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Euler-Bernoulli Equation Today

Mirjana Filipovic

***Abstract*—Special attention is paid to the motion of the flexible links in the robotic configuration. The elastic deformation is a dynamic value which depends on the total dynamics of the robot system movements. The Euler-Bernoulli equation (based on the known laws of dynamics) should be supplemented with all the forces that are participating in the formation of the elasticity moment of the considered mode according to the requirements of the motion complexity of elastic robotic systems. This yields the difference in the structure of Euler-Bernoulli equations for each mode. The stiffness matrix is a full matrix as well as damping matrix. Mathematical model of the actuators also comprises coupling between elasticity forces. Particular integral which defined Daniel Bernoulli should be supplemented with the stationary character of elastic deformation of any point of the considered mode, caused by the present forces. General form of the mechanism elastic line is a direct outcome of the system motion dynamics, and cannot be described by one scalar equation but by three equations for position and three equations for orientation of every point on that elastic line. Simulation results are shown for a selected robotic example involving the simultaneous presence of elasticity of the gear and of the link (two modes), as well as the environment force dynamics.**

I. Introduction

MODELING and control of elastic robotic systems has been a challenge to researchers in the last four decades.

Mathematical model of a mechanism with one degree of freedom (DOF), with one elastic gear was defined by Spong [1] in 1987. Based on the same principle, the elasticity of gears is introduced in the mathematical model in this paper, as well in papers [2]-[6]. However, when the introduction of link flexibility in the mathematical model is concerned, it is necessary to point out some essential problems in this domain.

In our paper we do not use “assumed modes technique”, proposed by Meirovitch in [7] (and used from all authors until today [8]-[13] etc.).

The first detailed presentation of the procedure for creating reference trajectory was given in [14].

The reference trajectory is calculated from the overall dynamic model when the robot tip is tracking a desired trajectory in a reference regime in the absence of disturbances as in papers [2], [3], [5], and [6]. Elastic deformation (of flexible links and elastic gears) is a quantity which is, at least, partly encompassed by the reference trajectory. It is assumed that all elasticity characteristics in the system (both of stiffness and damping) are "known" at least partly and at that level they can be included into the process of defining the reference motion.

LMA (“Lumped-mass approach”) is a method which defines motion equation at any point of considered mechanism. If any link of the mechanism is elastic then we can also define motion equation at any point of presented link. We don’t know exactly when this approach was stated. It defines dynamic equation in any point of mechanism during movement. The LMA [15]-[18] gives the possibility to analyze the motion of any point of each mode. Papers with this research topic (approach) were rare in robotics journals in the last three decades.

EBA (“Euler-Bernoulli approach”) assumes the use of Euler-Bernoulli equations which appeared in 1750. EBA [8]-[13] etc, gives the possibility to analyze a flexible line form of each mode in the course of task realization. The EBA is an approach that is still in the focus of researchers’ interest and it was analyzed most often in the last decades.

Relationship between the LMA and EBA has been established in papers [2], [3], [5], [6].

We consider that EBA and LMA are two comparative methods addressing the same problem but from different aspects.

The Euler Bernoulli equation as well as its solution were used in the literature [8]-[13] etc, published until now as defined [7]. In the meantime, from 1750 when the Euler Bernoulli equation was published until today our knowledge, especially in the robotics, the oscillation theory and the elasticity theory, has progressed significantly. As a consequence, this paper points out the necessity of the extension of the Euler Bernoulli equation as well as its solution from many aspects.

In the previous literature [8]-[13] etc, the general solution of the motion of an elastic robotic system has been obtained by considering flexural deformations as transversal oscillations that can be determined by the method of particular integrals of D. Bernoulli. We consider that any elastic deformation can be presented by superimposing D. Bernoulli’s particular solutions of the oscillatory character and stationary solution of the forced character. See papers [2], [3], [5], [6].

“Assumed modes technique” [7] was used by all authors in the last 40 years to form Euler Bernoulli equation of beam. In our paper we form Euler Bernoulli equation but we do not use “assumed modes technique” in contrast to our contemporaries. We think that the “assumed modes technique” was and still can be useful in some other research areas but it is used in a wrong way in robotics, theory of oscillations and theory of elasticity. We assume that the elastic deformation as well as circular frequency of each mode of elastic element is consequences of the overall dynamics motion of the robotic system.

Let us emphasize once again that in this paper we propose a mathematical model solution that includes in its root the possibility for simultaneous analyzing both present

Mirjana Filipovic, Mihajlo Pupin Institute, Volgina 15, 11060 Belgrade, Serbia, (phone: +381 11 2771 024; fax: +381 11 2776 583; e-mail: mira@robot.imp.bg.ac.yu).

phenomena – the elasticity of gears and the flexibility of links and the idea originated from [18], but on the new principles.

Our future work should be directed to the implementation of the gears elasticity and the flexibility of links on any model of rigid robot and also on the model of reconfigurable rigid robot as given in [19] or any other type of mechanism. The mechanism would be modeled to contain elastic elements and to generate vibrations, which are used for conveying particulate and granular materials in [20].

In Sections II we define a general form of the equation of flexible line of a complex robotic system of arbitrary configuration, using Euler-Bernoulli equation. We give the new interpretation of the Euler-Bernoulli's equation. Section III analyzes the movement dynamics of a multiple DOF elastic robotic pair with elastic gear and flexible link in the presence of the second mode and environment force. Section IV gives some concluding remarks.

II. Interpretation of the Euler-Bernoulli Equation

Equation of the elastic line of beam bending is of the following form:

$$\hat{M}_{1,1} + \beta_{1,1} \cdot \frac{\partial^2 \hat{y}_{1,1}}{\partial \hat{x}_{1,1}^{\ 2}} = 0 . \qquad (1)$$

where $\hat{M}_{1,1}(Nm)$ is the load moment, in these source equations encompassing only inertia, $\beta_{1,1} \cdot \frac{\partial^2 \hat{y}_{1,1}}{\partial \hat{x}_{1,1}^{\ 2}} = \hat{\varepsilon}_{1,1}$ bending moment, $\beta_{1,1}(Nm^2)$ is the flexural rigidity.

General solution of motion, i.e. the form of transversal oscillations of flexible beams can be found by the method of particular integrals of D. Bernoulli, that is:

$$\hat{y}_{to1,1}(\hat{x}_{1,1}, t) = \hat{X}_{1,1}(\hat{x}_{1,1}) \cdot \hat{T}_{to1,1}(t) . \qquad (2)$$

See Fig. 1. The symbol "^" denotes generally the quantities that are related to an arbitrary point of the elastic line of the mode, for example: $\hat{y}_{1,1}, \hat{x}_{1,1}, \hat{\varepsilon}_{1,1}$. The same quantities that are not designated by "^" are defined for the mode tip, for example: $y_{1,1}, x_{1,1}, \varepsilon_{1,1}$.

By superimposing the particular solutions (2), any transversal oscillation can be presented in the following form:

$$\hat{y}_{to1}(\hat{x}_{1,j}, t) = \sum_{j=1}^{\infty} \hat{X}_{1,j}(\hat{x}_{1,j}) \cdot \hat{T}_{to1,j}(t) . \qquad (3)$$

Equations (1)-(3) were defined under the assumption that the elasticity force is opposed only by the inertial force proper. Besides, it is supposed by definition that the motion in (1) is caused by an external force $F_{1,1}$, suddenly added and then removed. The solution (2)-(3) of D. Bernoulli satisfies these assumptions.

$\vartheta_{1,1}$ is the bending angle of the mode; $\omega_{1,1}$ is the rotation angle of the tip of the mode, see [21]. Bernoulli presumed the horizontal position of the observed body as its stationary

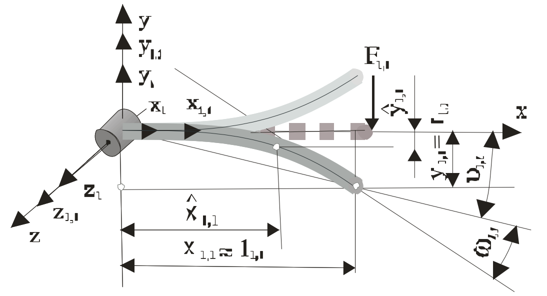

Fig. 1. Idealized motion of elastic body according to D. Bernoulli.

state (in this case it matches the position x - axis, see Fig. 1). At such presumption, the oscillations happen just around the x - axis. If Bernoulli, at any case, had included the gravity force G in its (1), the situation would have been more real. Then the stationery body position would not have matched the x - axis position, but the body position would have been little lower and the oscillations would have happened around the new stationery position (as presented in the Fig. 2).

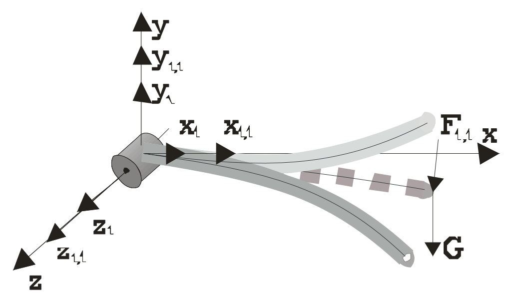

Fig. 2. The motion of elastic body in case of presence gravity force.

All marks are the same as in papers [2], [3], [5], [6].

Equations (1)-(3) need a short explanation that, we think, should be assumed, but which is missing from the original literature. Euler and Bernoulli wrote (3) based on 'vision'. They did not define the mathematical model of a link with an infinite number of modes, which has a general form of (4), but they did define the motion solution (shape of elastic line) of such a link, which is presented in (3). They left the task of link modeling with an infinite number of modes to their successors. Transversal oscillations defined by (4) describe the motion of elastic beam to which we assigned an infinite number of DOFs (modes), and which can be described by a mathematical model composed of an infinite number of equations, in the form:

$$\hat{M}_{1,j} + \hat{\varepsilon}_{1,j} = 0, \quad j = 1, 2, ..., j, ...\infty . \qquad (4)$$

Dynamics of each mode is described by one equation. The equations in the model (4) are not of equal structure as our contemporaries, authors of numerous works, presently interpret it. We think that the coupling between the modes involved leads to structural diversity among the equations in the model (4). This explanation is of key importance and is necessary for understanding our further discussion.

Under a mode we understand the presence of coupling between all the modes present in the system. We analyze the system in which the action of coupling forces (inertial, Coriolis', and elasticity forces) exists between the present modes. To differentiate it from "mode shape" or "assumed mode", we could call it a coupled mode or, shorter, in the text to follow, a mode. This yields the difference in the structure of Euler-Bernoulli equations for each mode.

The Bernoulli solution (2)-(3) describes only partially the nature of motion of real elastic beams. More precisely, it is only one component of motion. Euler-Bernoulli equations (1)-(3) should be expanded from several aspects in order to be applicable in a broader analysis of elasticity of robot mechanisms. By supplementing these equations with the expressions that come out directly from the motion dynamics of elastic bodies, they become more complex.

As known, a flexible deformation of a body under consideration may be caused by: *disturbance* forces, which cause the oscillatory nature of motion, *stationary* forces, which cause the stationary nature of motion.

By superposing the particular solution of oscillatory nature, and the stationary solution of forced nature, any flexible deformation of a considered mode may be presented in the following general form:

$$\hat{y}_{1,1} = \hat{X}_{1,1}(\hat{x}_{1,1}) \cdot (\hat{T}_{st1,1}(t) + \hat{T}_{to1,1}(t)) . \qquad (5)$$

$\hat{T}_{st1,1}$ is the stationary part of flexible deformation caused by stationary forces that vary continuously over time. $\hat{T}_{to1,1}$ is the oscillatory part of flexible deformation as in (3).

Component $\hat{X}_{1,1}(\hat{x}_{1,1})$ describes a possible geometrical relation between $\hat{y}_{1,1}$ and $\hat{x}_{1,1}$. Component $\hat{T}_{st1,1} + \hat{T}_{to1,1}$ describes the dependence of flexure $\hat{y}_{1,1}$ on force, which is the only time-varying quantity in expression (5). By combining the particular solution of the oscillatory nature of motion, the stationary solution of the forced nature of motion and the geometry of flexible line of the mode considered, we may obtain the general solution of the motion of the first mode.

By superposing solutions (5), any flexible deformations of a flexible link with an infinite number of degrees of freedom may be presented in the following form:

$$\hat{y}_1(\hat{x}_{1,j}, t) = \sum_{j=1}^{\infty} \hat{X}_{1,j}(\hat{x}_{1,j}) \cdot (\hat{T}_{st1,j}(t) + \hat{T}_{to1,j}(t)) . \quad (6)$$

The motion of the considered robotic system mode is far more complex than the motion of the body presented in Fig. 1. This means that the equations that describe the robotic system (its elements) must also be more complex than the (1)-(3), formulated by Euler and Bernoulli. This fact is overlooked, and the original equations are widely used in the literature to describe the robotic system motion. This is very inadequate because valuable pieces of information about the complexity of the elastic robotic system motion are thus lost. Hence, it should be especially emphasized the necessity of expanding the source equations for the purpose of modeling robotic systems, and this should be done in the following way:

* based on the known laws of dynamics, (1) is to be supplemented by all the forces that participate in the formation of the bending moment of the considered mode. It is assumed that the forces of coupling (inertial, Coriolis, and elastic) between the present modes are also involved, which yields structural difference between (1) in the model (4),

* Equations (2)-(3) are to be supplemented by the stationary character of the elastic deformation caused by the forces involved.

* Damping is an omnipresent flexibility characteristic of real systems, so that it is naturally included in the Euler-Bernoulli equation.

Now $\hat{\varepsilon}_{1,1} = \beta_{1,1} \cdot \dfrac{\partial^2(\hat{y}_{1,1} + \eta_{1,1} \cdot \dot{\hat{y}}_{1,1})}{\partial \hat{x}^2_{1,1}}$ is a bending moment, $\eta_{1,1}$ is a factor characterizing the share of damping in the total flexibility characteristic.

Model of the elastic line of complex elastic robotic system is given in the matrix form by the following Euler-Bernoulli equation (see [5], [6]):

$$\hat{H} \cdot \frac{d^2 \hat{y}}{dt^2} + \hat{h} + j_e^T \cdot F_{uk} + z \cdot \Theta \cdot \varepsilon + \hat{\varepsilon} = 0 . \qquad (7)$$

The robotic system having m links (each of them containing n_i modes). If we define $k = n_1 + n_2 + \ldots + n_m$ then we have that matrix characterizing the $\hat{H} \in R^{kxk}$ - inertia, $\hat{h} \in R^{kx1}$ - centrifugal, gravitational and Coriolis forces, $j_e^T \in R^{kx6}$ - mapping the effect of the dynamic contact force F_{uk}, $\Theta \in R^{kxk}$ - robot configuration, $z \in R^{kxk}$ - mutual influence of the forces of elastic modes of all the links. Equation (7) represents the equation of motion of the elastic line of the overall robotic system. It is known that the robot configuration can substantially influence the mutual position of elastic lines of particular links. Solution of the system (7) and dynamic motor motion, i.e. the form of its elastic line for all the links involved in the presence of the dynamics (angle) of rotation of each motor $\overline{\theta}$, as well as by taking into account the robotic configuration, i.e. the angle α between the axes z_{i-1} and z_i.

$$\begin{array}{l} \hat{y} = \hat{a}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \overline{\theta}, \alpha, t) \\ \hat{x} = \hat{b}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \overline{\theta}, \alpha, t) \\ \hat{z} = \hat{c}(\hat{x}_{i,j}, T_{sti,j}, T_{sti,j}, \overline{\theta}, \alpha, t) \\ \hat{\psi} = \hat{d}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \overline{\theta}, \alpha, t) \\ \hat{\xi} = \hat{e}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \overline{\theta}, \alpha, t) \\ \hat{\varphi} = \hat{f}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \overline{\theta}, \alpha, t) \end{array} . \qquad (8)$$

The equation of motion of all the forces at the point of each mode tip of any link can be defined from (7) by setting the boundary conditions. Vector equation of all the forces involved for each mode tip of any link is:

$$H\frac{d^2y}{dt^2}+h+j_e^T\cdot F_{uk}+z\cdot\Theta\cdot\varepsilon+\varepsilon=0\,. \qquad (9)$$

In order to describe the behavior of a robotic system, we have to add to the mathematical model of all the motors written in a vector form. The motor moment is opposed by the bending moment of the first elastic mode that comes after the motor, and also in part, by the bending moments of the other elastic modes that are connected in series after the given motor. All the modes after the motor, due to their position, influence the dynamics of motor motion. The effect of the first mode bending moment is defined by the factor $+1/2^0$, of the second by $-1/2^1$, of the third by $+1/2^2$, of the fourth by $-1/2^3$, of the fifth by $+1/2^4$... Mathematical model of all motors are:

$$\begin{aligned} &u=R\cdot i+C_E\cdot\dot{\theta} \\ &C_M\cdot i=I\cdot\ddot{\theta}+B_u\cdot\dot{\theta}-S\cdot(z_m\cdot\varepsilon+\varepsilon_m)\end{aligned}\,. \qquad (10)$$

$R_1[\Omega]$ is the rotor circuit resistance; $i_1[A]$ is the rotor current; $C_{E1}\ [V/(rad/s)]$ and $C_{M1}\ [Nm/A]$ are the proportionality constants of the electromotive force and moment, respectively; $B_{u1}\ [Nm/(rad/s)]$ is the coefficient of viscous friction; $I_1\ [kgm^2]$ is the inertia moments of the rotor and reducer; S_1 is the expression defining the reducer geometry; new structures of the matrix z and also z_m appear as a consequence of the coupling between the modes of particular links. The robot tip motion is defined by the sum of the stationary and oscillatory motion of each mode tip plus the dynamics of motion of the motor powering each link, as well by the included robot configuration. From (9)-(10) we can calculate the position y,x,z and orientation ψ,ξ,φ of each mode tip, of each link, and finally, of the robot tip motion.

III. Example

Here we have one more innovation concerning the known considerations. In robotics the reference trajectory is defined in purely kinematics way i.e. geometric and now in the presence of the elasticity elements we can include also the elastic deformation values at the reference level i.e. at the level of knowing the elasticity characteristics during the reference trajectory defining.

There are two aspects in defining the reference trajectory of the motor angle (see [2], [3], [5], and [6]), viz.:

1) Elastic deformation is considered as a quantity which is not encompassed by the reference trajectory.

2) Elastic deformation is a quantity which is at least partly encompassed by the reference trajectory.

A robot starts from a point "A" (Fig. 3) and moves toward a point "B" in the predicted time $T=2(s)$. Dynamics of the environment force is included into the dynamics of system's motion [22]. The adopted velocity profile is trapezoidal, with the period of acceleration/deceleration of $0.2\cdot T$.

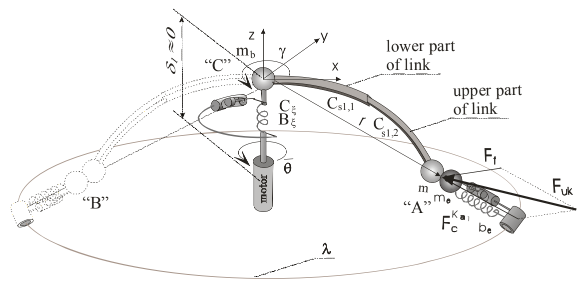

Fig. 3. Robot mechanism.

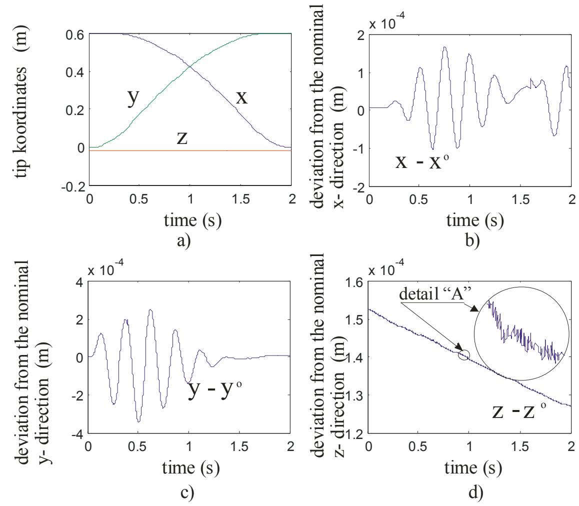

Fig. 4. The tip coordinates and the position deviation from the reference level.

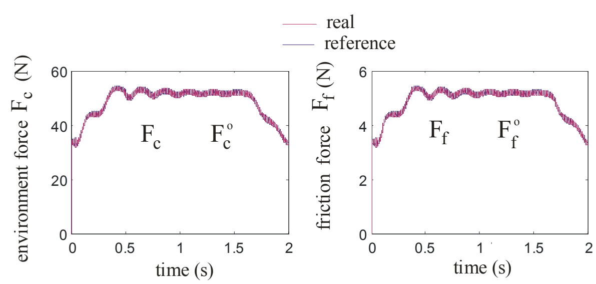

Fig. 5. The environment force dynamics.

$dt=0.000053335\ (s)$, all other characteristics of the system and environment are the same as in papers [3].

Elastic deformation is a quantity which is at least partly encompassed by the reference trajectory also. The characteristics of stiffness $C_\xi=1.8143\cdot10^3[Nm/rad]$ and damping $B_\xi=10[Nm/(rad/s)]$ of the gear in the real and reference regimes are not the same and neither are the stiffness $C_{s1,1}=6.1569\cdot10^4[N/m]$, $C_{s1,2}=1.873\cdot10^3[N/m]$ and damping $B_{s1,1}=0[N\cdot s/m]$, $B_{s1,2}=6[N\cdot s/m]$

characteristics of the link. $C_\xi = 0.2 \cdot C^o_\xi$, $B_\xi = 0.2 \cdot B^o_\xi$, $C_{s1,1} = 0.99 \cdot C^o_{s1,1}$, $B_{s1,1} = 570 + B^o_{s1,1}$, $C_{s1,2} = 0.99 \cdot C^o_{s1,2}$, $B_{s1,2} = 0.99 \cdot B^o_{s1,2}$.

The only disturbance in the system is the partial lack of the knowledge of all flexibility characteristics.

As can be seen from Fig. 4 in its motion from point "A" to point "B" the robot tip tracks well the reference trajectory in the space of Cartesian coordinates. As a position control law for controlling local feedback was applied, the tracking of the reference force was directly dependent on the deviation of position from the reference level (see Fig. 5).

The elastic deformations that are taking place in the

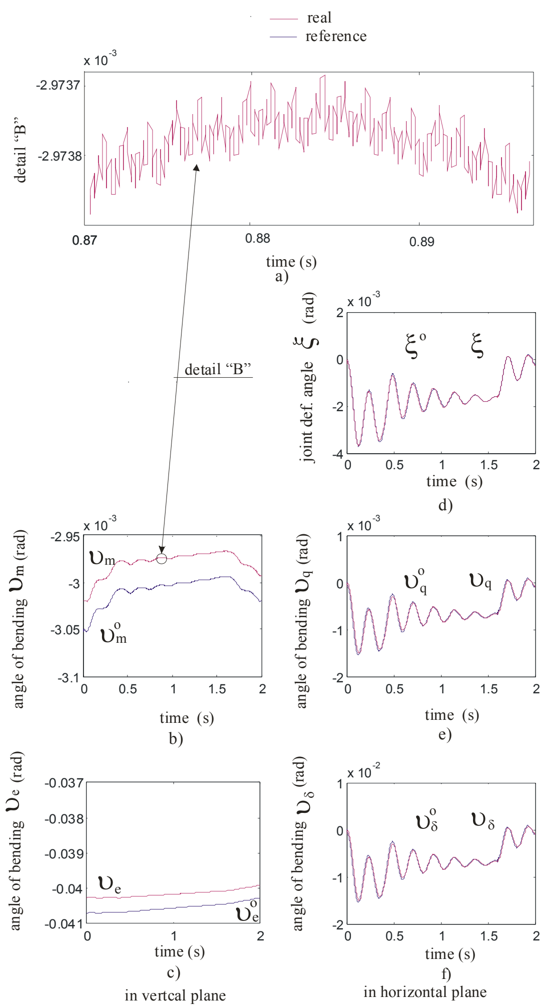

Fig. 6. The elastic deformations.

vertical plane angle of bending of the lower part of the link (the first mode) ϑ_m and the angle of bending of the upper part of the link (the second mode) ϑ_e, as well as elastic deformations taking place in the horizontal plane, the angle of bending of the lower part of the link (the first mode) ϑ_q, the angle of bending of the upper part of the link (the second mode) ϑ_δ and the deflection angle of gear ξ are given in Fig. 6.

The rigidity of the second mode is about ten times lower compared with that of the first mode, it is then logical that the bending angle for the second mode is about ten times larger compared to that of the first mode.

A more significant lack of knowledge of damping characteristics of the link (the second mode) causes small deviations of all quantity from the reference in the course of robotic task realization.

Let us show the special significance of results from Figs. 6a. These figure exhibits the wealth of different amplitudes and circular frequencies of the present modes of elastic elements. We have oscillations within oscillations. This confirms that we have modeled all elastic elements as well as high harmonics (in this case two harmonics of considered link).

IV. Conclusion

Based on the Euler-Bernoulli equation, we defined the equation of elastic line of a complex robotic system. We demonstrated that the equation of motion of all the forces involved at any point follows directly from the Euler-Bernoulli equation. If we define boundary conditions for the mode tip as the most interesting point on the elastic line, we obtain the equation of motion at that point, what is classical form of the mathematical model of the elastic robotic system considered. The reference trajectory depends on the level of knowing elasticity characteristics. The estimated elasticity characteristics may be included into the reference trajectory, and thus into the control law.

Euler-Bernoulli equation has been expanded from several aspects:

1) Euler-Bernoulli equation (based on the known laws of dynamics) should be supplemented with all the forces that are participating in the formation of the bending moment of the considered mode, what causes the difference in the structure of these equations for each mode.

2) Structure of the stiffness (and damping) matrix must also have the elements outside the diagonal, because of the existence of strong coupling between the elasticity forces involved.

3) Damping is an omnipresent elasticity characteristic of real systems, so that it is naturally included in the Euler-Bernoulli equation.

4) General form of the transversal elastic deformation is defined by superimposing particular solutions of oscillatory character (solution of Daniel Bernoulli) and stationary solution of the forced character (which is a consequence of the forces involved).

5) General form of the elastic line is a direct outcome of the dynamics of system motion and cannot be represented by one scalar equation but three equations are needed to define the position and three equations to define the orientation of each point on the elastic line.

Structure of the mathematical models of actuators: With elastic robotic systems, the actuator torque is opposed by the bending moment of the first elastic mode, which comes after the motor, and partly by the bending moments of other modes, which are connected in series after the motor considered. All modes coming after the motor, because of their position, exert influence on the dynamics of motor motion. The mathematical model of the actuators in our paper is connected to the rest of the mechanism via the equivalent elasticity moment.

Elastic deformation is a consequence of the overall dynamics of the robotic system, what is essentially different from the method that was used until today, which purports usage of "assumed modes technique".

All this has been presented for a relatively simple robotic system that offered the possibility of analyzing the phenomena involved. Through the analysis and modeling of an elastic mechanism we made an attempt to give a contribution to the development of this area.

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