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The Dynamical Servo Control Problem for the Acrobot Based on Virtual Constraints Approach

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Abstract— **This paper discusses the dynamical servo control problem for the Acrobot - a mechanics with two links while just one actuator applied to the second link. The dynamical servo control aims to access generic points in the state space not just the equilibrium points. Stabilizing of generic points is impossible and a possible way is to stabilize some periodic orbits passing through the desired points. Virtual constraints are used to generate such orbits. By analyzing the integration of the zero dynamics of the Acrobot subjected to virtual constraints, conditions of whether there is an orbit passing through the desired point and the orbit function are found. A cascade control strategy is proposed to stabilize both the virtual constraint and the orbit function which decides the system behavior. Simulation results show the effectiveness of the control law.**

I. INTRODUCTION

Underactuated systems are systems with less actuators than degree of freedom. It represents a large class of systems in real world, like space crafts, mobile robots, and flexible link robots. It is difficult to control than fully actuated system. So control of underactuated systems is a hotspot in control research(see [1] [2] [4]). The Acrobot as shown in Fig. 1 is a typical underactuated system but not so complex in structure. It is an ideal bench mark for the control research of underactuated mechanical systems.

Former researchers have carried out a series of researches. Spong studied the swing up control problem for the Acrobot in [1] and divided it into two stages: swing up stage realized by partial feedback linearization method and balance stage realized by LQR method. Xin XIN in his paper [2] proposed an energy based control law to swing up the Acrobot by constructing a Lyapunov function with mechanical energy of the system, angle and velocity of the second link. The proposed control law can stabilizes the Acrobot to a special orbit passing through the upward position with zero velocity, which makes it easier to switch to the balance controller.

The paper discusses dynamical servo control problem for the Acrobot. Different from the traditional problem of stabilizing underactuated systems to a fixed equilibrium point or a special trajectory as [1], [2], dynamical servo control aims to drive the system to a desired generic state instantaneously.

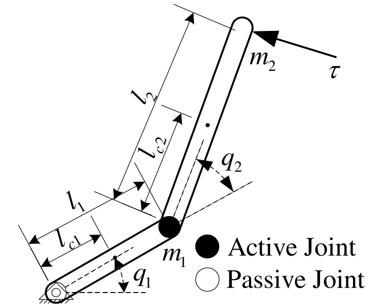

Fig. 1. Structure of the Acrobot system

The dynamical servo control can be used as a part of complex tasks, such as switching from swinging up phase to balancing phase or switching from swinging phase to flying phase as a monkey does. Stabilizing of generic points is impossible and a possible way is to stabilize periodic orbits passing through the desired points.

Some related problems have been studied. Matthew D. Berkemeier and Ronald S. Fearing studied the inverted trajectory tracking problem of the Acrobot. In [5] they choose an output function which limits the Acrobot to a special manifold and generates groups of periodic orbits. Anton S. Shiriaev studied characters of underactuated systems which were imposed by virtual constrains and gave explicit formulas of the integrate curves and periodic orbits in [3][6] and proposed LQR based method to stabilize the orbits. They have applied their approach to the Acrobot system in [4][7] and the Furuta Pendulum system in [8].

Virtual constraints are used to limit the original system to a particular manifold. These constraints don't exist in physical system but are introduced by the feedback control. Anton gives the formulas of the general integrate curves but don't give whether there is a limit cycle passing through a specific point.

In order to realize dynamical servo control, the paper first introduces a specific virtual constraint into Acrobot system and gets the first integral of the virtual limit system. Based on the fact that all limit cycles must contain at least one equilibrium point. Then several theories are given to determine whether there are limit cycles about an equilibrium point, and further more, a method is proposed to determine range of limit cycles about an equilibrium point.

Finally a cascade controller is designed to stabilize both the virtual constraint and the limit cycle function. Numerical simulation shows the effectiveness of the proposed control strategy.

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II. PROBLEM FORMULATION

A. Dynamical model

With the structure shown in Fig. 1 and the notation and conventions shows in TABLE I, the equations of motion of the Acrobot are:

$$d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + h_1 + \varphi_1 = 0 \quad (1)$$
$$d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + h_2 + \varphi_2 = \tau \quad (2)$$

where:

$$\begin{array}{ll} d_{11} = \theta_1 + \theta_2 + 2\theta_3 \cos q_2 & \varphi_2 = \theta_5 g \cos(q_1 + q_2) \\ d_{12} = d_{21} = \theta_2 + \theta_3 \cos q_2 & \theta_1 = m_1 l_{c1}^2 + m_2 l_1^2 + I_1 \\ d_{22} = \theta_2 & \theta_2 = m_2 l_{c2}^2 + I_2 \\ h_1 = -\theta_3 \dot{q}_2 \sin q_2 (2\dot{q}_1 + \dot{q}_2) & \theta_3 = m_2 l_1 l_{c2} \\ h_2 = \theta_3 \dot{q}_1^2 \sin q_2 & \theta_4 = m_1 l_{c1} + m_2 l_1 \\ \varphi_1 = \theta_4 g \cos q_1 + \theta_5 g \cos(q_1 + q_2) & \theta_5 = m_2 l_{c2} \end{array}$$

d_{11}, d_{22} are the self inertial acceleration item; d_{12}, d_{21} are couple inertial acceleration item; h_1, h_2 are the Coriolis and centrifugal force item; ϕ_1, ϕ_2 are the gravitational loading force item.

TABLE I

NOTATIONS

m_1, m_2	:	mass of the two links
l_1, l_2	:	length of the two links
l_{c1}, l_{c2}	:	length from the joints to the COG of the two links
I_1, I_2	:	moment of inertia of the two links
g	:	acceleration of gravity
τ	:	the input torque

B. Dynamical servo control

***Definition** 1:* For system $\dot{x} = f(x,u), x(t_0) = x_0$, given a reference trajectory $\gamma(t)$ and tracking error δ, suppose there exists a control law $u(t)$.

If $\exists T_0$ when $t > T_0$, $|x(t) - \gamma(t)| < \delta$ holds, then it is called servo control;

If $\exists T_0$ when $t_i > T_0, i = 1,2,3,,,, |x(t_i) - \gamma(t_i)| < \delta$ holds, then it is called dynamical servo control.

The difference between traditional servo control and dynamical servo control is that the former one forces the system states to converge to the given trajectory continually; while the later one forces only at some discrete moment (periodically) the system state to fall on the given trajectory. This makes it possible for the underactuated systems.

Simply for the Acrobot, the dynamical servo control means that the system can arrive at the given position(i.e. $x_d = \left[\begin{array}{cccc} q_{1d} & q_{2d} & \dot{q}_{1d} & \dot{q}_{2d} \end{array}\right]$). In this paper we just consider the situation with $x_d = \left[\begin{array}{cccc} q_{1d} & q_{2d} & 0 & 0 \end{array}\right]$.

III. LIMIT CYCLE ANALYSIS

A. Virtual constraints

To realize dynamical servo control for the Acrobot with the desired point is $x_d = \left[\begin{array}{cccc} q_{1d} & q_{2d} & 0 & 0 \end{array}\right]$. There are a lot of virtual constraints but this paper just considers the following virtual constraint:

$$q_1 = q_{1d} \quad (3)$$

This virtual constraint describes an Acrobot with a fixed first link and a swinging second link. Taking time derivative of q_1 and $\dot{q}_1$ along (3), one gets

$$\dot{q}_1 = \ddot{q}_1 = 0 \quad (4)$$

Substitute (3) and (4) into (1), one can get

$$d_{12}\ddot{q}_2 + h_1 + \varphi_1 = 0 \quad (5)$$

Expand (5) and yield

$$\alpha(q_{1d}, q_2)\ddot{q}_2 + \beta(q_{1d}, q_2)\dot{q}_2^2 + \gamma(q_{1d}, q_2) = 0 \quad (6)$$

where:

$$\begin{array}{rcl} \alpha(\cdot) & = & \theta_2 + \theta_3 \cos q_2 \\ \beta(\cdot) & = & -\theta_3 \sin q_2 \\ \gamma(\cdot) & = & \theta_4 g \cos q_{1d} + \theta_5 g \cos(q_{1d} + q_2) \end{array}$$

The equation (6) is the zero dynamics of the Acrobot with virtual constraints (3). The first integral of (6) has been invested by Anton S. Shiriaev in [6]. In fact there is a class of equations like (6) generated by different virtual constraints not only (3). So the followings results are not limited to (3).

Introducing $Y = \dot{q}_2^2$ then one can get

$$\ddot{q}_2 = \frac{d\dot{q}_2}{dt} = \frac{d\dot{q}_2}{dq_2}\frac{dq_2}{dt} = \frac{d\dot{q}_2}{dq_2}\dot{q}_2 = \frac{1}{2}\frac{d\dot{q}_2^2}{dq_2} = \frac{1}{2}\frac{dY}{dq_2} \quad (7)$$

Then (6) can be rewrite as

$$\frac{1}{2}\alpha(q_{1d}, q_2)\frac{dY}{dq_2} + \beta(q_{1d}, q_2)Y + \gamma(q_{1d}, q_2) = 0 \quad (8)$$

Assume that $\alpha(q_{1d}, q_2) > 0$, and denote

$$\begin{array}{rcl} \beta_0(\cdot) & = & 2\dfrac{\beta(q_{1d}, q_2)}{\alpha(q_{1d}, q_2)} \\ \gamma_0(\cdot) & = & -2\dfrac{\gamma(q_{1d}, q_2)}{\alpha(q_{1d}, q_2)} \end{array}$$

Then one gets

$$Y'_{q_2} + \beta_0(q_{1d}, q_2)Y - \gamma_0(q_{1d}, q_2) = 0 \quad (9)$$

Denote the integral factor as:

$$I = e^{\int_{q_{20}}^{q_2} \beta_0(q_{1d}, x)dx} \quad (10)$$

Multiply both sides of the equation by I, there is

$$\frac{\partial(IY)}{\partial q_2} = I\gamma_0(q_{1d}, q_2) \quad (11)$$

The solution of (11) is

$$IY = \int_{q_{20}}^{q2} I\gamma_0(q_{1d}, x)dx + Y_0 \quad (12)$$

or

$$Y = \dot{q}_2^2 = I^{-1}\int_{q_{20}}^{q2} I\gamma_0(q_{1d}, x)dx + Y_0 I^{-1} \quad (13)$$

By directly calculation one can get:

$$I = \left(\frac{\theta_2 + \theta_3 \cos q_2}{\theta_2 + \theta_3 \cos q_{20}}\right)^2 \quad (14)$$

and

$$Y = \frac{-2g}{(\theta_2 + \theta_3 \cos q_2)^2} [F(q_2) - F(q_{20}) + C] \quad (15)$$

where:

$$\begin{array}{rcl} C & = & \dfrac{(\theta_2 + \theta_3 \cos q_{20})^2 \dot{q}_{20}^2}{-2g} \\ F(q_2) & = & (\theta_2\theta_4 \cos q_{1d} + 0.5\theta_3\theta_5 \cos q_{1d}) q_2 \\ & + & (\theta_2\theta_5 + \theta_3\theta_4) \cos q_{1d} \sin q_2 \\ & + & \theta_2\theta_5 \sin q_{1d} \cos q_2 \\ & + & 0.25\theta_3\theta_5 \sin(q_{1d} + 2q_2) \end{array}$$

B. Limit cycle determine condition

Suppose there exists a feedback controller which can make (1) holds, i.e. system states are limit on the manifold $\Omega = \Omega(q_2, \dot{q}_2)\left|_{(q_1=q_{1d}, \dot{q}_1=0)}\right.$. Equation (13) describes the relationship between q_2 and $\dot{q}_2$ on the manifold.

If there exists a limit cycle passing though the desired point $[q_{2d}, 0]$ then the dynamical servo control can be realized by stabilizing the limit cycle.

From Poincare-Bendixson theorem(see [9] for more info), a limit cycle must contain at least one equilibrium point, so if there is no equilibrium point in a specific area, then there must exist no limit cycle in this area.

***Theorem** 1:* consider the Acrobot system under virtual constraints (3), denote $\lambda = \theta_4/\theta_5$, then:

If $\lambda \leq 1$, for any q_{1d} there exists equilibrium points in Ω;

If $\lambda > 1$, only for q_{1d} which meets $|\cos q_{1d}| \leq 1/\lambda$, there exists equilibrium points in Ω.

or if $\lambda > 1$ and $|\cos q_{1d}| > 1/\lambda$, there exists no equilibrium points in Ω.

proof: The closed loop equilibrium point is decided by:

$$\varphi_1 = \theta_4 g \cos q_1 + \theta_5 g \cos(q_1 + q_2) = 0$$

For a given q_{1d}, that is:

$$\cos(q_{1d} + q_2) = -\lambda \cos q_{1d} \quad (16)$$

Equation (16) has a solution only when $|\lambda \cos q_{1d}| \leq 1$. So if $\lambda \leq 1$, then $|\lambda \cos q_{1d}| \leq 1$ holds, (16) always has a solution; if $\lambda > 1$, only when $|\cos q_{1d}| \leq 1/\lambda$, (16) may have a solution.

***Theorem** 2:* for equilibrium points on Ω denoted as

$$q = [q_1, q_2, \dot{q}_1, \dot{q}_2] = q^* = [q_{1d}, q_2^*, 0, 0] \quad (17)$$

For simplicity, denote $\partial f/\partial q_2$ as f' and $\partial^2 f/\partial q_2^2$ as f''.Under the assumption $\alpha > 0$, if $\gamma_0'\left|_{q=q^*}\right. > 0$, then q^* is a focus; if $\gamma_0'\left|_{q=q^*}\right. < 0$ then q^* is a saddle point.

proof: The first integral of the zero dynamics (6) shows as (13). Introduce a function:

$$U = Y - I^{-1} \int_{q_{20}}^{q2} I\gamma_0(q_{1d}, x) dx - Y_0 I^{-1} \quad (18)$$

This function describes a special trajectory when $U = 0$. Calculate the Hessian of $U(q_2, \dot{q}_2)$, one gets

$$H(U) = \begin{bmatrix} \dfrac{\partial^2 U}{\partial q_2^2} & \dfrac{\partial^2 U}{\partial q_2 \partial \dot{q}_2} \\ \dfrac{\partial^2 U}{\partial \dot{q}_2 \partial q_2} & \dfrac{\partial^2 U}{\partial \dot{q}_2^2} \end{bmatrix} = \begin{bmatrix} -\gamma_0' & 0 \\ 0 & 2 \end{bmatrix} \Big|_{q=q^*} \quad (19)$$

So if $-\gamma_0'\left|_{q=q^*}\right. > 0$, from the Lagrange-Dirichlet theorem, q^* is a focus; if$-\gamma_0'\left|_{q=q^*}\right. < 0$, q^* is a saddle point.

Since

$$\gamma_0 = -2\frac{\gamma}{\alpha} \Leftrightarrow \gamma_0' = -2\frac{\gamma'\alpha - \alpha'\gamma}{\alpha^2}$$

So the determine condition can be expressed by

$$-\gamma_0'\left|_{q=q^*}\right. \vee 0 \Leftrightarrow \gamma'\left|_{q=q^*}\right. \vee 0$$

For $\gamma(\cdot) = \theta_4 g \cos q_{1d} + \theta_5 g \cos(q_{1d} + q_2)$, the simplified determine condition is:

$$\gamma'\left|_{q=q^*}\right. \vee 0 \Leftrightarrow \sin(q_{1d} + q_2^*) \wedge 0 \quad (20)$$

i.e. if $\sin(q_{1d} + q_2^*) < 0$, q^* is a focus, and vice versa.

***Remark** 1:* the same results can be achieved by another method. The trajectory on the phase plane $(q_2, \dot{q}_2^2)$ can be described by (13). Considering the fact that if there is a limit cycle about an equilibrium point, there must exist a point $\hat{q}_2$ makes $Y'_{q_2}\left|_{q_2=\hat{q}_2}\right. = 0$ and $Y''_{q_2}\left|_{q_2=\hat{q}_2}\right. < 0$. This means that the trajectory in phase plane $(q_2, \dot{q}_2^2)$ is semi-closed and deflexed.

Direct calculation shows that:

$$Y'_{q_2} = -\beta_0 I^{-1} \left(\int I\gamma_0 dx + Y_0 \right) + \gamma_0 \quad (21)$$

$$Y''_{q_2} = (\beta_0^2 - \beta_0') I^{-1} \left(\int I\gamma_0 dx + Y_0 \right) - \beta_0\gamma_0 + \gamma_0' \quad (22)$$

Substitute (21) into (22)

$$\frac{\gamma_0(\beta_0^2 - \beta_0')}{\beta_0} - \beta_0\gamma_0 + \gamma_0' < 0 \Leftrightarrow \frac{\beta_0\gamma_0' - \gamma_0\beta_0'}{\beta_0} < 0$$

Suppose the limit cycle is closely near the equilibrium point, so $\hat{q}_2 \approx q_2^*$, then one gets

$$\frac{\beta_0\gamma_0' - \gamma_0\beta_0'}{\beta_0} < 0 \Leftrightarrow -\gamma_0'\left|_{q=q^*}\right. > 0$$

C. Range of limit cycles

For a limit cycle, there must exist two zero crossing points(one is the initial point $[q_{20}, 0]$) which makes $Y = 0$. The two points can be written as:

$$q_L = [q_{2L}, 0], q_R = [q_{2R}, 0]$$

and $Y(q_2) \geq 0\left|_{q_2 \in [q_{2L}, q_{2R}]}\right.$. The negative value of Y is impossible because in real situation $Y = \dot{q}_2^2 \geq 0$.

Assume $Y_0 = 0$, and denote

$$P(q_2) = \int_{q_{20}}^{q2} p(x) dx = \int_{q_{20}}^{q2} I\gamma_0 dx \quad (23)$$

Then the roots of $Y = 0$ are same with the roots of $P(q_2) = 0$ considering $I^{-1} > 0$. Since $P'(q_2) = I\gamma_0$, so $P'(q_2) = 0$ equals $\gamma_0 = 0$. Noticing that points which meet $\gamma_0 = 0$ are the equilibrium points, the relationship between equilibrium points and function $P(q_2)$ is that they are the locally maximum or minimum point of $P(q_2)$. Considering $P''(q_2) = I\gamma_0'$, the maximum points($P''(q_2) < 0$) are focus points, and there must exist limit cycles around them.

As we seen in Fig. 2 no matter from any point in the span $[q_{2L}, q_{2R}]$, there always exists a limit cycle. To determine all the spans on the whole phase space is complex. The paper

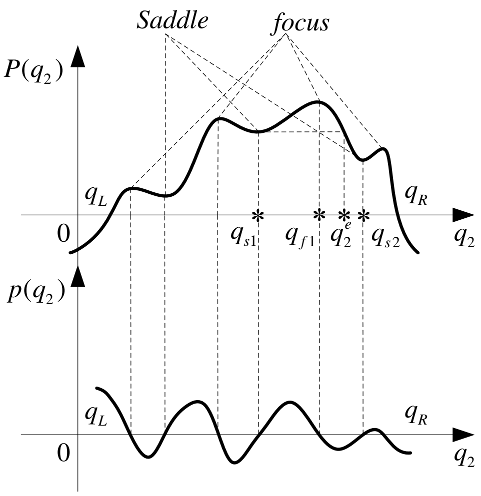

Fig. 2. Relationship between the equilibrium points and limit cycles

just considers such a span $[q_{2L}, q_{2R}]$ in which there is one focus and two saddle points. The relationship is:

$$q_{2L} \leq q_{s1} < q_{f1} < q_{s2} \leq q_{2R} \tag{24}$$

where q_{f1} is the focus point and q_{s1}, q_{s2} are the saddle points.

***Theorem** 3:* for system with (1), (2), (3), (13), consider a span as (24).

If $P(q_{s1}) > P(q_{s2})$ then the minimum range of limit cycle is $[q_{s1}, q_2^e]$where $q_2^e \in [q_{f1}, q_{s2}]$ owns the same value with q_{s1};

If $P(q_{s1}) < P(q_{s2})$ then the minimum range of limit cycle is $[q_2^e, q_{s1}]$where $q_2^e \in [q_{s1}, q_{f1}]$ owns the same value with q_{s2};

proof: The results are obviously. Since q_{f1} is the maximum point in the span, so for any initial value q_{20} in the range, equation $P(q_2, q_{20}) = 0$ always has another root q_{21}. These two points form a semi-closed cycle in the phase plane $(q_2, \dot{q}_2^2)$, which is a closed cycle i.e. a limit cycle in the phase plane $(q_2, \dot{q}_2)$.

Taking $q_{1d} = -1.2$ as an example, using parameters listed in TABLE II, the analysis process is consisted of the following 3 steps.

1) Calculate the equilibrium points
Substitute $q_{1d} = -1.2$ into (16), one can get

$$q_2^* = \begin{cases} 2k\pi - 0.9454 \\ 2k\pi - 2.9377 \end{cases}, k \in Z$$

2) Determine which one of they are focus or saddle
Considering theorem 2 and (20), $\sin(-1.2-0.9454) = -0.8394 < 0$ proves $[-1.2, -0.9454]$ is a focus; $\sin(-1.2-2.9377) = 0.8394 > 0$ proves $[-1.2, -2.9377]$is a saddle.

3) Choose an span to calculate the range
The chosen span is

$$q_{s1} = -2.9377, q_{f1} = -0.9454, q_{s2} = 3.3455$$

Substitute them into (23), one can find $P(q_{s1}) > P(q_{s2})$ and $P(q_{s1}) = P(-0.1863)$. From theorem 3, the range is $[-2.9377, -0.1863]$.

Fig. 3 clearly shows that the trajectories starting from range $[-2.9377, -0.1863]$ form semi-closed orbits in phase plane $(q_2, \dot{q}_2^2)$.

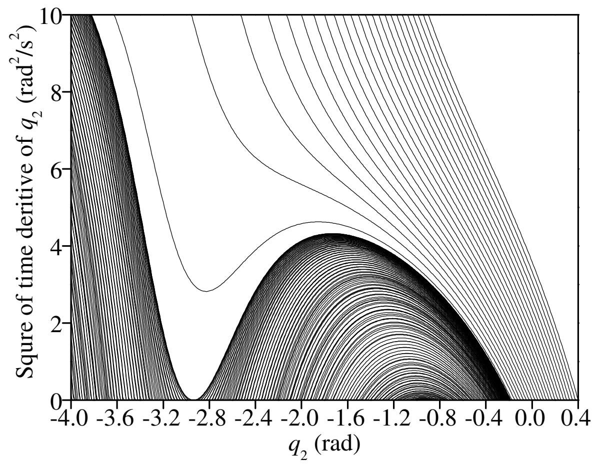

Fig. 3. Limit cycles on the $q_2, \dot{q}_2^2$ plane. The Trajectories initial from -4.0 to 0.4, while the limit cycles just exist in range from -2.9377 to -0.1863.

IV. STABILIZE CONTROL OF PERIOD ORBITS

A. Virtual constraints stabilization transformation

Introduce a function:

$$y = q_1 - q_{1d} \tag{25}$$

Take time derivative of y twice then there is

$$\ddot{y} = \ddot{q}_1 \tag{26}$$

Substitute (1) into (2), one gets

$$\ddot{q}_1 = \frac{-d_{12}\tau - d_{22}(h_1 + \varphi_1) + d_{12}(h_2 + \varphi_2)}{d_{11}d_{22} - d_{12}d_{21}} \tag{27}$$

Substitute (27) into (26) and yields

$$\ddot{y} = \frac{-d_{12}\tau - d_{22}(h_1 + \varphi_1) + d_{12}(h_2 + \varphi_2)}{d_{11}d_{22} - d_{12}d_{21}} \tag{28}$$

Introduce a virtual input

$$\begin{aligned} v &= K_1\dot{y} + K_2 y + \\ &+ \frac{-d_{12}\tau - d_{22}(h_1 + \varphi_1) + d_{12}(h_2 + \varphi_2)}{d_{11}d_{22} - d_{12}d_{21}} \end{aligned} \tag{29}$$

where K_1 and K_2 are positive constants. So there is

$$\ddot{y} + K_1\dot{y} + K_2 y = v \tag{30}$$

This closed loop subsystem (30) has a transfer function

$$G(s) = \frac{Y(s)}{v(s)} = \frac{1}{s^2 + K_1 s + K_2} \tag{31}$$

The subsystem is stable with proper K_1, K_2 and when $v \to 0, y \to 0$. I.e. the virtual constraint (3) holds. Then the zero dynamics can be written as

$$\begin{aligned} d_{11}v + d_{12}\ddot{q}_2 + h_1 + \varphi_1 &= 0 \\ \ddot{y} + K_1\dot{y} + K_2 y &= v \end{aligned} \tag{32}$$

B. Limit cycle stabilization controller design

When $v \rightarrow 0, y \rightarrow 0$, (32) describe the manifold Ω, the next task is to drive system state to the desired limit cycle.

Consider the function (18) and construct a Lyapunov function as

$$V = \frac{1}{2}U^2 \tag{33}$$

Take time derivative of V

$$\dot{V} = U\dot{U} \tag{34}$$

Take time derivative of U

$$\dot{U} = \dot{q}_2 \left(2\ddot{q}_2 + I^{-1}\beta \dot{q}_2 \int I\gamma_0 dx - \gamma_0\right) \tag{35}$$

From (32) one can get

$$\ddot{q}_2 = -\frac{d_{11}v + h_1 + \varphi_1}{d_{12}} \tag{36}$$

By (33)-(36), one gets

$$\dot{V} = U\dot{q}_2 \left(g(q,\dot{q})v + f(q,\dot{q})\right) \tag{37}$$

where:

$$g(q,\dot{q}) = -2\frac{d_{11}}{d_{12}}$$
$$f(q,\dot{q}) = -2\frac{h_1 + \varphi_1}{d_{12}} + I^{-1}\beta \dot{q}_2 \int I\gamma_0 dx - \gamma_0$$

Choose the control law as

$$v = \left(-K_3 |U| \operatorname{sign}(U\dot{q}_2) - f(\cdot)\right)/g(\cdot)$$

where K_3 is a positive constant. This will make

$$\dot{V} = -K_3 U^2 |\dot{q}_2|$$

So with such a control law the Lyapunov function V will converge to zero. The item $|U|$ aims to make the $v \rightarrow 0$ when $U \rightarrow 0$.

On the whole the feedback control law is

$$\begin{aligned}\tau &= \frac{d_{12}d_{21} - d_{11}d_{22}}{d_{12}} \left(\frac{-K_3 |U| \operatorname{sign}(U\dot{q}_2) - f(\cdot)}{g(\cdot)}\right) + \\ &- \frac{d_{12}d_{21} - d_{11}d_{22}}{d_{12}} (K_1\dot{y} + K_2 y) + \\ &+ \frac{d_{22}(h_1 + \varphi_1) - d_{12}(h_2 + \varphi_2)}{d_{12}}\end{aligned}$$

V. NUMERICAL SIMULATION

In this section the simulation results are given to prove the existence of the limit cycles and the effectiveness of the proposed control strategy. Simulations are performed by Matlab/Simulink. Same system parameters as [1] are used and listed in TABLE II.

TABLE II

PARAMETERS OF THE ACROBOT

m_1	m_2	l_1	l_2	l_{c1}	l_{c2}	I_1	I_2	g
kg	kg	m	m	m	m	kgm^2	kgm^2	m/s^2
1	1	1	2	0.5	1	1/12	1/3	9.81

The control parameters are $K_1 = 20, K_2 = 100, K_3 = 30$. The initial condition is $[q_1, q_2, \dot{q}_1, \dot{q}_2] = \left[-\frac{\pi}{2}, 0, 0, 0\right]$.

Choosing the desired point $q_d = [-1.2, -0.8, 0, 0]$, the range of limit cycle for $q_{1d} = -1.2$ has been proved to be $[-2.9377, -0.1863]$, so there must exist a limit cycle crossing the desired point. The simulation results are shown in Fig. 4-Fig. 6.

Fig. 4 shows the angle of the two links. q_1 converges to the desired value -1.2 rad, while q_2 converges to a periodic trajectory. Fig. 5 shows the phase portrait of q_2 and $\dot{q}_2$ along solution of the closed loop system. The system states directly converge to a limit cycle crossing $[q_2, \dot{q}_2] = [-0.8, 0]$. Fig. 6 shows the limit cycle function U.

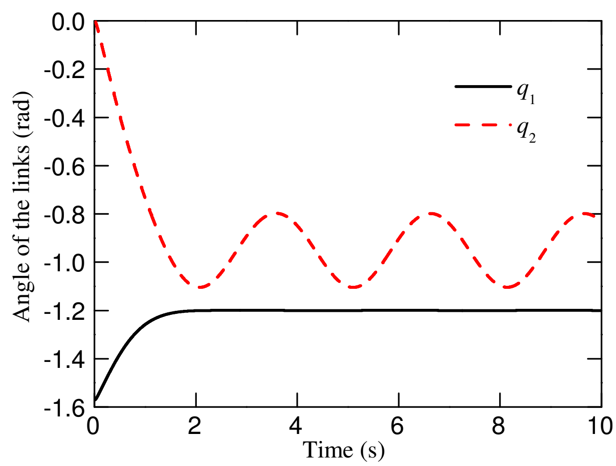

Fig. 4. Angle of the two links. The desired point is $q_{1d} = -1.2, q_{2d} = -0.8$.

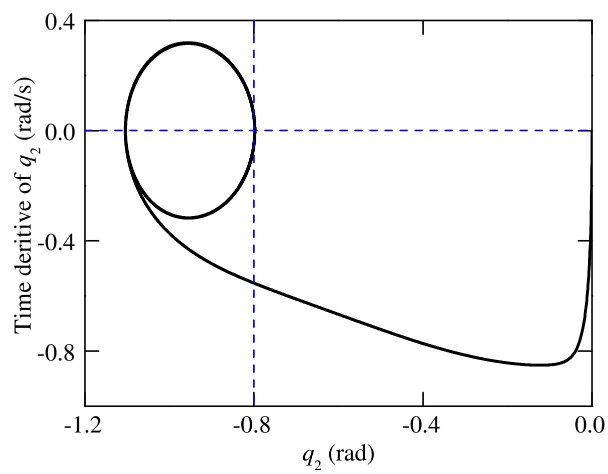

Fig. 5. Phase portrait of q_2 and $\dot{q}_2$ along solution of the closed loop system. The desired point is $q_{1d} = -1.2, q_{2d} = -0.8$.

To check the possibility of dynamical servo control, the next three desired points are given.

$$\begin{aligned} q_{d1} &= [-1.4, -0.8, 0, 0] \\ q_{d2} &= [-1.2, -0.8, 0, 0] \\ q_{d3} &= [-1.2, -0.3, 0, 0] \end{aligned}$$

The existence of limit cycles passing through those points can be determined using same method as before. The switch time is 25s and 50s, the simulation results are shown in Fig. 7- Fig. 8.

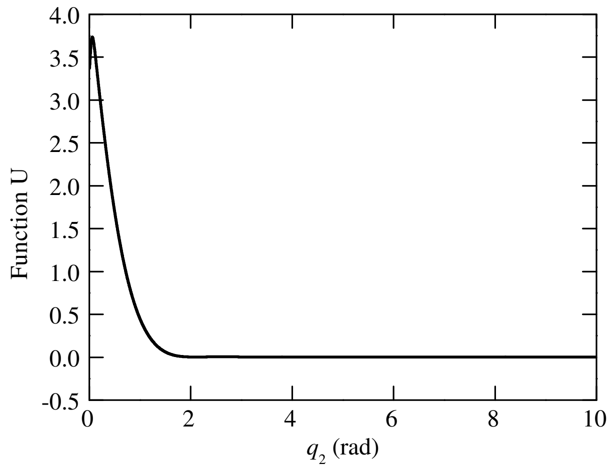

Fig. 6. Function U. The desired point is $q_{1d} = -1.2, q_{2d} = -0.8$.

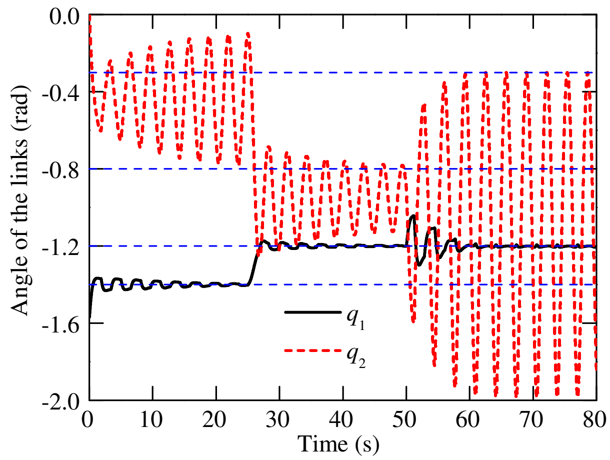

Fig. 7. Angle of the two links. The desired points are q_{d1}, q_{d2}, q_{d3} and the switch times are 25s and 50s.

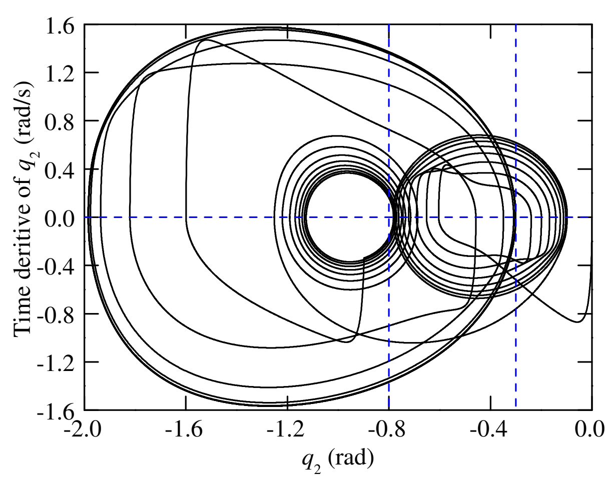

Fig. 8. Phase portrait of q_2 and $\dot{q}_2$ along solution of the closed loop system. The desired points are q_{d1}, q_{d2}, q_{d3} and the switch times are 25s and 50s.

Fig. 7 shows the angle of the two links. q_1 follows the desired value -1.4 rad and -1.2 rad, while q_2 follows the desired value -0.8 rad and -0.3 rad but in a periodic way. Fig. 8 shows the phase portrait of q_2 and $\dot{q}_2$ along solution of the closed loop system. Three limit cycles crossing the desired points and the transition process between them can be seen clearly from Fig. 8.

VI. CONCLUSIONS

In this paper the dynamical servo control problem for the Acrobot is discussed through virtual constraints approach. The definition of dynamical servo control problem is given by comparing with the traditional servo control problem. The dynamical servo control aims to access generic points in the state space not just the equilibrium points. A possible way is to stabilize some periodic orbits passing through the desired points. Virtual constraints are used to generate such orbits.

Then conditions of whether there are limit cycles about an equilibrium point are given and method to determine whether there is an orbit passing through the desired point is proposed. Finally a cascade control strategy is proposed to stabilize both the virtual constraint and the orbit function which decides the system behavior. The virtual constraint controller provides a virtual input for the external subsystem meanwhile can guarantee the inner stability of the subsystem. Simulation results show that the control strategy is effective.

One simple virtual constraint is considered in this paper, in fact, there are a broad of virtual constraints can be used to realize the dynamical servo control problem. And in this paper the desired points are limited with no velocity, so there are still many problems worth to study.

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