark** 6 (Flexibility of F-formations):* A consequence of Proposition 3 is that the F-formations are *not rigid*. This is due to the fact that we are dealing with robots described by 3 configuration variables, but function F introduces only 2 constraints for each robot: this results in a residual degree of freedom for each vehicle. More precisely, a formation is rigid if the set $\mathcal{F}_r$ is composed by only one equivalence class: this is not true in our case, since $\mathcal{F}_r$ is a manifold of dimension n.

***Definition** 11 (Formation internal dynamics):* Suppose F is a regular and rototranslation invariant constraint function such that, $\forall\, \xi \in \Gamma$,

$$\det(\partial_{x_1} F(\xi), \partial_{y_1} F(\xi), \dots, \partial_{x_n} F(\xi), \partial_{y_n} F(\xi)) \neq 0\,. \tag{10}$$

Set $\beta_i = \theta_i - \theta_0$. By the implicit function theorem, there exists a $\mathcal{C}^1$ diffeomorphism $\gamma : T^n \longrightarrow \Gamma$, $\beta = (\beta_1, \ldots, \beta_n) \rightsquigarrow \gamma(\beta)$, where T^n is the n-torus. Let us identify $\mathcal{F}_r$ with Γ and $\forall \zeta \in \Gamma$, $\forall \omega_0 \in \mathcal{C}([0, +\infty), \mathbb{R})$, let $\Phi_{\omega_0} : [0, +\infty) \times \Gamma \longrightarrow \Gamma$, $(t, \zeta) \rightsquigarrow \Phi_{\omega_0}(t, \zeta)$, be the map given by (7). Then for any $\bar{\beta} \in T^n$, set,

$$\beta(t, \bar{\beta}, \omega_0) = \gamma^{-1}(\Phi_{\omega_0}(t, \gamma(\bar{\beta}))) \ , \forall\, t \geq 0,$$

which belongs to $\mathcal{C}^1([0, +\infty), \mathbb{R}^n)$. Then $\beta(t, \bar{\beta}, \omega_0)$ is the unique solution of the following system,

$$\begin{cases} \dot{\beta}(t) = h(\beta(t), \omega_0), \\ \beta(0) = \bar{\beta}, \end{cases} \tag{11}$$

where, $\forall\, t \in [0, +\infty)$, $\forall\, \beta \in T^n$: $h(\beta, \omega_0) = \frac{d}{d\tau}(\gamma^{-1}(\Phi_{\omega_0}(\tau, \gamma(\beta))))(0)$. Equation (11) is called *formation internal dynamics*.

System (11) represents the motion of the followers in the reference frame of the leader. Note that the equilibria of function h are the configurations that are constant in the leader's reference frame.

The following theorem states that if property (12) (see below) holds, then the displacement and the relative orientation of each follower with respect to the leader's reference frame are fixed if and only if the robots either move along circular paths or parallel straight lines. The proof of the theorem is omitted due to space limitations.

***Theorem** 1:* Suppose that F is a regular rototranslation invariant constraint function and that:

$$\begin{cases} \mathcal{F} \text{ does not contain any point of the set:} \\ \Big\{ \xi \in \mathcal{X} \mid \langle \begin{pmatrix} x_i - x_0 \\ y_i - y_0 \end{pmatrix}, \tau(\theta_0) \rangle = 0, \\ \quad \theta_i \in \{\theta_0, \theta_0 + \pi\}, \forall\, i = 1, \ldots, n \Big\}. \end{cases} \tag{12}$$

Suppose that an $(n+1)$-tuple of robots $\mathcal{R}$ is in F-formation. The following properties are equivalent:

$i)$ $[\![\mathcal{R}]\!]$ is constant.

$ii)$ ω_0 is constant, moreover:

$a)$ if $\omega_0 = 0$, then $\theta_0(t) = \bar{\theta}_0$, $\forall\, t \geq 0$, and $\forall\, i = 1, \ldots, n$, $\theta_i(t) = \{\bar{\theta}_0, \bar{\theta}_0 + \pi\}$, $\forall\, t \geq 0$.

$b)$ if $\omega_0 \neq 0$, then there exists $(\bar{x}, \bar{y})^T \in \mathbb{R}^2$ such that $\forall\, i = 0, \ldots, n$:

$$\langle \begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} - \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}, \tau(\theta_i(t)) \rangle = 0, \ \forall\, t \geq 0. \qquad \blacksquare$$

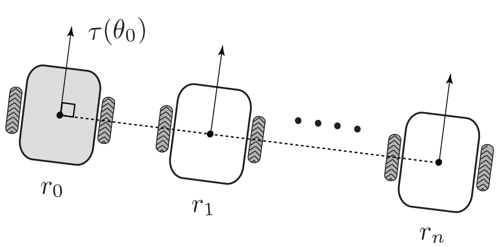

Fig. 2. Forbidden configuration in the manifold $\mathcal{F}$.

From a geometric viewpoint, condition (12) means that the manifold $\mathcal{F}$ must not contain configurations in which all followers are placed on the straight line passing through the leader and orthogonal to the leader's heading direction $\tau(\theta_0)$, with the robots all oriented in the same direction (see Fig. 2).

***Definition** 12 (Nice constraint function):* Let F be a constraint function. F is called a *nice* constraint if it satisfies the following properties:

$i)$ F is regular,

$ii)$ F is rototranslation invariant,

$iii)$ F verifies property (12),

$iv)$ F verifies property (10).

The following theorem provides a sufficient condition for the existence of the equilibrium configurations introduced in Theorem 1, when the angular velocity ω_0 of the leader is constant and sufficiently small. For any $i = 1, \ldots, n$, let $\Pi_i : \mathcal{X} \longrightarrow \mathbb{R}^2$ be the linear function defined by $\Pi_i(\xi) = (x_i, y_i)^T$.

***Theorem** 2:* Let F be a nice constraint function. Set,

$$\bar{\rho} = \sup\{\|\Pi_i(\xi)\| \mid \forall\, \xi \in \Gamma, \forall\, i = 1, \ldots, n\}. \tag{13}$$

Then $\bar{\rho} > 0$ and for any $\bar{\omega}$ such that,

$$-1/\bar{\rho} < \bar{\omega} < 1/\bar{\rho}, \tag{14}$$

there exists an $(n+1)$-tuple of robots $\mathcal{R}$ such that $[\![\mathcal{R}]\!]$ is constant, $\omega_0(t) = \bar{\omega}$ and $|\theta_i(t) - \theta_0(t)| < \arcsin(\bar{\omega}\bar{\rho})$, $\forall\, t \geq 0$.

Proof: First of all $\bar{\rho} > 0$, since F verifies property (12). Suppose that $\bar{\omega} \neq 0$. Since the n-torus T^n is compact and (10) holds, there exists a map $\gamma : T^n \longrightarrow \Gamma$ such that $\beta = (\beta_1, \ldots, \beta_n) \rightsquigarrow \gamma(\beta)$, which is a global diffeomorphism. Let $J : [-\pi, \pi]^n \longrightarrow T$ be the canonical continuous immersion and $\alpha : \Gamma \longrightarrow [-\pi, \pi]^n$ be the map defined by $\alpha(\xi) = (\arg(\Pi_1(\xi) - (0, 1/\bar{\omega})^T) + \pi/2, \ldots, \arg(\Pi_n(\xi) - (0, 1/\bar{\omega})^T) + \pi/2)$. Note that α is a continuous well defined map since, $\Pi_i(\xi) - (0, 1/\bar{\omega})^T \neq 0$, $\forall\, \xi \in \Gamma$, $\forall\, i = 1, \ldots, n$, because, by hypothesis (14), vector $(0, 1/\bar{\omega})^T \notin \bigcup_i \Pi_i(\Gamma)$. Then define the continuous map $\alpha \circ \gamma \circ J : [-\pi, \pi]^n \longrightarrow [-\arcsin(\bar{\omega}\bar{\rho}), \arcsin(\bar{\omega}\bar{\rho})]^n$, where $\bar{\beta} =$, (the fact that the image set is $[-\arcsin(\bar{\omega}\bar{\rho}), \arcsin(\bar{\omega}\bar{\rho})]^n$ can be obtained from (14) by simple geometrical considerations). This map has a fixed point, i.e., there exists $\bar{\beta} = (\bar{\beta}_1, \ldots, \bar{\beta}_n)$ such that,

$$\bar{\beta}_i = \arg(\Pi_i(\gamma(\bar{\beta}))) - (1, 1/\bar{\omega})^T + \pi/2, \ \forall\, i = 1, \ldots, n. \tag{15}$$

Let $r_0 = (x_0, y_0, \theta_0)$ be the leader robot which has the constant controls $v_0(t) = 1$, $\omega_0(t) = \bar{\omega}$, $\forall\, t \geq 0$ and initial condition $x_0(0) = 0$, $y_0(0) = 0$, $\theta_0(0) = 0$, and set $\forall\, i = 1, \ldots, n$, $r_i(t) = (x_i(t), y_i(t), \theta_i(t))$, where $(x_i(t), y_i(t))^T = (x_0(t), y_0(t))^T + R(\theta_0(t))\Pi_i(\xi(\bar{\beta}))$, and $\theta_i(t) - \theta_0(t) = \bar{\beta}_i$. First of all, r_i are robots, $\forall\, i = 1, \ldots, n$.

In fact, by (15) it follows that:

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} &= R(\theta_0(t))(1,0)^T + \bar{\omega} R(\theta_0(t)+\pi/2)\Pi_i(\gamma(\bar{\beta})) \\ &= \bar{\omega}\, R(\theta_0(t)+\pi/2)\,\Pi_i(\gamma(\bar{\beta}) - (0,\, 1/\bar{\omega})^T) \\ &= \rho_i\, \bar{\omega}\, R(\theta_0(t)+\pi/2) R(\bar{\beta}_i - \pi/2)(1,\, 0)^T \\ &= \rho_i\, \bar{\omega}\, R(\theta_i(t))(1,\, 0)^T,\end{aligned}$$

where $\rho_i = \|\Pi_i(\gamma(\bar{\beta})) - (0,\, 1/\bar{\omega})^T\|$. Then $\mathcal{R} = (r_0, r_1, \ldots, r_n)$ is an $(n+1)$-tuple of robots which is in F-formation and $[\![R]\!]$ is constant by Remarks 3 and 5, since, $H_{-r_0(t)}(\mathcal{R}(t)) = \gamma(\bar{\beta}) \in \Gamma$, by construction. ∎

Let us recall the following definition [15, Prop. III.2]:

***Definition** 13 (Cooperative system):* System (11) is *cooperative* on $\mathcal{B}$ with control in $W \subset \mathbb{R}$ if, $\forall\, \beta \in \mathcal{B}$, $\forall\, \omega_0 \in W$, $\partial_{\beta_j} h_i(\beta, \omega_0) \geq 0$, $\forall\, i \neq j$ and $\partial_{\omega_0} h_i(\beta, \omega_0) \geq 0$, $\forall\, i$.

The condition of the next proposition ensures that the formation internal dynamics (11) lies in the interior of a box when ω_0 is bounded: this means that the variations in shape of the formation are limited. The symbol $\preceq$ is used to denote componentwise inequality between vectors.

***Proposition** 4:* Let $\beta_l, \beta_u \in \mathcal{B}$ and $\omega_l, \omega_u \in W$ be such that

$$h(\beta_l, \omega_l) = 0,\ h(\beta_u, \omega_u) = 0\ ,$$

and $\beta_l \preceq \beta_u$, $\omega_l \leq \omega_u$. Set $\mathcal{S} = \{\beta \in \mathcal{B} \,|\, \beta_l \preceq \beta \preceq \beta_u\}$. If system (11) is cooperative in $\mathcal{S}$ and if $\omega_0(t) \in [\omega_l, \omega_u]$, $\forall\, t \geq 0$, then $\mathcal{S}$ is invariant for system (11).

Proof: Is a consequence of the definition of cooperative controlled systems (see Sect. 3 and Proposition III.2 of [15]). ∎

III. Some examples

In this section, the theory presented in Sect. II is applied to a specific constraint function, which induces the type of hierarchical formation introduced in [14]. More precisely, we suppose that for $i = 1, \ldots, n$, the robot r_i follows the (relative leader) r_{l_i}, where $l_i \in \{0, 1, \ldots, i-1\}$, in such a way that r_{l_i} is constant in the relative frame of r_i, that is the distance between r_{l_i} and r_i is equal to a constant d_i and r_i sees r_{l_i} with a constant visual angle ϕ_i (see Fig.(3) (a)). To this end, given $d_1, \ldots, d_n > 0$, $\phi_1, \ldots, \phi_n : |\phi_i| < \frac{\pi}{2}$, we set $\forall i = 1, \ldots, n$

$$F_i(\xi_{l_i}, \xi_i) = \begin{pmatrix} x_{l_i} \\ y_{l_i} \end{pmatrix} - \begin{pmatrix} x_i \\ y_i \end{pmatrix} - d_i\, \tau(\theta_i + \phi_i). \qquad (16)$$

F is a regular constraint function since (1) and (3) are satisfied: in fact $\det(\cos\theta_i\, \partial_{x_i} F_i + \sin\theta_i\, \partial_{y_i} F_i,\, \partial_{\theta_i} F_i) = d_i \cos\phi_i$ and $|\phi_i| < \pi/2$ by hypothesis. Clearly F is rototranslation invariant and it verifies properties (12) and (10). Therefore F is a nice constraint function and we can apply the theory developed in Sect. II. In particular, from Proposition 1, we have that if $\mathcal{R}$ is an $(n+1)$-tuple of robots which is in F-formation at the initial time (i.e., $F(\mathcal{R}(0)) = 0$), then for any trajectory of the leader, there exist and are unique the controls $v_f = (v_1, \ldots, v_n)$, $\omega_f = (\omega_1, \ldots, \omega_n)$

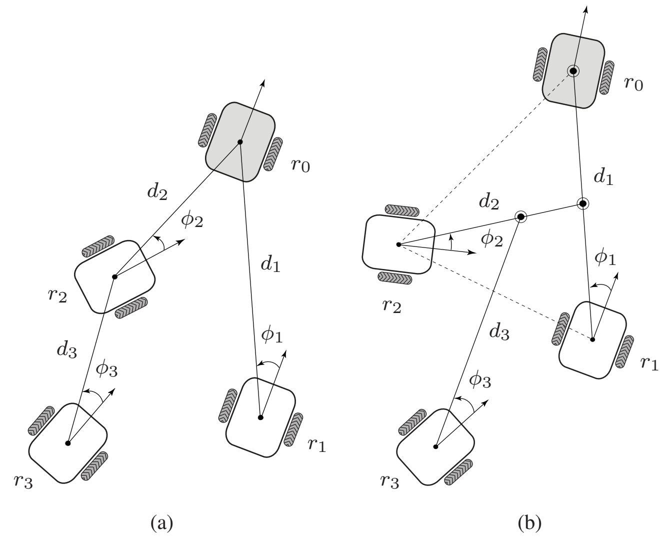

Fig. 3. (a) Sample formation with 4 robots; (b) General formation with 4 robots and $\lambda_{i,k} = 1/i$, $i = 1, 2, 3$, $k = 0, \ldots, i-1$.

for the followers such that $\mathcal{R}$ is in F-formation for any $t \geq 0$ (i.e., $F(\mathcal{R}(t)) = 0$, $\forall\, t \geq 0$). These controls are given by:

$$v_i\ =\ v_{l_i} \frac{\cos(\theta_{l_i} - \theta_i - \phi_i)}{\cos\phi_i}, \quad \omega_i\ =\ v_{l_i} \frac{\sin(\theta_{l_i} - \theta_i)}{d_i \cos\phi_i}.$$

In particular from Theorem 1 we have that an $(n+1)$-tuple of robots $\mathcal{R}$ in F-formation has fixed configurations in the leaders reference frame (i.e., $[\![\mathcal{R}]\!]$ is constant) if and only if all robots move along straight lines or circles. From Theorem 2 we have that such a fixed relative configuration exists when the angular velocity of the leader ω_0 is constant and $|\omega_0| < 1/\bar{\rho}$, where $\bar{\rho}$ is given by (13).

Let us define the weighted digraph $\mathcal{G} = (V, E, Q)$ associated to (16), where $V = \{0, 1, \ldots, n\}$ is the set of indices of the $n+1$ robots, $E = \{(l_i,\, i),\, i = 1, \ldots, n\}$ and Q is an $(n+1) \times (n+1)$ weighted adjacency matrix with the following properties: for $k, j = 0, \ldots, n$, the entry $q_{kj} = d_j$ if $(k,\, j) \in E$, and $q_{kj} = 0$ otherwise. By definition $\mathcal{G}$ is a *directed tree* and Q is *upper triangular*.

With the previous graph theoretic notions in hand, it is easy to prove that $\bar{\rho} \leq \max\{\text{weighted dist}_{\mathcal{G}}(0, u) \,|\, u \in V\} =$ weighted depth of the directed tree $\mathcal{G}$ [4, p. 20].

Let us choose $\beta_i = \theta_0 - \theta_i$: the formation internal dynamics is then given by, $\forall i = 1, \ldots, n$,

$$\dot{\beta}_i = h_i(\beta, \omega_0) = \omega_0 + \frac{v_{l_i}(\beta)}{d_i \cos\phi_i}\, \sin(\beta_{l_i} - \beta_i). \qquad (17)$$

where, $\forall i = 1, \ldots, n$

$$v_i(\beta) = v_{l_i}(\beta)\, \frac{\cos(\beta_i - \beta_{l_i} - \phi_i)}{\cos\phi_i}\ . \qquad (18)$$

Let us assume that $\omega_0(t) \in [a,\, b]$, where a, b are such that, $\forall\, i = 1, \ldots, n$,

$$-\frac{\pi}{2} < \arcsin(a\bar{\rho}) - \phi_i < \arcsin(b\bar{\rho}) - \phi_i < \frac{\pi}{2}\ , \qquad (19)$$

(this condition can always be satisfied by choosing $|a|$ and $|b|$ sufficiently small). Let β_a, β_b be the equilibrium values for the formation internal dynamics when $\omega_0 = a$ and $\omega_0 = b$, respectively (these equilibria exist by Theorem 2). Then $h(\beta_a, \omega_a) = h(\beta_b, \omega_b) = 0$ and, since, by Theorem 2 and (19) $|\beta_i - \beta_{l_i} - \phi_i| < \frac{\pi}{2}$, $\forall i = 1, \dots, n$, then, by (18), $v_i(\beta) > 0$, $\forall i = 1 \dots, n$. Therefore system (17) is cooperative on $\mathcal{S} = \{\beta \,|\, \beta_a \preceq \beta \preceq \beta_b\}$ with control ω_0 in $[a, b]$ and, by Proposition 4, set $\mathcal{S}$ is invariant for (17).

Note that we have already addressed the problem of determining an invariant set for (17) in [14, Th. 1] using a different method, that did not rely on the notion of cooperativity.

Let us now consider the following generalization of (16). This time robot r_i is required to follow an assigned *convex combination* of the positions of the preceding $i-1$ robots at a fixed distance $d_i > 0$ and with a fixed visual angle $|\phi_i| < \pi/2$:

$$F_i(\xi_0, \xi_1, \dots, \xi_i) = \sum_{k=0}^{i-1} \lambda_{i,k} \begin{pmatrix} x_k \\ y_k \end{pmatrix} - \begin{pmatrix} x_i \\ y_i \end{pmatrix} - d_i\, \tau(\theta_i + \phi_i), \tag{20}$$

where for any $i = 1, \dots, n$, $\lambda_{i,0}, \dots, \lambda_{i,i-1} > 0$ and $\sum_{k=0}^{i-1} \lambda_{i,k} = 1$, (see Fig. 3(b)). This particular function may be useful to describe formations occurring in nature, such as, e.g., bird flocks, where is believed that each animal follows an average of the position of the preceding birds. F in (20) inherits the same properties of (16): in fact, it is a nice constraint function. The unique control inputs for the followers are in this case: $v_i = \frac{1}{\cos\phi_i} \sum_{k=0}^{i-1} \lambda_{i,k}\, v_k \cos(\theta_k - \theta_i - \phi_i)$ and $\omega_i = \frac{1}{d_i \cos\phi_i} \sum_{k=0}^{i-1} \lambda_{i,k}\, v_k \sin(\theta_k - \theta_i)$. This time we can bound $\bar{\rho}$ as follows,

$$\bar{\rho} \leq \max_{i=1,\dots,n} \Big[d_i + \sum_{k=0}^{i-1} \lambda_{i,k}\, \bar{\rho}_k \Big], \tag{21}$$

where $\bar{\rho}_0 = 0$. Let us select $\beta_i = \theta_0 - \theta_i$ and set $\beta_0 = 0$: the formation internal dynamics is then given by,

$$\dot{\beta}_i = \omega_0 - \frac{1}{d_i \cos\phi_i} \sum_{k=0}^{i-1} \lambda_{i,k}\, v_k(\beta) \sin(\beta_i - \beta_k).$$

In the simulation experiment shown in Fig. 4, the formation of Fig. 3(b) with $(d_i, \phi_i) = (1, \pi/4)$, $i = 1, 2, 3$, has been considered. We have set $\lambda_{i,k} = 1/i$, $i = 1, 2, 3$, $k = 0, \dots, i-1$: in this way robot i follows exactly the average of the positions of the preceding $i-1$ vehicles in the formation. The leader r_0 moves along a circular trajectory with constant curvature $\omega_0(t) = 0.25$ rad/s (a circle of radius 4 m). At the initial time $t = 0$ the robots are in formation and all aligned with the x-axis (i.e., $\theta_i(0) = 0$). The robots asymptotically reach an equilibrium configuration in which they all move along circles with the same center and each follower occupies a fixed position in the leader's reference frame. This equilibrium condition exists as a consequence of Theorem 2, since in this case, applying (21), we obtain $1/\bar{\rho} = 0.546$.

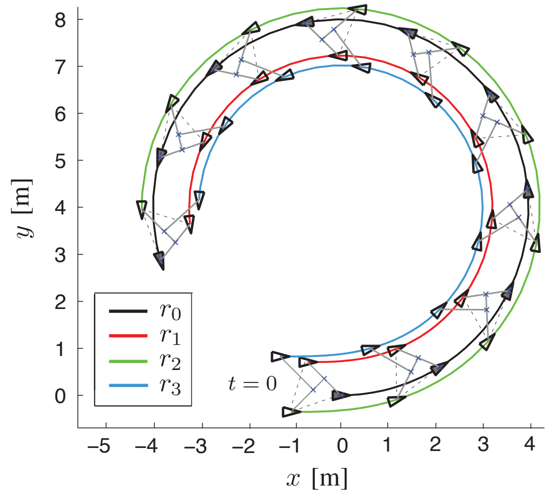

Fig. 4. Trajectory of the robots for the general formation in Fig. 3(b).

IV. Conclusions and Future Work

The paper defines general formations of unicycle robots. One of the robots plays the role of the leader and the formation is induced through a constraint function F that depends on the pose of the vehicles. We have studied the *flexibility* of such formations with respect to the leader's reference frame by introducing the notion of *formation internal dynamics*, have characterized its equilibria and given sufficient geometric conditions for their existence (Theorems 1 and 2). The theory is illustrated on two constraint functions F, one of which induces the hierarchical formation type introduced in [14].

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