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Quotient Kinematics Machines: Concept, Analysis and Synthesis

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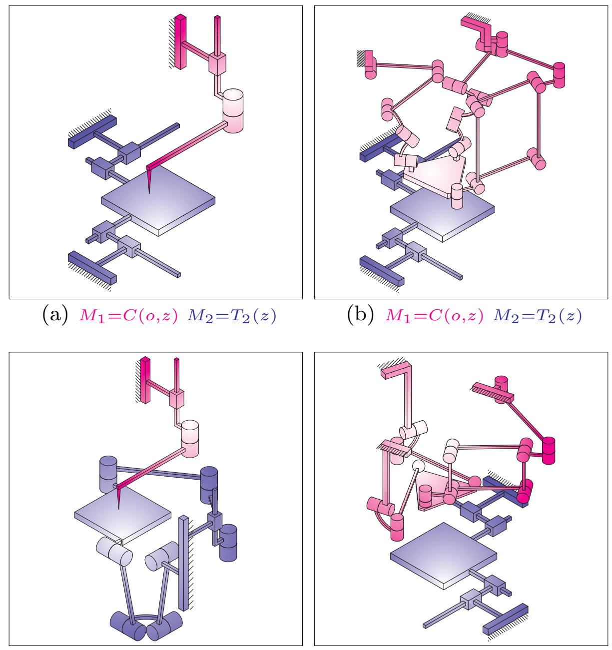
***Abstract*— In this paper, we identify a class of structurally distinguished machines, called *quotient kinematics machines* (QKM). A QKM realizes a motion task, typically characterized by a subgroup G of rigid transformation group $SE(3)$, through coordinated motion of two mechanisms called modules. One is referred to as a *subgroup module* H and the other a complementary or *quotient module* G/H of H in G. Since QKM can retain both large workspace/rotation range of SKMs and speed/accuracy of PKMs by appropriate choice of modules, it is often implemented in high end machine design for semiconductor die/wire-bonding and 5-axis machining, etc. To promote QKM technology beyond occasional studies and applications, we use differential geometric techniques to develop a rigorous and precise treatment of QKMs, including: (i) modeling and analysis of QKMs; (ii) classification and synthesis of QKMs; (iii) PKM realization of quotient modules.**

I. Introduction

A. Problem statement

In multi-axis machine and multi-DoF (Degree of Freedom) manipulator design, a mechanism's end-effector link can go through multiple degrees of translations and rotations in a continuous finite range. If the range of motion agrees with a Lie subgroup G (or a submanifold N) of the rigid transformation group $SE(3)$ on a neighborhood U of identity motion $e \in SE(3)$, we say the mechanism has (or generate) a *motion type* ([1], or *motion pattern* [2]) of Q ($Q = G$ or N).

Given a desired motion type Q, *mechanism synthesis* refers to an interconnection of rigid links through a basic set of joints (or pairs) so that one particular movable link, identified as the end-effector, has the motion type of Q. Commonly used joints include the *lower pairs*: revolute joint $\mathcal{R}$, prismatic joint $\mathcal{P}$, helical joint $\mathcal{H}$, cylindrical joint $\mathcal{C}$, spherical joint $\mathcal{S}$, and *composite pairs*: universal joint $\mathcal{U}$, omni wrist $\mathcal{O}$ [3] parallelogram $\mathcal{P}_a$, ball-joint parallelogram P_a^*, spatial parallelogram $\mathcal{U}^*$ [4]. Lower pairs are all subgroup generators of $SE(3)$; while composite pairs in general generate submanifolds of $SE(3)$. We have used the same notation for Lie subgroups and submanifolds as in [1].



(a) $M_1{=}C(o,z)$ $M_2{=}T_2(z)$ (b) $M_1{=}C(o,z)$ $M_2{=}T_2(z)$

(c) $M_1{=}C(o,z) M_2{=}X(z)/C(o,z)$ (d) $M_1{=}X(z)/T_2(z)$ $M_2{=}T_2(z)$

Fig. 1. A collection of X QKMs:(a) $(C, T_2(z))$ QKM with a C SKM and a $T_2(z)$ PKM;(b) $(C, T_2(z))$ QKM with a C PKM and a $T_2(z)$ PKM;(c) $(C, X/C)$ QKM with a C SKM and a X/C PKM; and,(d) $(T_2(z), X/T_2(z))$ QKM with a $T_2(z)$ PKM and a $X/T_2(z)$ PKM.

Three types of machines can be synthesized using basic joints: the *serial kinematics machine* (SKM), the *parallel kinematics machine* (PKM) and the *hybrid kinematics machine* (HKM). Their pros and cons are well understood in principle.

Aside from the three types of traditional kinematics machines, there is a fourth type of machines, which we call the *quotient kinematics machines* (QKM). A QKM $Q(M_1, M_2)$ consists of two mechanisms M_1, M_2 called *modules*, both end-effectors of which act in unison so that one generates a motion type of Q w.r.t. the other. Given a Lie subgroup G of $SE(3)$ for instance, its QKM usually consists of a *subgroup module* that generates a

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subgroup H of G, and a *quotient module* that generates a submanifold M of G, which can be intuitively explained as a quotient by eliminating H from G. We refer to such motion type a *quotient motion type*, denoted by G/H. Fig.1 shows 4 examples of $X(z)$ QKMs.

By splitting a large DoF motion task into two with less DoFs, a QKM can retain advantages of SKMs and PKMs while avoiding their shortfalls. QKMs are also structurally rich, with different choices of subgroup/quotient modules and SKM/PKM/HKM realizations. QKMs is also suited for modular and reconfigurable designs, since off-the-shelf modules can be used in QKM design, leading to low cost and high reliability design. Different modules can be combined to synthesize task motions with various requirements.

B. Literature review

Although QKMs have been practised in industries for some time, there appears to be very few literatures available, providing a formal and systematic treatment of QKM as a distinct kinematics structure. Tsai and Joshi proposed a $SE(3)(T(3), S(o))$ QKM (which they misinterpreted as a HKM) which completely splits translation from rotation [5]. In other words, there is no *parasitic rotation* in the translation module and no *parasitic translation* in the rotation module.

For 5-axis machine tool design, Bohez enumerated all QKMs that split a $\mathcal{PPPRR}$ SKM into two SKM modules, and conducted workspace properties comparison among such QKMs [6]. However, the motion type of 5-axis machining task is not explicitly given, causing the enumeration results to be incomplete. Besides, no PKM/HKM modules are considered in [6]. Recently, there is a surge of study and application on design of 5-axis QKMs with PKM modules [7], [8], [9], [10], [11]. But the current state of study on QKMs is on a occasional basis and largely incomplete.

C. Organization of this paper

We present in this paper a rigorous and precise treatment of QKMs. Rigorous modeling, analysis and classification of QKM and systematic synthesis of QKM is given in Section II; PKM realization of quotient modules is given in Section III. Finally, conclusions and future works are given.

II. QKM Concept and Synthesis

In this section, we give a formal treatment of QKM classification and synthesis. We assume that readers are familiar with basic concepts of differentiable manifold and Lie group theory (see [12], [13], [14], [15] for more detailed treatment). Key concepts to be grasped include the special Euclidean group $SE(3)$ and its Lie subgroups and submanifolds; For Category I submanifolds and category II submanifolds and its application to mechanism synthesis, please refer to [1].

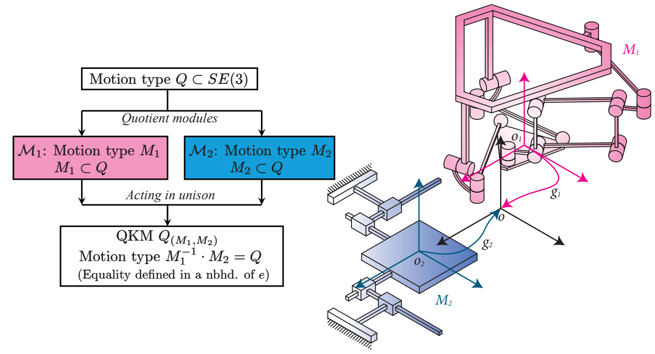


Fig. 2. A QKM consists of two motion modules M_1 and M_2 acting in unison. Their relative rigid displacement is given by $g_1^{-1} \cdot g_2 \in M_1^{-1} \cdot M_2$ or $g_2^{-1} \cdot g_1 \in M_2^{-1} \cdot M_1$.

A. QKM concept and quotient manifolds

Definition 1: (Quotient Kinematics Mechanism) A quotient kinematics mechanism (QKM) consists of two mechanism modules M_1 and M_2 acting in unison, as shown in Fig. 2. Let M_1 and M_2 also denote the set of rigid displacements generated by the end-effectors of the respective mechanisms, expressed in a common coordinate frame O. The QKM, denoted $Q(M_1, M_2)$, is said to have the motion type of Q if the set of relative displacements, $M_2^{-1} \cdot M_1$ or $M_1^{-1} \cdot M_2$, agree with Q (or a conjugate member of Q) in a neighborhood of the identity e:

$$M_1^{-1} \cdot M_2 = Q \ (\text{or } M_2^{-1} \cdot M_1 = Q) \tag{1}$$

$\diamondsuit$

Remark 1: In Def.1, M_i^{-1} stands for the kinematics inverse of M_i, given by:

$$M^{-1} \triangleq \{g \in SE(3) | g^{-1} \in M\} \tag{2}$$

It is not difficult to verify $T_eM = T_eM^{-1}$. $\diamondsuit$

In respect of Def.1, we are ready to review the $X(z)$ QKM example given earlier in a more rigorous manner.

Example 1: $X(z)$ **QKM** As shown in Fig. 1, there are several choices for the two modules M_1, M_2. In Fig. 1(a) and (b), both modules are subgroup modules; in Fig. 1(c) and (d), one module is a subgroup module, and the other a quotient module. For the QKM shown in Fig. 1(a):

$$C(o, z)T_2(z) = T_2(z)C(o, z) = X(z) \tag{3}$$

can be easily verified, or equivalently (by inverse function theorem, [12]):

$$T_eC(o, z) \oplus T_eT_2(z) = T_eX(z) \tag{4}$$

We see that kinematics inverse of a subgroup module does not change the QKM's motion type. Besides, (3) implies that the two modules can be interchanged.

The QKM shown in Fig.1(d) consists of a subgroup module $T_2(z)$, and a quotient module $M_1 = X(z)/T_2(z)$ which is a spatial version of a 4-bar linkage. By analyzing the constraint wrench system, $T_eM_1 = T_eC(q, z)$.

$$T_eT_2(z) \oplus T_eM_1 = T_eM_1^{-1} \oplus T_eT_2(z) = T_eX(z) \tag{5}$$

is verified. But $M_1 \neq C(q,z)$ since there exists parasitic translation in the rotation generated by a 4-bar linkage.

The quotient module $M_2 = X(z)/C(o,z)$ shown in Fig.1(c) is also contained in $X(z)$. Its verification can be conducted in a similar way. $\diamondsuit$

Generalizing from this example, we see that the two modules of a QKM can be inverted (*inverse property*) and interchanged (*symmetry property*). Moreover, when Q is a Lie subgroup G and $M_1^{-1}, M_2 \subset G$, (1) is equivalent to a more convenient linear algebraic condition:

$$T_e M_1 \oplus T_e M_2 = \mathfrak{g} \tag{6}$$

We have shown by Example 1 that a quotient module $M = G/H$ is defined by:

$$H \cdot M = M \cdot H = G \tag{7}$$

or equivalently:

$$\mathfrak{h} \oplus T_e M = \mathfrak{g} \tag{8}$$

It is standard linear algebra ([16]) that (8) is equivalent to $T_e M$ being isomorphic to the quotient space $\mathfrak{g}/\mathfrak{h}$ by the natural projection π:

$$\pi : \mathfrak{g} \mapsto \mathfrak{g}/\mathfrak{h}, \pi(\hat{v}) = \hat{v} + \mathfrak{h} \tag{9}$$

$T_e M$ is called a *representative* of $\mathfrak{g}/\mathfrak{h}$. In other words, given a basis $\{\hat{v}_i\}_{i=1}^r$ of $T_e M$ and a basis $\{\hat{w}_j\}_{j=1}^n$ of $\mathfrak{h}$ where $\dim \mathfrak{h} = n$, $r = \dim \mathfrak{g} - \dim \mathfrak{h}$, $\{\hat{v}_i, \hat{w}_j\}_{i,j=1,1}^{r,n}$ forms a basis of $\mathfrak{g}$. This gives us a systematic way ([1]-Lemma.1-4) to classify all representatives of $\mathfrak{g}/\mathfrak{h}$ along with an ordered basis $\{\hat{v}_i\}_{i=1}^r$. Using canonical coordinates of the 1^{st} and 2^{nd} kind of $SE(3)$, we immediately have two systematic ways to generate two special classes of quotient modules, namely:

1) 1^{st} *canonical submanifolds*:

$$\{e^{\sum_{i=1}^r v_i\theta_i} | \theta_i \in (-\varepsilon, \varepsilon), i = 1, \ldots, r\} \tag{10}$$

2) 2^{nd} *canonical submanifolds*:

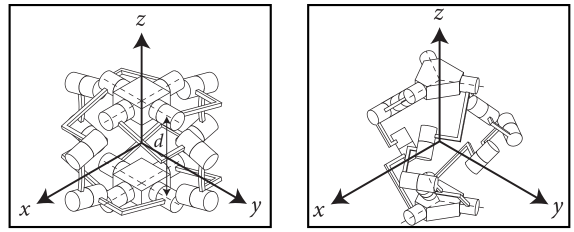
$$\{e^{v_1\theta_1} \cdots e^{v_r\theta_r} | \theta_i \in (-\varepsilon, \varepsilon), i = 1, \ldots, r\} \tag{11}$$

2^{nd} canonical submanifolds or POE correspond to motion types of a SKM, i.e. manifolds that is synthesizable by a SKM. Note that Cat. 2 submanifolds are POEs themselves. For example:

$$\begin{aligned} X(x) \cdot X(y) &= T(3)U(o,x,y) \\ &= \{e^{\hat{e}_1\theta_1} \cdots e^{\hat{e}_5\theta_5} | \theta_i \in \mathbb{R}\} \end{aligned} \tag{12}$$

In fact, most works on PKM type synthesis are concerned with PKM and subchains with POE motion types [17], [18], [19], [20], [21], [1]. Its systematic generation is also clear by [1]-Lemma.1-4.

1^{st} canonical submanifolds are largely overlooked in mechanism synthesis, and thus deserves more explanation here. Recall that the motion type of omni wrist [3],



(a) (b)

Fig. 3. Omni wrist and 3-$\mathcal{RSR}$ PKM

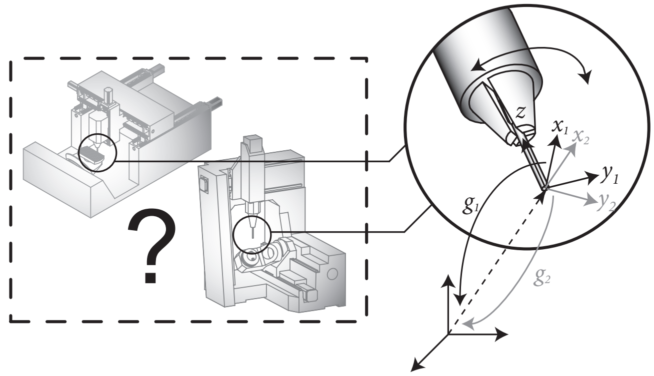


Fig. 4. Configuration of a 5-axis machine with spindle symmetry: $g_1 \sim g_2$ iff $g_1^{-1}g_2 \in R(o,z)$

as shown in Fig.3(a), is:

$$O(o,dz) \triangleq \left\{ \left[\begin{array}{cc} e^{2\hat{x}\theta_1 + 2\hat{y}\theta_2} & -\frac{1}{2}d(e^{\hat{x}\theta_1+\hat{y}\theta_2} - I)^2 e_3 \\ 0 & 1 \end{array} \right] \middle| \theta_i \in (-\varepsilon, \varepsilon) \right\} \tag{13}$$

The rotation part of $O(o,dz)$ (ignoring parasitic translation) is in the form of a 1^{st} canonical submanifold, which we denote by $\tilde{U}$:

$$\tilde{U}(o,z) \triangleq \left\{e^{\hat{e}_4\theta_1 + \hat{e}_5\theta_2} | \theta_i \in (-\varepsilon, \varepsilon)\right\} \tag{14}$$

$\tilde{U}$ is equivalent to zero torsion rotation [22]. However, previous works ([22], [23], [24]) did not use the notion of motion type and fail to discover their underlying relationship. Other examples of 1^{st} canonical submanifolds includes the motion type of the symmetric 3-$\mathcal{RSR}$ PKM [25], as shown in Fig.3(b),whose motion type is:

$$P(o,z) \triangleq \left\{e^{\hat{e}_3\theta_1 + \hat{e}_4\theta_2 + \hat{e}_5\theta_3} | \theta_i \in (-\varepsilon, \varepsilon)\right\} \tag{15}$$

and the QKM consisting of a 3-$\mathcal{PRS}$ Z3 PKM module and a $T_2(z)$ module, whose motion type is:

$$T(3) \cdot \tilde{U}(o,z) = \left\{e^{\sum_{i=1}^5 \hat{e}_i\theta_i} | \theta_i \in (-\varepsilon, \varepsilon)\right\} \tag{16}$$

General representatives M of G/H defined by (7) can be thought of as hypersurfaces of 2^{nd} canonical submanifolds. For example, the motion type $O(o,dz)$ of omni wrist is a submanifold of $P(o,z)$ (15), which is the exponential image of a surface in span(e_3, e_4, e_5).

Besides quotient modules of QKMs, motion types defined by a quotient space also arise in the case of 5-axis machining. As shown in Fig.4, the tool spindle of a 5-axis machine has an axial symmetry characterized by $R(o,z)$, two configurations g_1, g_2 of the spindle are considered equivalent if $g_1^{-1}g_2 \in R(o,z)$. Thus the configuration space of 5-axis machining is the 5 dimensional quotient manifold $SE(3)/R(o,z)$ [26]. The motion type of a 5-axis machine is such a submanifold M of $SE(3)$, that for each configuration $[g]$, M contains a configuration g' such that $g' \in [g]$. Considering the motion type of 5-axis machine, we are looking for submanifolds M of G that are locally in one-to-one correspondence with G/H, that is M such that $\pi|_M : M \mapsto G/H$ is a diffeomorphism (smooth and invertible, and the inverse is also smooth). By inverse function theorem [12], this is equivalent to (8). Thus a 5-axis machine is a quotient motion generator of $SE(3)/R(o,z)$. In particular, $T(3) \cdot U(o,x,y)$ and $T(3) \cdot \tilde{U}(o,z)$ are two particular rep.s of $SE(3)/R(o,z)$.

B. *Classification and synthesis of QKM*

Motivated by the potential application of QKM in 5-axis machine design, we propose in this subsection synthesis of general QKMs.

Proposition 1: (QKM classification and synthesis)

The following statements are true:

S1) if $H \subset G$ are subgroups of $SE(3)$, then $G(H, G/H)$ is referred to as *normal* QKM synthesis;

S2) if $N_1 \subset T(3), N_2 \subset S(p)$, then $N_1 \cdot N_2(N_1, N_2)$ is a QKM with pure translation/rotation modules;

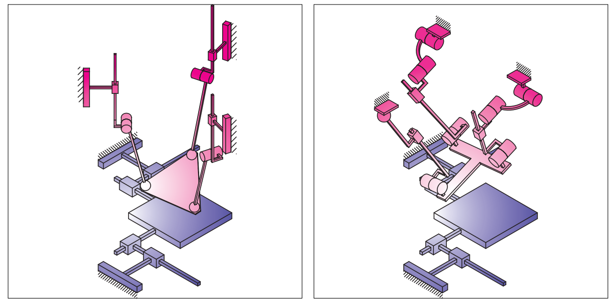
S3) if G_1, G_2 and $H = H_1 \cdot H_2 \triangleq G_1 \cap G_2$ are subgroups of $SE(3)$, then $G_1 \cdot G_2(\{G_1/H_1\}, \{G_2/H_2\})$ is referred to as *dependent* QKM synthesis;

S4) if $H \subset G' \subset G$ are subgroups of $SE(3)$, then $\{G/H\}(\{G/H'\}, \{H'/H\})$ is referred to as *generalized* QKM synthesis. In particular, $\{G/H'\}$ is referred to as *expansion* of denominator, $\{H'/H\}$ is referred to as *reduction* of the numerator.

Example 2: (translational quotient modules) Consider synthesis of 5-axis QKM using the generalized QKM synthesis (S4). Given $G = SE(3)$ and $H = R(o,z)$, we can choose $G' = S(o)$, then the resulting QKM is $(\{SE(3)/S(o)\}, \{S(o)/R(o,z)\})$. The $\{SE(3)/S(o)\}$ module can be referred to as a *translational quotient module* (TQM). Most industrial serial robots use a TQM module to position its wrist, such as the spherical type SKM $(R(o,z)R(o,x)T(y) \in \{SE(3)/S(p)\})$ shown in [27]-Fig. 8.6. The other type, as illustrated by the TY head of ESEC used for rapid positioning of the bonding head [28], corresponds to $T(x)R(o,z)R(o,y) \in \{SE(3)/S(p)\}$.

$\{X(v)/R(o,v)\}, v \in \mathbb{R}^3$ is a particular subclass of $\{SE(3)/S(o)\}$, with parasitic rotation of $R(o,v)$ but not that of $S(o)$. For example, the Heckett SKM400 3-axis milling machine tool [29] has a motion type of $R(p,x)P_a(y,v_1)P_a(y,v_2) \in \{X(x)/R(o,x)\}$.

Example 3: $\{SE(3)/PL(z)\}$ modules Consider the QKMs of $(\{SE(3)/PL(z)\}, \{PL(z)/R(o,z)\})$ type.



(a) $M_1 = \{SE(3)/PL(z)\}$ $M_2 = T_2(z)$ (b) $M_1 = \{SE(3)/PL(z)\}$ $M_2 = T_2(z)$

Fig. 5. A collection of 5-axis mechanisms:(a) a QKM consisting of a Z3 PKM with a decoupled XY PKM table;and (b) a QKM consisting of a Exechon PKM with a decoupled XY PKM table

$\{SE(3)/PL(z)\}$ quotient module is usually identified with the $1T2R$ Cat.1 submanifold $T(z)U(o,x,y)$, whose module synthesis is considered in [30], [9], [11].

The Sprint Z3 PKM, as shown in Fig.5(a), is another $\{SE(3)/PL(z)\}$ module. Its kinematics is studied in [22], and its motion type M_1 is given by:

$$M_1 = \left\{ \left[\begin{array}{c|c} e^{(\hat{e}_1 \cos\theta + \hat{e}_2 \sin\theta)a} & \begin{array}{c} \frac{r}{2}\cos 2\theta(1-\cos a) \\ -\frac{r}{2}\sin 2\theta(1-\cos a) \\ z \end{array} \\ \hline 0 & 1 \end{array} \right] \right| \\ \left. \theta \in [0, 2\pi], z \in (-\varepsilon, \varepsilon), a \in (-\varepsilon, \varepsilon) \right\} \tag{17}$$

from which we get $T_e M_1 = \text{span}(\hat{e}_3, \hat{e}_4, \hat{e}_5)$. Thus (8) is satisfied:

$$T_e M_1 \oplus T_e PL(z) = T_e M_1 \oplus se(2) = se(3) \tag{18}$$

Note that M_1 is obviously not equal to $T(z)U(o,x,y)$ due to existence of parasitic translation in $T_2(z)$. If we let $\theta_1 = a\cos\theta$ and $\theta_2 = a\sin\theta$, the rotation matrix of $Z3$ are elements from $\tilde{U}(o,z)$. This implies that $(M_1, T_2(z)) \in (\{SE(3)/PL(z)\}, \{PL(z)/R(o,z)\})$ is a $T(3) \cdot \tilde{U}(o,z)$ QKM.

The Exechon PKM (Fig.5(b)), though being used as a TQM in [31], is also a $\{SE(3)/PL(z)\}$ as we shall see. For ease of computation, take the kinematics inverse of Exechon, its motion type M_1^{-1} is given by equating $T_2(x) \cdot U(o,x,y)$ with $T_2(y)S(-re_3)$:

$$M_1^{-1} = \left\{ \left[\begin{array}{c|c} e^{\hat{e}_1\theta_1} e^{\hat{e}_2\theta_2} & \begin{array}{c} 0 \\ r\sin\theta_1\sin\theta_2 \\ z \end{array} \\ \hline 0 & 1 \end{array} \right] \middle| \theta_i \in (\varepsilon, \varepsilon) \right\} \tag{19}$$

M_1^{-1} translates along z and rotates like a universal joint, with parasitic translation in $T(y)$. This implies that $(M_1, T_2(z)) \in (\{SE(3)/PL(z)\}, \{PL(z)/R(o,z)\})$ is a $T(3) \cdot U(o,x,y)$ QKM. It is also interesting to note that the inverted Exechon is an overconstrained version of a 3-$\mathcal{PRS}$ PKM with two coplanar subchains. ♢

III. PKM realization of quotient modules

Given a quotient motion type $\{G/H\}$, its PKM synthesis problem amounts to: (i) *find submanifolds M_i's that contain a rep.* $Q \in \{G/H\}$; (ii) *verify that* $\cap M_i = Q$; and (iii) *generate subchains for each* M_i. In this subsection, we propose two methods to synthesize quotient PKMs. They are both based on the expansion and reduction rules in Prop.1-S4, which can be readily used to find M_i's. Then we can use VMC/FMC of [1]-Prop.6 to verify the resulting PKM. The subchain synthesis is already a mature topic in most cases [1] and thus shall not be a focus of our study.

Algorithm 1: indirect synthesis The *indirect synthesis method* refers to PKM synthesis with an explicit rep. Q of $\{G/H\}$:

Input) $H \subset G \subset SE(3), r \geq \dim G/H$, and a set of basic joints, $\mathcal{B}$;

A1) Specify a rep. $Q \in \{G/H\}$;

A2) Apply expansion and reduction rule: specify $G_i \supseteq G$ and $H_i \subseteq H$, and rep.s $M_i \in \{G_i/H_i\}$ such that $Q \subset M_i$;

A3) Verify the *velocity matching condition* (VMC, [1]):

$$T_eQ = T_eM_1 \cap \cdots \cap T_eM_r \qquad (20)$$

or, equivalently the *force matching condition* (FMC, [1]):

$$T_e^*Q^\perp = T_e^*M_1^\perp + \cdots + T_e^*M_r^\perp \qquad (21)$$

where

$$T_e^*Q^\perp \triangleq \left\{ f \in \mathbb{R}^6 | \langle f, \xi \rangle = 0, \forall \xi \in T_eQ \right\} \qquad (22)$$

denotes the subspaces of constraint (reciprocal) forces (wrenches) for T_eQ.

A4) if VMC or FMC is satisfied, $M_1 \| \cdots \| M_r$ is a parallel motion generator of $Q \in \{G/H\}$. Synthesize M_i subchains using $\mathcal{B}$, e.g. method used in [1]. If VMC and FMC fails, go back to A2).

Example 4: indirect synthesis of $\{SE(3)/PL(z)\}$ module Choose the motion type of $Z_3 \in \{SE(3)/PL(z)\}$ for instance. Since $Z_3 \subset T_2(y)\tilde{U}(re_1, z)$ due to zero torsion, the PKM $M_1 \| M_2 \| M_3$ with:

$$\begin{cases} M_1 = T_2(y)\tilde{U}(re_1, z) \\ M_2 = I_{g_0}(T_2(y)\tilde{U}(re_1, z)) \\ M_3 = I_{g_0^2}(T_2(y)\tilde{U}(re_1, z)) \end{cases}, g_0 = \begin{bmatrix} e^{\hat{e}_3 \frac{2\pi}{3}} & 0 \\ 0 & 1 \end{bmatrix} \qquad (23)$$

is an overconstrained version of Z3. An immediate mechanical realization is given by a PKM with 3 symmetric $\mathcal{P}\mathcal{P}_a\mathcal{O}$ subchains. Due to the parasitic translation of omni wrist which is z-axial symmetric, its motion type is given by:

$$M = \left\{ \left[\begin{array}{c|c} e^{(\hat{e}_1 \cos\theta + \hat{e}_2 \sin\theta)a} & \begin{matrix} x \\ y \\ z \end{matrix} \\ \hline 0 & 1 \end{array} \right] \middle| \begin{array}{c} \theta \in [0, 2\pi], \\ a \in (-\varepsilon, \varepsilon), \\ z \in (-\varepsilon, \varepsilon) \end{array} \right\}$$
$$x = \frac{r}{2}\cos 2\theta(1-\cos a) - d\sin\theta(1-\cos\frac{a}{2})\sin\frac{a}{2}$$
$$y = -\frac{r}{2}\sin 2\theta(1-\cos a) + d\cos\theta(1-\cos\frac{a}{2})\sin\frac{a}{2} \qquad (24)$$

in reference to (13) and (17). The PKM is overconstrained and thus less prone to singularities. It also has a maximal tilting angle of $\pm 95°$ in all directions due to the adoption of omni wrists. For insufficiency of space, we do not give a picture here. ♢

Algorithm 2: direct synthesis The *direct synthesis method* refers to PKM synthesis without an explicit rep. Q of $\{G/H\}$:

Input) $H \subset G \subset SE(3), r \geq \dim G/H$, and a set of basic joints, $\mathcal{B}$;

A1) Apply expansion and reduction rule: specify $G_i \supseteq G$ and $H_i \subseteq H$, and rep.s $M_i \in \{G_i/H_i\}$ such that $Q \subset M_i$. In addition, $\cap_{i=1}^r M_i$ should also be verified for (local) submanifold property (see p. 75, [12]);

A2) Verify the *velocity matching condition* (VMC, [1]):

$$T_eQ \oplus \mathfrak{h} = (T_eM_1 \cap \cdots \cap T_eM_r) \oplus \mathfrak{h} = \mathfrak{g} \qquad (25)$$

or, equivalently the *force matching condition* (FMC, [1]):

$$T_e^*Q^\perp \cap \mathfrak{h}^{*\perp} = \mathfrak{g}^{*\perp} \qquad (26)$$

A3) if VMC or FMC is satisfied, $M_1 \| \cdots \| M_r$ is a parallel motion generator of $Q \in \{G/H\}$. Synthesize M_i subchains using $\mathcal{B}$, e.g. method used in [1]. If VMC and FMC fails, go back to A1).

Example 5: direct synthesis of $\{X(z)/T_2(z)\}$ modules To directly synthesize a $\{X(z)/T_2(z)\}$ PKM, we first consider expansion of numerator, the only case of which being $G_i = SE(3)$. Given $\forall M_i \in \{SE(3)/T_2(z)\}$, it is not difficult to see that $M_i \cap X(z)$ is the desired rep. $Q \in \{X(z)/T_2(z)\}$:

$$\forall M_i \in \{G_i/H\} \text{ with } G \subset G_i, Q \triangleq M_i \cap G \in \{G/H\} \qquad (27)$$

For example, the motion type of a revolute-spherical dyad ($\mathcal{RS}$ subchain, [32]) is a rep. of $\{SE(3)/T_2(z)\}$. Using (27), we have:

$$Q = R(o,x)S(de_2) \cap X(z) = P_a(x, de_2)R(de_2, z) \qquad (28)$$

If we let $M_1 = R(o,x)S(de_2)$ and $M_2 = X(z)$, we get a PKM $M_1 \| M_2$ with motion type $Q \in \{X(z)/T_2(z)\}$. Then we consider reduction of denominator by letting $H_i = T(v_i), v_i \perp z, i = 1, 2$ such that $T_2(z) =$

$T(v_1)T(v_2)$. For example, $M_i = T(z)R(p_i, z)R(p_i + l_1 v_i, z), i = 1, 2$, then:

$$M_1 \cap M_2 = T(z) \cdot N \tag{29}$$

where N is the 1 dimensional submanifold of a four-bar linkage $\mathcal{R}(p_1, z)\mathcal{R}(p_1+l_1v_1, z)\mathcal{R}(p_2+l_2v_2, z)\mathcal{R}(p_2, z)$. Finally, a combined expansion and reduction effort can be considered. We let $M_i = U(p_i, v_i, z)S(l_i\hat{e}_3v_i), v_i \perp z, i = 1, 2$ and $M_3 = X(z)$. $M_1 \| M_2$ is a $\{SE(3)/T_2(z)\}$ PKM by reduction rule since the universal-spherical dyad $\mathcal{U}(p_i, v_i, z)\mathcal{S}(p_i+l_i\hat{e}_3v_i)$ is a rep. of $\{SE(3)/T(\hat{e}_3v_i)\}, i = 1, 2$. Thus $M_1\|M_2\|M_3$ is a $\{X(z)/T_2(z)\}$ PKM by expansion rule. A mechanical realization of the PKM is given in Fig.1(d). The $\{X(z)/C(o, z)\}$ PKM shown in Fig.1(c) is synthesized in a similar manner. $\diamondsuit$

IV. Conclusions

In this paper, we have given a systematic treatment of QKMs. Its classification and systematic synthesis is solved using the notion of quotient motion type. PKM realization of quotient modules are systematically solve for the first time. Our study gives a unified understanding of the many QKM case studies, and offers a systematic way to synthesize novel QKMs.

Our future work is focused on systematic synthesis of quotient PKMs, and further study on parasitic motions.

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