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Scalable and Convergent Multi-Robot Passive and Active Sensing

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***Abstract*— A major barrier preventing the wide employment of mobile networks of robots in tasks such as exploration, mapping, surveillance, and environmental monitoring is the lack of efficient and scalable multi-robot passive and active sensing (estimation) methodologies. The main reason for this is the absence of theoretical and practical tools that can provide computationally tractable methodologies which can deal efficiently with the highly nonlinear and uncertain nature of multi-robot dynamics when employed in the aforementioned tasks.**

In this paper, a new approach is proposed and analyzed for developing efficient and scalable methodologies for a general class of multi-robot passive and active sensing applications. The proposed approach employs an estimation scheme that switches among linear elements and, as a result, its computational requirements are about the same as those of a linear estimator. The parameters of the switching estimator are calculated off-line using a *convex* optimization algorithm which is based on optimization and approximation using Sum-of-Squares (SoS) polynomials. As shown by rigorous arguments, the estimation accuracy of the proposed scheme is equal to the *optimal* estimation accuracy plus a term that is inversely proportional to the number of estimator's switching elements (or, equivalently, to the memory storage capacity of the robots' equipment). The proposed approach can handle various types of constraints such as communication and computational constraints as well as obstacle avoidance and maximum speed constraints and can treat both problems of passive and active sensing in a unified manner. The efficiency of the approach is demonstrated on a 3D active target tracking application employing flying robots.

I. INTRODUCTION

Despite the significant impact that groups of mobile robots can have on duties that currently require human participation, their potential has not yet been realized. The main reason for this is the absence of theoretical and practical tools that can provide computationally tractable methodologies which can deal efficiently with the highly nonlinear and uncertain nature of multi-robot dynamics when employed in the aforementioned tasks. As a matter of fact, the majority of techniques and methods for multi-robot sensing and estimation applications are based on local approximations of the overall system dynamics (team of robots + measurement model + the external environment): for instance, in *Passive Sensing (PS)* applications such as robot position and orientation estimation using robot-to-robot distance measurements, passive target tracking, localization, mapping and Simultaneous Localization And Mapping (SLAM), the majority of existing approaches employ Extended Kalman Filter (EKF) techniques which are based on linearization of the overall system dynamics, see e.g. [16], [13], [6] and the references therein. A similar situation is also present in *Active Sensing (AS)* applications such as active target tracking, Combined Localization and Active Target Tracking (CLATT) and combined SLAM and Exploration (SLAM-E), where the objective is to generate the robots' trajectories so that estimation accuracy is optimized. In most of the existing approaches, the trajectory generators are usually based on convex or local approximations (relaxations) of non-convex optimization problems, see e.g. [16], [8], [10] and the references therein. However, linearization (in case of PS) or relaxation (in case of AS) may have a significant or even devastating effect on the overall system efficiency: poor estimation accuracy or even divergence of the estimator in the case of EKF and getting trapped into local minima in case of trajectory generation.

Attempts that have been made to employ techniques that avoid the usage of linearization or convex/local relaxations face the well-known problem of curse of dimensionality: for instance, the usage of polynomial EKF, see e.g. [7], [12], for estimation purposes or the usage of dynamic programming techniques for trajectory generation, see e.g. [8], require the implementation of algorithms that scale poorly with the number of the robot team members and, as a result, their deployment in large-scale, real-life applications is formidable.

In this paper, we propose and analyze a new approach that has the potential to overcome the above mentioned shortcomings of the existing methodologies. The proposed approach adopts a general framework that can treat in a unified manner:

- both PS and AS multi-robots problems;
- a large class of multi-robot sensing applications such as relative pose estimation using only inter-robot measurements, target tracking, localization, SLAM, SLAM-E and CLATT.

Contrary to the approaches that are based on linearization or local approximation/relaxation techniques, the proposed approach uses the full nonlinear model of the overall multi-robot system dynamics in order to construct an estimator and/or a trajectory generator that fulfills the computational, communication, etc requirements imposed by the particular multi-robot application. Moreover, it guarantees stable, convergent and efficient performance of the overall estimation process. The main attributes of the proposed approach can be summarized as follows:

- The proposed approach employs an estimation scheme

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that switches among linear elements and, as a result, its computational requirements are about the same as those of a linear estimator. The parameters of the switching estimator are calculated off-line using a *convex* optimization algorithm which is based on optimization and approximation using Sum-of-Squares (SoS) polynomials. Stable and convergent estimator's performance is guaranteed, overcoming the short-coming of many existing methodologies where there is always the possibility of estimator error divergence.

• The estimation efficiency (accuracy) of the proposed estimator is equal to the *optimal* estimation accuracy plus a term that is inversely proportional to the number of estimator's switching elements (or, equivalently, to the memory storage capacity of the robots' equipment).

• All the aforementioned properties of the proposed approach are retained in case where the multi-robot sensing application is subject to communication constraints (e.g. the overall estimation process should be implemented in a distributed manner where each single robot uses only a portion of the available sensor information) or other types of constraints such as obstacle avoidance and maximum speed constraints. As a matter of fact, all these types of constraints can be straightforwardly handled by the proposed approach.

We close this section by noticing *that due to space limitations, the proofs of the theoretical results as well as simulation experiments on a variety of different multi-robot PS and AS applications, are not presented here. The interested reader can download the full-version of the paper [9], where detailed theoretical analysis as well as a description and evaluation of the simulation experiments are presented.*

A. Notations and Preliminaries

w.p.1 denotes "with probability 1". $\dim(x)$ denotes the dimension (length) of the vector x. For a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x (i.e., $|x| = \sqrt{x^\tau x}$), while for a matrix $A \in \mathbb{R}^{n^2}$, $|A|, \mathrm{tr}(A)$ denotes the induced matrix norm and the trace, respectively. For a symmetric matrix A, the notation $A \succ 0$ ($A \succeq 0$) is used to denote that A is a positive definite (resp. positive semidefinite) matrix. If $P \succ 0$, then $|x|_P = \sqrt{x^\tau P x}$. The notation $\mathrm{vec}\,(A, B, C, \ldots,)$, where $A, B, C, \ldots$ are scalars, vectors or matrices, is used to denote a vector whose elements are the entries of $A, B, C, \ldots$ (taken column-wise). For a compact subset $\mathcal{X} \subset \mathbb{R}^n$, a smooth function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ and a collection of nonnegative integers A we define $\|f\|^A_{\mathcal{X}} = \sup_{j\in A} \sup_{x\in\mathcal{X}} \left|\frac{\partial^j f}{\partial x^j}(x)\right|$. For a smooth function $V(x_1, x_2)$ where x_i are vectors, the following notation is also used: $V_{x_i}(x_1, x_2) = \frac{\partial V}{\partial x_i}(x_1, x_2), V_{x_i x_i}(x_1, x_2) = \frac{\partial^2 V}{\partial x_i^2}(x_1, x_2)$. For a function $f : \Re_+ \mapsto \Re_+$, we say that $f(a) = \mathcal{O}(a)$ if $f(a) \leq ca, \forall a \in \Re_+$, for some positive constant c independent of a. For two positive integers L, n, we will use the following notation:

$$\mathcal{P}_n(L) = \frac{(L+n)!}{L!n!} + n - 1 \tag{1.1}$$

The following definition will be finally needed in the paper.

Definition 1: Fix the positive integer L; the notation $z(x) = \mathcal{M}^L_n(x)$ will be used to denote that the vector function $z : \mathbb{R}^n \mapsto \mathbb{R}^{\mathcal{P}_n(L)}$ is defined as follows:

$$z(x) = \left[\sqrt{x_1}, \ldots, \sqrt{x_n}, x_1, x_2, \ldots, x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \ldots, x_n^L\right]^\tau$$

where a_i are nonnegative integers satisfying $\sum_{i=1}^n a_i \in \{1, \ldots, L\}$.

II. Active Estimator Design

In general, the dynamics of a team of M robots performing an active sensing (estimation) task can be represented by a set of stochastic nonlinear state-space differential equations given as follows:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + g_\omega(x)\omega \\ y &= h(x) + h_\xi(x)\xi \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^n$ denotes the state vector that is associated to the particular problem, $u \in \mathbb{R}^m$ denotes the vector of control inputs to the robots, $y \in \mathbb{R}^k$ denotes the vector of all available sensor measurements, f, g, h, g_ω, h_ξ are smooth nonlinear functions of appropriate dimensions, and ω, ξ correspond to vectors of zero-mean, unity-variance Gaussian processes. There are two different estimation problems associated with (2.1):

(PS) *Passive Sensing:* Given the time-histories of the available signals $y(s), u(s),\ s \in [0)$, construct an estimator $\hat{x}(t) = \mathcal{PS}(y(s), u(s))$ such that the estimation error[1] accuracy $\mathcal{E}(t) = E\left[|x(t) - \hat{x}(t)|^2\right]$ converges to as small values as possible.

(AS) *Active Sensing:* Given the time-history of the available signal $y(s), u(s),\ s \in [0)$, construct a combined estimator/trajectory generator $[\hat{x}(t), u(t)] = \mathcal{AS}(y(s), u(s))$ such that the estimation error accuracy $\mathcal{E}(t) = E\left[|x(t) - \hat{x}(t)|^2\right]$ converges to as small values as possible.

Please note that the fundamental difference between PS and AS is that while in the first case the robot trajectories $x(t)$ or, equivalently, the robot control inputs $u(t)$ are pre-specified or calculated on-line based on a procedure that is external to the estimation process, in the later case the robot trajectories are designed in the aim of optimizing the estimator's accuracy; as expected, if AS is employed, the estimator's accuracy can be significantly increased as compared to the PS case. In the rest of the section, we will concentrate in the AS problem and come back to the PS problem in the next section.

Remark 1: Active target tracking, CLATT and SLAM-E are some of the standard multi-robot estimation problems that belong to the AS category as defined above. In all these problems, the state vector x is defined according to

$$x = \left[{x^{(1)}}^\tau, \ldots, {x^{(M)}}^\tau, {x_T^{(1)}}^\tau, \ldots, {x_T^{(K)}}^\tau\right]^\tau$$

[1]It should be emphasized that convergence of $\mathcal{E}(t)$ to small values is the main but not the only criterion imposed in AS or PS applications; the rate of convergence, the worst-case performance of $\mathcal{E}(t)$ as well as – in the case of AS – the stability of the robot trajectories are additional criteria that are taken into account in AS and PS designs. The above issue will be clarified further later in this and the next sections.

where $x^{(i)}$ denotes the i-th robot's pose (position and orientation) vector and $x_T^{(i)}$ denotes the position of the i-th target (in the case of active target tracking and CLATT) or of the i-th landmark or feature (in the case of SLAM-E). Note that in the case of SLAM-E, $x_T^{(i)}$ is typically assumed constant (stationary environment) while in the case of active target tracking and CLATT, the ith target's position is usually assumed to evolve according to a stochastic linear model of the form $\dot{x}_T^{(i)} = F^{(i)}x_T^{(i)} + G^{(i)}\omega$ for some known constant matrices $F^{(i)}, G^{(i)}$. The sensor measurements vector y comprises robot-to-target or robot-to-landmark distance or bearing measurements, robot-to-robot distance or bearing measurements as well as IMU and/or GPS and/or odometer measurements, etc employed for robot localization. The interested reader is referred to e.g. [8], [10], [16], [14] and the references therein for more details on these two problems. $\diamond$

Remark 2: Typically, in multi-robot PS and AS applications the functions g_ω and h_ξ are assumed to be constant matrices in which case the system (2.1) dynamics reduce to a (nonlinear) additive Gaussian model. The more general – than the additive Gaussian model – formulation (2.1) allows, however, to deal with cases where the noise and/or disturbances have a multiplicative effect on the overall system dynamics, like applications involving odometry readings where it is more realistic to assume a multiplicative noise model for the effect of these readings, see e.g. [12].

The AS estimator for generating $\hat{x}(t)$ is assumed to take the following form:

$$\begin{array}{rcl} \dot{\hat{x}} & = & f(\hat{x}) + g(\hat{x})u + u_o \\ \hat{y} & = & h(\hat{x}) \end{array} \tag{2.2}$$

where u_o is an auxiliary time-varying vector which – together with u – is available for design. Assuming that the estimator (2.2) is employed, the *optimal AS estimator design* associated to (2.1) can be formulated as a stochastic optimal control problem described according to

$$\min_{(u(s),u_o(s),s\in[0,\infty))} \mathcal{J} \tag{2.3}$$

where

$$\begin{array}{rcl} \mathcal{J} & = & J(x(0),\hat{x}(0)) \\ & = & E\left[\int_0^\infty \left(|u(s)|_R^2 + |u_o(s)|_{R_o}^2 + |x(s)-\hat{x}(s)|_Q^2\right) ds\right] \end{array}$$

with $R \succ 0, R_o \succ 0, Q \succ 0$ being user defined matrices. Note that the above cost-criterion is in form similar to the cost criteria used in Linear Gaussian Quadratic Estimation and Control; similar to these cost criteria, different choices for the positive-definite matrices R, R_o, Q can be used to balance the trade-off between efficient steady-state and efficient transient performance. It is worth noticing that the proposed approach does not require the selection of the matrices R, R_o, Q; the definition of these matrices as well as of the cost criterion $\mathcal{J}$ were made only for analysis purposes.

We will make the following assumption regarding the optimal choices for u, u_o:

(A1) The optimal u^*, u_o^* (minimizing $\mathcal{J}$) satisfy

$$\begin{array}{rcl} u^*(t) & = & k_c(Y)\,(y - h(\hat{x})) \\ u_o^*(t) & = & k_o(Y)\,(y - h(\hat{x})) \end{array} \tag{2.4}$$

for some smooth functions k_c, k_o, where Y is calculated according to

$$\begin{array}{rcl} \dot{Y}_1 & = & -aY_1 + y,\ Y_1(0) = 0 \\ & \vdots & \\ \dot{Y}_p & = & -aY_p + Y_{p-1},\ Y_p(0) = 0 \\ Y & = & \mathrm{vec}(\hat{x}, Y_1, \ldots, Y_p), \end{array}$$

with a being a positive user-defined constant and p a positive integer satisfying $p \geq n/k + 1$.

It should be emphasized that in the deterministic case – i.e. the case where ω, ξ are either bounded-amplitude or bounded-energy signals – it can be seen that assumption (A1) holds: the deterministic case involves the design of a stable observer for system (2.1) whose dimension is generically sufficient to be equal to $2n+1$, see e.g. [1] and the references therein. The observer (estimator) structure assumed in (2.2), (2.4) can be seen that is capable of representing any smooth observer of dimension $2n+1$ (the proof of such a claim is pretty straightforward and is based on the fact that k_c, k_o can be chosen to be arbitrary functions). For the more general (stochastic) case treated in this paper, it is not known – to the best of our knowledge – whether an estimator of the form (2.2), (2.4) can represent the optimal estimator; however, experimental investigations with estimators of the form (2.2), (2.4) indicate that they are capable of producing quite satisfactory performance. Nevertheless, all the results of this paper can be easily extended to the case where a different structure for the estimator (2.2), (2.4) is assumed as long as this estimator remains linear[2] wrt y.

We return to the stochastic optimal control problem (2.3). By adopting a stochastic dynamic programming framework, we let $\tilde{V}$ denote the *optimal "cost-to-go" function*, see e.g. [15], defined according to

$$\tilde{V}(x(t),\hat{x}(t)) = \min J(x(t),\hat{x}(t)) \tag{2.5}$$

$$J(x(t),\hat{x}(t)) = E\left[\int_t^\infty \left(|u(s)|_R^2 + |u_o(s)|_{R_o}^2 + |x(s)-\hat{x}(s)|_Q^2\right) ds\right] \tag{2.6}$$

It is a well-known fact that $\tilde{V}$ can be obtained as a solution of a partial differential equation [the Hamilton-Jacobi-Bellman (HJB) equation associated with (2.1), (2.2), (2.5); see e.g. [15]]. The HJB is not possible, in general, to be solved on-line (in real-time). Moreover, the solution $\tilde{V}$ of the HJB equation is not defined in the usual sense of smooth solutions: the solution $\tilde{V}$, if exists, is typically defined by using the notion of *viscosity* solutions, see [4]. However, if

[2] The assumption that the estimator is linear wrt y is crucial for our analysis: if such an assumption does not hold, then the overall estimator's dynamics become nonlinear wrt the noise term ξ, in which case it is not possible to apply the arguments in the proof of Theorem 1 (see the Appendix).

$\tilde{V}$ admits a *viscosity* solution it can be approximated to any degree of accuracy by a smooth solution V; this is all that we need for our analysis.

We will impose the following assumptions regarding the optimal "cost-to-go" function and the associated optimal signals u^*, u_o^* defined in (A1).

(A2) For all admissible initial conditions, the optimization problem (2.5) – or, equivalently, the associated HJB equation – admits a unique viscosity solution $\tilde{V}$ satisfying $\tilde{V}(x,x) = 0$, $\tilde{V}(x,\hat{x}) > 0$ if $x \neq \hat{x}$.

(A3) Suppose that the admissible initial conditions satisfy $(x(0), \hat{x}(0)) \in \mathcal{X}_0$ where $\mathcal{X}_0 \subset \mathbb{R}^{2n}$ is a compact subset. Then, under the optimal $u^*(t), u_o^*(t)$ defined in (A1), $(x(t), \hat{x}(t)) \in \bar{\mathcal{X}} \supset \mathcal{X}_0$ and $Y(t) \in \bar{\Psi}$ for all $t \in [0, \infty)$, where $\bar{\mathcal{X}} \subset \mathbb{R}^{2n}$, $\bar{\Psi} \subset \mathbb{R}^{\dim(Y)}$ are compact subsets.

Assumption (A2) requires that the problem at hand makes sense, i.e. that for all admissible $x(0), \hat{x}(0)$, there exists a control strategy $u(t) = k_c(Y)(y - h(\hat{x}))$ that renders system (2.1) stochastically observable; the proof of the above claim is quite technical and is based on converse Lyapunov arguments such as the ones in [3], [5]. Assumption (A3) states that if the initial states $x(0), \hat{x}(0)$ belong to a bounded subset, then $x(t), \hat{x}(t), Y(t)$ will remain bounded. From a mathematical point of view, assumption (A3) is very strict for general systems of the form (2.1). However, in the multi-robot applications of our interest, such an assumption is *practically valid* as long as the robot trajectories remain within a pre-specified area. It has to be emphasized that the assumption that the system states remain bounded is a typical assumption made in the nonlinear estimation and filtering literature, see e.g. [2]. In the full version of the paper [9] we show how this assumption can be relaxed to allow $\bar{\mathcal{X}}, \bar{\Psi}$ to become unbounded.

Based on assumptions (A1)-(A3), we can see that the following Lyapunov-type result holds.

Proposition 1: Fix the positive matrices R, R_o, Q and let V be a smooth approximation of $\tilde{V}$ satisfying

$$V(x,x) = 0, \quad \|\tilde{V}(x,\hat{x}) - V(x,\hat{x})\|_{\bar{\mathcal{X}}}^{\{0,1,2\}} = \varepsilon \tag{2.7}$$

for some positive constant ε (note that ε can be made arbitrarily small). Then, there exist positive constants λ_i such that for all $(x, \hat{x}) \in \bar{\mathcal{X}}, Y \in \bar{\Psi}$, the following is valid:

$$\lambda_1 |x - \hat{x}|^2 \leq V(x,\hat{x}) \leq \lambda_2 |x - \hat{x}|^2 \tag{2.8}$$

$$\begin{aligned} \mathcal{L}V(x,\hat{x}) &\triangleq V_x^\tau f(x) + V_{\hat{x}}^\tau f(\hat{x}) \\ &\quad + \frac{1}{2}\mathrm{tr}\left\{ g_\omega^\tau(x) V_{xx} g_\omega(x) \right\} + \mathcal{A}(x,\hat{x}) \\ &\quad + \left(V_x^\tau g(x) + V_{\hat{x}}^\tau g(\hat{x})\right) k_c(Y)(y - h(\hat{x})) \\ &\quad + V_{\hat{x}}^\tau k_o(Y)(y - h(\hat{x})) \\ &\leq -\lambda_3 |x - \hat{x}|^2 + \lambda_4 |\xi|^2 + \mathcal{O}(\varepsilon) \end{aligned} \tag{2.9}$$

where $\mathcal{A}(x,\hat{x}) = \frac{1}{2}\mathrm{tr}\Big\{ \left(g(\hat{x}) k_c(Y) h_\xi(x) + k_o(Y) h_\xi(x)\right)^\tau V_{\hat{x}\hat{x}} \left(g(\hat{x}) k_c(Y) h_\xi(x) + k_o(Y) h_\xi(x)\right) \Big\}$.

In order to understand the meaning of the above Proposition and its relation to the optimal control problem (2.3) we note that (2.8), (2.9) imply that

$$\mathcal{E}^*(t) \leq \frac{\lambda_2}{\lambda_1} e^{-\frac{\lambda_3}{\lambda_2}} \left[|x(0) - \hat{x}(0)|^2 \right] + \frac{\lambda_2}{\lambda_3 \lambda_1} \left\{ \lambda_4 + \mathcal{O}(\varepsilon) \right\} \tag{2.10}$$

where $\mathcal{E}^*(t)$ denotes the estimation error accuracy under the optimal signals $u^*(t), u_o^*(t)$ [the proof of (2.10) is along the same lines as the proof of (2.14) in Theorem 1]. In other words, any optimal AS estimator – parametrized by R, R_o, Q – satisfies inequality (2.10) for some positive constants λ_i that depend on the particular choice for R, R_o, Q. As a result, different choices for the matrices R, R_o, Q in the optimal control problem (2.3) result in different values for the constants λ_i and, thus, in different transient and steady state characteristics for the optimal AS estimator.

We are now ready to present the proposed approach. Let $\epsilon_i, i = 1,2,3,4$ be four user-defined positive constants and let $\mathcal{X}, \Psi$ be two compact supersets of $\bar{\mathcal{X}}, \Psi$, respectively, which are assumed to be sufficiently large so that they contain all possible solutions $(x(t), \hat{x}(t))$ and $\Psi(t)$ generated by the proposed AS estimator. Also, fix a positive integer L and a strictly increasing function $\mathcal{R} : \mathbb{Z}_+ \mapsto \mathbb{Z}_+$ [e.g. $\mathcal{R}$ may be chosen according to $\mathcal{R}(L) = \mathcal{P}_n(L)$ where $\mathcal{P}_n(L)$ is defined in (1.1)] and consider a partition of $\mathcal{R}(L)$ disjoint subsets $\Psi_j,\ j \in \{1, \ldots, L\}$ satisfying $\bigcup_{j=1}^{\mathcal{R}(L)} \Psi_j = \Psi$ and designed so that they minimize $\int_\Psi \left| \psi - \sum_{j=1}^{\mathcal{R}(L)} \bar{\psi}_j \phi_j(\psi) \right|^2 d\psi$ where ϕ_j denotes the indicator function defined according to

$$\phi_j(Y) = \begin{cases} 1 & \text{if } (Y) \in \Psi_j \\ 0 & \text{otherwise} \end{cases}$$

and $\bar{\psi}_j$ denotes the centroid of Ψ_j, i.e. $\bar{\psi}_j = \int_\Psi x \phi_j(\psi) d\psi / \int_\Psi \phi_j(\psi) d\psi$. Then the proposed scheme for the calculation of u, u_o is as follows:

$$\begin{aligned} u(t) &= \textstyle\sum_{j=1}^{\mathcal{R}(L)} \theta_c^{(j)} \phi_j(Y)(y - h(\hat{x}(t))) \\ u_o(t) &= \textstyle\sum_{j=1}^{\mathcal{R}(L)} \theta_o^{(j)} \phi_j(Y)(y - h(\hat{x}(t))) \end{aligned} \tag{2.11}$$

where the matrices $\theta_o^{(j)}$ and $\theta_c^{(j)}$ are calculated as the solutions of the convex optimization problem (2.12) described in Table I.

The following Theorem summarizes the convergence properties of the proposed scheme.

Theorem 1: Let assumptions (A1)-(A3) hold. Then, there exists a positive integer $\bar{L}$ such that the following holds for all $L \geq \bar{L}$: Suppose that the user-defined parameters $\epsilon_i, i = 1,2,3,4$ satisfy

$$\epsilon_1 = \lambda_1 - \nu_1, \epsilon_2 = \lambda_2 + \nu_2, \epsilon_3 = \lambda_3, \epsilon_4 = \lambda_4$$

for some positive constants $\lambda_i, i = 1,2,3,4$ that are associated to the optimal control problem (2.3) as described in Proposition 1 and some positive constants $\nu_i = \mathcal{O}(1/L), i = 1,2$. Then, the estimator (2.2), (2.11) satisfies with probabil-

Table I: AS and PS Estimator Design

$$\min_{\theta_c,\theta_o,P} \quad \sum_{i=1}^{\mathcal{N}} \mu_i \tag{2.12}$$

$$\begin{aligned}
\text{s.t.} \quad & (\text{here } i = 1,\ldots,\mathcal{N}) \\
\mu_i \;\geq\; & \mathcal{V}_x^{[i]\tau} f(x^{[i]}) + \mathcal{V}_{\hat{x}}^{[i]\tau} f(\hat{x}^{[i]}) + \frac{1}{2}\mathrm{tr}\Big\{ g_\omega^\tau(x^{[i]}) \mathcal{V}_{xx}^{[i]} g_\omega(x^{[i]}) \Big\} + \bar{\mathcal{A}}(x^{[i]},\hat{x}^{[i]}) \\
& + \left(\mathcal{V}_x^{[i]\tau} g(x^{[i]}) + \mathcal{V}_{\hat{x}}^{[i]\tau} g(\hat{x}^{[i]}) \right) \kappa_c^{[i]}(y^{[i]} - h(\hat{x}^{[i]})) + \mathcal{V}_{\hat{x}}^{[i]\tau} \kappa_o^{[i]}(y^{[i]} - h(\hat{x}^{[i]})) \\
& + \epsilon_3 \left| x^{[i]} - \hat{x}^{[i]} \right|^2 + \epsilon_4 \left| \xi^{[i]} \right|^2 \\
\mu_i \;\geq\; & 0 \\
\mathcal{V}^{[i]} \;\geq\; & -\epsilon_1 \left| x^{[i]} - \hat{x}^{[i]} \right|^2, \quad \mathcal{V}^{[i]} \leq -\epsilon_2 \left| x^{[i]} - \hat{x}^{[i]} \right|^2 \\
P \;\succeq\; & 0
\end{aligned}$$

where

- $\mathcal{N}$ is a positive integer satisfying $\mathcal{N} \gg N$ where $N = \frac{1}{2}\mathcal{P}_n(L) \times (\mathcal{P}_n(L)+1) + \mathcal{R}(L) \times (k+n)$.
- $z(x,\hat{x}) = \mathcal{M}_{3n}^L(\mathrm{vec}(x-\hat{x},x,\hat{x}))$;
- $\mathcal{V}(P;x,\hat{x}) = z^\tau(x,\hat{x})Pz(x,\hat{x})$, $\mathcal{V}^{[i]} = \mathcal{V}(P;x^{[i]},\hat{x}^{[i]})$, $\mathcal{V}_x^{[i]} = \mathcal{V}_x(P;x^{[i]},\hat{x}^{[i]})$, etc.
- $\kappa_o(\theta_o;Y) = \sum_{j=1}^{\mathcal{R}(L)} \theta_o^{(j)}\phi_j(Y),\ \kappa_c(\theta_c;Y) = \sum_{j=1}^{\mathcal{R}(L)} \theta_c^{(j)}\phi_j(Y),\ \kappa_o^{[i]} = \sum_{j=1}^{\mathcal{R}(L)} \theta_o^{(j)}\phi_j(Y^{[i]}),\ \kappa_c^{[i]} = \sum_{j=1}^{\mathcal{R}(L)} \theta_c^{(j)}\phi_j(Y^{[i]})$.
- $(x^{[i]},\hat{x}^{[i]})$ is a collection of $\mathcal{N}$ random pairs generated according to a uniform random distribution in $\mathcal{X}$; similarly, $Y^{[i]}$ is a collection of $\mathcal{N}$ random vectors generated according to a uniform random distribution in Ψ and $\xi^{[i]}$ the corresponding sensor noise vectors.
- $\bar{\mathcal{A}}(x^{[i]},\hat{x}^{[i]}) = \frac{1}{2}\mathrm{tr}\Big\{ \left(g(\hat{x}^{[i]})\kappa_c^{[i]} h_\xi(x^{[i]}) + \kappa_o^{[i]} h_\xi(x^{[i]}) \right)^\tau \mathcal{V}_{\hat{x}\hat{x}}^{[i]} \left(g(x^{\hat{[i]}})\kappa_c^{[i]} h_\xi(x^{[i]}) + \kappa_o^{[i]} h_\xi(x^{[i]}) \right) \Big\}$.
- $\theta_o = \mathrm{vec}(\theta_o^{(1)},\ldots,\theta_o^{(\mathcal{R}(L))}),\ \theta_c = \mathrm{vec}(\theta_c^{(1)},\ldots,\theta_c^{(\mathcal{R}(L))})$).

In the case of PS estimator design, the first constraint in (2.12) is replaced by the following constraint:

$$\begin{aligned}
\mu_i \;\geq\; & \max_{u:|u|\leq u^{\max}} \Big\{ \mathcal{V}_x^{[i]\tau} f(x^{[i]}) + \mathcal{V}_{\hat{x}}^{[i]\tau} f(\hat{x}^{[i]}) + \frac{1}{2}\mathrm{tr}\Big\{ g_\omega^\tau(x^{[i]}) \mathcal{V}_{xx}^{[i]} g_\omega(x^{[i]}) \Big\} + \bar{\mathcal{A}}^{PS}(x^{[i]},\hat{x}^{[i]}) \\
& + \left(\mathcal{V}_x^{[i]\tau} g(x^{[i]}) + \mathcal{V}_{\hat{x}}^{[i]\tau} g(\hat{x}^{[i]}) \right) u + \mathcal{V}_{\hat{x}}^{[i]\tau} \kappa_o^{[i]}(y^{[i]} - h(\hat{x}^{[i]})) + \epsilon_3 \left| x^{[i]} - \hat{x}^{[i]} \right|^2 + \epsilon_4 \left| \xi^{[i]} \right|^2 \Big\}
\end{aligned} \tag{2.13}$$

where $\bar{\mathcal{A}}^{PS}(x^{[i]},\hat{x}^{[i]}) = \frac{1}{2}\mathrm{tr}\Big\{ \left(\kappa_o^{[i]} h_\xi(x^{[i]}) \right)^\tau \mathcal{V}_{\hat{x}\hat{x}}^{[i]} \left(\kappa_o^{[i]} h_\xi(x^{[i]}) \right) \Big\}$.

ity 1:

$$\begin{aligned}
\mathcal{E}(t) \;\leq\; & \frac{\lambda_2 + \mathcal{O}(1/L)}{(\lambda_1 - \mathcal{O}(1/L))} e^{-\frac{\lambda_3}{\lambda_2 + \mathcal{O}(1/L)}} \left[|x(0) - \hat{x}(0)|^2 \right] \\
& + \frac{\lambda_2 + \mathcal{O}(1/L)}{\lambda_3 \left(\lambda_1 - \mathcal{O}(1/L) \right)} \left\{ \lambda_4 + \mathcal{O}(\varepsilon) + \mathcal{O}(1/L) \right\}
\end{aligned} \tag{2.14}$$

In simple words, Theorem 1 states that if the user-defined constants ϵ_i are appropriately selected and the user-defined integer L is sufficiently large, then the proposed estimator can approximate – with arbitrary accuracy – the performance of an optimal AS estimator [minimizing $\mathcal{J}$ in (2.3) for some matrices R, R_o, Q]. Note that the integer L determines both the size of the monomial z used in the optimization problem (2.12) and the number $\mathcal{R}(L)$ of estimator's switching elements. Several comments are in order:

• Note that the number $\mathcal{R}(L)$ of estimator's switching elements has a negligible effect on the computational requirements of the proposed scheme. Similarly, the dimension of the "monomial-like" vector $z(x,\hat{x})$, although it increases exponentially with the L, it does not have any effect on the computational requirements of the proposed scheme as the vector $z(x,\hat{x})$ is employed only in the *off line* computations used to solve the optimization problem (2.12).

• As already noticed above, the solution of the – convex but still quite computationally demanding – optimization problem (2.12) is constructed off-line. This is quite similar to existing PS or AS approaches employing reinforcement learning (or neuro-dynamic programming), particle filters and their extensions/variants. However, there are two fundamental differences between the existing approaches and the proposed one: (a) extensive, carefully-designed and, sometimes, quite elaborate simulations of the overall multi-robot/exogenous environment system are required in the case of existing approaches, while, for the case of the proposed approach, simulation of the measurement model (for the generation of the signals $Y^{[i]}$) is only required; (b) due to the fact that the proposed method is based on Lyapunov-stability and standard control-based principles, the robustness of the proposed scheme (due to e.g. uncertainties in the functions f, g, g_ω, h, h_ξ) can be easily guaranteed and analyzed, something that it is not possible, in general, for the existing approaches. As a matter of fact, inequality (2.14) can be used in a similar fashion as in simple first order closedloop linear systems in order to choose the constants ϵ_i so that robustness is guaranteed.

• Theorem 1 provides no constructive method for the choice of the constants ϵ_i. As a result, different choices for these constants may have to be tried and the resulting AS estimator should be analyzed using (2.14): by noticing that the term $\mathcal{O}(\varepsilon) + \mathcal{O}(1/L)$ in (2.14) can be conservatively estimated – by using the solution of the optimization problem (2.12) – according to $\mathcal{O}(\varepsilon) + \mathcal{O}(1/L) \approx \max_{i=1}^{N} \mu_i$, we can replace in inequality (2.14) the term $\mathcal{O}(\varepsilon) + \mathcal{O}(1/L)$ by $\max_{i=1}^{N} \mu_i$ in order to estimate the effect of a particular choice for the constants ϵ_i to the AS estimator's efficiency.

III. Passive Estimator Design

The results reported in the previous section for the case of AS estimator design can be easily modified to cope with the case of PS estimator design. Since, in the later case the control input $u(t)$ is determined by a process external to the estimation process, special attention has to be paid in order to deal with such a problem: the optimal control design in the PS case must be performed over all possible choices of *admissible* control inputs. Without loss of generality, we will assume that the class of admissible control inputs is defined according to $|u(t)| \leq u^{\max}$, for some known positive constant $u^{\max}$; other, more general or different cases for the class of admissible control inputs can be dealt similarly. Then, the optimal "control" problem (2.3) is modified for the PS case according to

$$\min_{(u_o(s), s\in[0,\infty))} \mathcal{J}^{(PS)} \tag{3.1}$$

$$\mathcal{J}^{(PS)} = \sup_{\substack{u(s),\, s\in[0,\infty) \\ |u(s)| < u^{\max}}} E\left[\int_0^\infty \left(|u_o(s)|^2_{R_o} + |x(s) - \hat{x}(s)|^2_Q\right) ds\right]$$

Working similarly to Proposition 1 and Theorem 1, we can establish the following result.

Theorem 2: Proposition 1 and Theorem 1 hold with the following modifications:

(a) $\mathcal{J}$ in (2.3) is replaced by $\mathcal{J}^{(PS)}$ defined in (3.1) and the constants $\lambda_i, i = 1, 2, 3, 4$ depend only on the matrices R_o, Q (since there is no matrix R in the PS case).

(b) The first constraint in the optimization problem (2.12) is replaced by the constraint (2.13) [cf. Table I].

IV. Extensions: Incorporating Constraints

The proposed approach can be straighfowardly modified in order to incorporate various types of constraints such as communication, obstacle avoidance, maximum speed constraints, etc. The reader is referred to the full version of the paper [9] for more details.

V. Simulations

The interested reader can download the full-version of the paper [9], where a detailed description and evaluation of the simulation experiments are presented.

VI. Conclusions

In this paper, a unified methodology for treating a large class of passive and active multi-robot sensing (estimation) applications has been proposed and analyzed. The main advantage of the proposed methodology is that it provides stable, convergent and efficient estimation performance.

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