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Computing Clothoid Segments for Trajectory Generation

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Abstract—**A new, simple, and fast method to compute the sharpness and curvature of a clothoid segment of a continuous curvature path is presented. When generating continuous curvature paths, clothoid segments are needed to connect straight segments to perfect arc segments. This algorithm is part of a new trajectory generator and motion planner that generates smooth, natural, drivable paths, using a minimal amount of steering to reach the desired ending position similar to the way a human would drive.**

Keywords—**Motion planning, trajectory generation, autonomous vehicle, mobile robot, nonholonomic system, clothoid.**

I. Introduction

Motion planning is the computation of the open loop controls to steer a mobile robot or autonomous vehicle from a specified initial state to a specified final state over a certain time interval [1]. Motion planning can also be called trajectory generation, since one result of motion planning can be a sequence of connected trajectory segments that include steering controls (curvature and sharpness) as well as velocity controls (velocity and acceleration/deceleration) embedded in the data structure of the trajectory.

Dubins showed that a length-optimal path connecting an initial state and a final state of a mobile robot consists of only straight line segments and circular arc segments of maximal allowable curvature [2]. Reeds and Shepp gave a complete characterization of possible shortest paths [3]. However, these paths are not continuous curvature. Since curvature is related to the position of the steering wheel, a discontinuity in curvature means that the steering wheel position must change instantaneously, which is not feasible.

Paths to be followed by an autonomous vehicle must have continuous position, heading, and curvature at all points along the planned path in order to be feasible. Fraichard and Sheuer showed that under the continuous curvature assumption that near minimal distance paths consist of straight lines, circular arc segments of maximum allowable curvature, and clothoid curves of maximum allowable sharpness [4]. Clothoids are used to connect the straight-line segments to the circular arc segments and are drivable because the steering wheel turns in a continuous motion. However, the paths planned by Fraichard and Sheuer always turn the steering wheel at the maximum allowable rate until the minimum turning radius is reached and thus all turns are performed as tight as possible.

As seen above, previous studies have only considered the planning of minimum or near minimum distance paths, which require turning at a maximal sharpness and curvature. Such driving is not natural, and at high speeds, may even be unsafe. We have been considering the generation of continuous curvature paths that use a minimal amount of steering to reach the desired ending position similar to the way a human would drive. Our motivation is to generate a smooth, natural, and easily drivable path between two points. Instead of constructing paths from maximal curvature circular arcs, maximal sharpness clothoids, and line segments, we are investigating a new method for constructing continuous curvature trajectories where the curvature and sharpness are the smallest values needed to achieve a given change of heading over a given distance traveled. This paper presents our solution to the important sub-problem of finding the clothoid trajectory on which a vehicle starting at zero curvature will travel a given certain forward distance while changing its heading a certain amount.

II. Definitions

Curvature, $\kappa(s)$, is defined to be $d\delta/ds$, the change of vehicle heading (also called *deflection*) δ (in radians) with respect to distance s traveled on the curve (in meters). Positive curvatures denote left turns and negative curvatures denote right turns. Turning radius is the reciprocal of curvature. *Sharpness* α, is defined to be $d\kappa/ds$, the change of curvature κ (in radians/meter) with respect to distance s traveled on the curve (in meters).

A clothoid of sharpness α is a curve whose curvature is proportional to the distance s traveled along the curve measured from the origin, as given by the equation

$$\kappa(s) = \alpha s \cdot \tag{1}$$

This property makes the clothoid useful as a transition curve on a path between a straight line and a circular arc because the curvature is continuous and a vehicle following the curve at constant speed will have a constant rate of angular acceleration.

Curvature diagrams plot the curvature, $\kappa(s)$, at a point on a curve versus the distance, s, $0 \le s \le L$, traveled from the origin along that curve, as shown in Fig. 1b and d for clothoid segments. The mathematical relations shown in (2) between curvature κ, arc position s, sharpness α, and deflection δ, may all be derived from the geometry of the triangular curvature diagrams in Fig. 1b and d.

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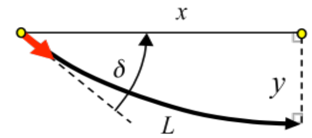
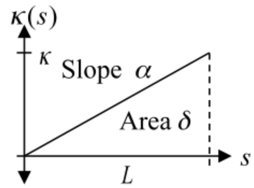
$$\kappa = \alpha s = \frac{2\delta}{s} = \sqrt{2\delta\alpha} \qquad \delta = \frac{\kappa s}{2} = \frac{\alpha s^2}{2} = \frac{\kappa^2}{2\alpha} \tag{2}$$

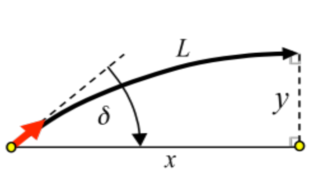
$$\alpha = \frac{\kappa}{s} = \frac{2\delta}{s^2} = \frac{\kappa^2}{2\delta} \qquad s = \frac{\kappa}{\alpha} = \frac{2\delta}{\kappa} = \sqrt{\frac{2\delta}{\alpha}}$$



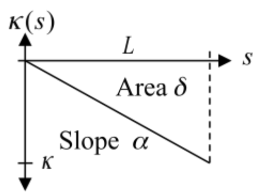
a. Clothoid Trajectory (left turning, δ>0)



b. Curvature Diagram for left turning path shown in (a.)



c. Clothoid Trajectory (right turning, δ<0)



d. Curvature Diagram for left turning path shown in (c.)

Fig. 1 Left and right turning clothoid segments

III. Computing a Clothoid Segment

We have been investigating a new method for constructing continuous curvature trajectories that start at zero curvature and that have the minimal max-curvature and minimal sharpness needed to achieve a given change of heading for a given forward distance traveled. The basis for our new algorithm is stated in the following theorem that develops a type of clothoid trigonometry for these curves and is used to compute the forward distance x and the lateral distance y traveled (see Fig. 1a,c).

Theorem for clothoid segments

When traveling on a clothoid segment that starts at zero curvature and deflects an angle of δ, these statements are true (refer to Fig. 1a and c for an illustration):

1. The following distances scale proportionally to the inverse of the magnitude of the ending curvature κ:
   the length L of the trajectory,
   the forward distance x traveled, and
   the lateral distance y traveled.
2. The pair wise ratios of these three distances are constants for a given angle δ.

<u>Proof</u>:

Referring to Fig. 2b, the Cartesian position of a point at distance s, from the origin along a clothoid (Euler spiral) of non-zero sharpness α is given parametrically [5] by:

$$x(s) = \sqrt{\frac{\pi}{|\alpha|}}\, C\!\left(\frac{\alpha s}{\sqrt{\pi|\alpha|}}\right), \quad y(s) = \sqrt{\frac{\pi}{|\alpha|}}\, S\!\left(\frac{\alpha s}{\sqrt{\pi|\alpha|}}\right). \tag{3}$$

where C(s) and S(s) are the Fresnel cosine and sine functions respectively, they being defined [7 (page 300, 7.3.1 and 7.3.2)] as:

$$C(s) = \int_0^s \cos\!\left(\frac{\pi}{2}u^2\right)du, \quad S(s) = \int_0^s \sin\!\left(\frac{\pi}{2}u^2\right)du \cdot \tag{4}$$

By adding a $sign(\alpha)$ factor to the x equation in (3), the new equation works for both the cases of forward left turns (Fig. 2b, $\alpha > 0$) and forward right turns (Fig. 2d, $\alpha < 0$) :

$$x(s) = sign(\alpha)\sqrt{\frac{\pi}{|\alpha|}}\, C\!\left(\frac{\alpha s}{\sqrt{\pi|\alpha|}}\right), \quad y(s) = \sqrt{\frac{\pi}{|\alpha|}}\, S\!\left(\frac{\alpha s}{\sqrt{\pi|\alpha|}}\right) \tag{5}$$

Equations (5) are stated on in terms of curve position s and sharpness, α. However, the motion planner prefers to work in terms of the total deflection δ and maximum curvature κ at the end of the curve where $s = L$. Using (2), sharpness α and length s parameters are substituted with deflection δ and curvature κ as shown in (6), (7), and (8) below.

$$\sqrt{\frac{\pi}{|\alpha|}} = \left\{\begin{matrix} \alpha > 0 : \sqrt{\dfrac{\pi}{\alpha}} = \dfrac{\sqrt{2\pi\delta}}{\kappa} \\ \alpha < 0 : \sqrt{\dfrac{\pi}{-\alpha}} = \dfrac{\sqrt{2\pi(-\delta)}}{-\kappa} \end{matrix}\right\} = \frac{\sqrt{2\pi|\delta|}}{|\kappa|} \tag{6}$$

$$sign(\alpha) = \frac{\alpha}{|\alpha|} = sign(\kappa) = \frac{\kappa}{|\kappa|} = sign(\delta) = \frac{\delta}{|\delta|} \tag{7}$$

Substituting out sharpness α, and the distance s traveled along the arc (5), the Cartesian position is rewritten as:

$$x\!\left(\frac{2\delta}{\kappa}\right) = sign(\delta)\frac{\sqrt{2\pi|\delta|}}{|\kappa|}\, C\!\left(\sqrt{\frac{2|\delta|}{\pi}}\right), \quad y\!\left(\frac{2\delta}{\kappa}\right) = \frac{\sqrt{2\pi|\delta|}}{|\kappa|}\, S\!\left(\sqrt{\frac{2|\delta|}{\pi}}\right). \tag{8}$$

An important and key observation here is that the curvature κ determines the size of the clothoid while the change of heading δ determines the shape of the clothoid. Therefore, we can factor $1/|\kappa|$ out of (8) and use it to scale the size of the clothoid without altering its shape:

$$x\!\left(\frac{2\delta}{\kappa}\right) = \frac{1}{|\kappa|}x(2\delta) \qquad y\!\left(\frac{2\delta}{\kappa}\right) = \frac{1}{|\kappa|}y(2\delta) \cdot \tag{9}$$

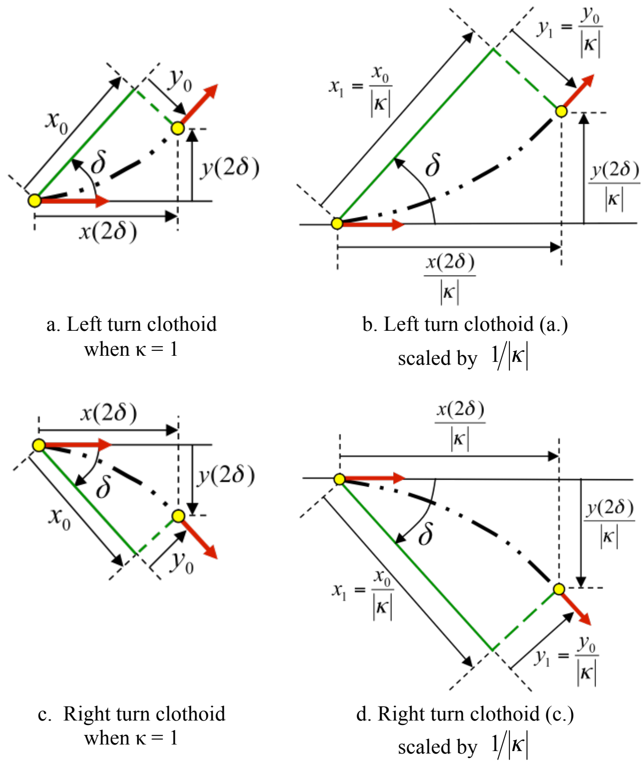
This is illustrated in Fig. 2. The shape of the clothoid is determined independent of curvature (setting curvature to a constant 1) by:

$$x(2\delta) = sign(\delta)\sqrt{2\pi|\delta|}\, C\!\left(\sqrt{\frac{2|\delta|}{\pi}}\right), \quad y(2\delta) = \sqrt{2\pi|\delta|}\, S\!\left(\sqrt{\frac{2|\delta|}{\pi}}\right). \tag{10}$$

Fig. 2 shows the similar clothoids. The clothoid in Fig. 2 a. (also c.) is produced from (10) and the clothoid in Fig. 2 b. (also d.) is produced from (8). Equations (9) describe the similarity relationship between the clothoids a. and b. (also c. and d.). Because the clothoids in Fig. 2 are similar, they share the same angles, including the same initial and final headings and their difference, the deflection angle δ. However, the distances x, and y, as well as the length of the curve are all scaled linearly by $1/|\kappa|$.

To find the forward and lateral distances traveled, shown as x and y respectively in Fig. 1a. (also c.), the clothoids in Fig. 2a. (also c.) must be scaled to the right size as in Fig. 2b. (also d.) and rotated by $-\delta$ to the standard position shown in Fig. 1a. (also c.). In summary:

1. Compute $x(2\delta)$ and $y(2\delta)$, which are only functions of δ.
2. Rotate $x(2\delta)$ and $y(2\delta)$ by $-\delta$ to put in standard position (like in Fig. 1).
3. Scale by the $1/|\kappa|$ to the correct size.



a. Left turn clothoid when κ = 1

b. Left turn clothoid (a.) scaled by $1/|\kappa|$

c. Right turn clothoid when κ = 1

d. Right turn clothoid (c.) scaled by $1/|\kappa|$

Fig. 2 Clothoid Scaling

The resulting forward and lateral distances, x_1 and y_1 respectively, are computed in (11).

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \frac{1}{|\kappa|}\begin{bmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{bmatrix}\begin{bmatrix} x(2\delta) \\ y(2\delta) \end{bmatrix}$$

$$= \frac{\sqrt{2\pi|\delta|}}{|\kappa|}\begin{bmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{bmatrix}\begin{bmatrix} \operatorname{sign}\delta \cdot C\left(\sqrt{\frac{2|\delta|}{\pi}}\right) \\ S\left(\sqrt{\frac{2|\delta|}{\pi}}\right) \end{bmatrix} \tag{11}$$

$$= \frac{\sqrt{2\pi|\delta|}}{|\kappa|}\begin{bmatrix} \operatorname{sign}\delta\cdot\cos\delta\cdot C\left(\sqrt{\frac{2|\delta|}{\pi}}\right) + \sin\delta\cdot S\left(\sqrt{\frac{2|\delta|}{\pi}}\right) \\ -\operatorname{sign}\delta\cdot\sin\delta\cdot C\left(\sqrt{\frac{2|\delta|}{\pi}}\right) + \cos\delta\cdot S\left(\sqrt{\frac{2|\delta|}{\pi}}\right) \end{bmatrix}$$

The arc length L is similarly a function of curvature and deflection (from (2) and letting $s = L$). Since κ and δ are always the same sign, L is always positive:

$$L = 2\delta/\kappa\,. \tag{12}$$

From (11) and (12), we can see that x_1 and y_1 and L are all inversely proportional to $|\kappa|$ (statement 1 of the theorem). Also, we can see that the ratio of any two of these distances is only a function of the deflection angle δ (statement 2 of the theorem). This completes the proof.

<u>End of proof</u>.

This theorem implies a kind of trigonometric relationship between the distances associated with clothoid trajectories. The "triangles" are shown in Fig. 1a and c. The "hypotenuse" of the triangle is the curve. Interestingly, both the curve and the forward distance x form right angles to the lateral distance y in these triangles (double right triangles!). By taking ratios of x_1 and y_1 (as given in (11)) to L (as given in (12)), the "sine" and "cosine" functions for the clothoid triangles in Fig. 1 are defined as:

$$\sin_C\delta = \frac{y_1}{L} = \begin{cases} \left(-\sin\delta\cdot C(\eta) + \cos\delta\cdot S(\eta)\right)/\eta\,, & \delta > 0 \\ 0, & \delta = 0 \\ -\sin_C(-\delta), & \delta < 0 \end{cases}$$

$$\cos_C\delta = \frac{x_1}{L} = \begin{cases} \left(\cos\delta\cdot C(\eta) + \sin\delta\cdot S(\eta)\right)/\eta, & \delta > 0 \\ 1\,, & \delta = 0 \\ \cos_C(-\delta), & \delta < 0 \end{cases} \tag{13}$$

where $\eta = \sqrt{2|\delta|/\pi}$, and their derivatives are :

$$\frac{d}{d\delta}\cos_C\delta = \sin_C\delta + \frac{(1-\cos_C\delta)}{\pi\eta^2} \qquad \frac{d}{d\delta}\sin_C\delta = -\cos_C\delta - \frac{\sin_C\delta}{\pi\eta^2}$$

These functions are plotted in Fig. 3. The clothoid cosine ratio can be used to quickly plan clothoid trajectory segments. These segments are the primitive pieces of the paths planned by our motion planner [8].

The method takes as inputs the forward distance x and the required deflection δ, and from these values, the exact clothoid for the segment can be determined. The clothoid is characterized by the total distance traveled on the curve $s = L$, the final curvature κ, and the sharpness of the clothoid α. These can all be easily computed using (14) (which is derived from (13), (12) and (2)):

$$L = \frac{x}{\cos_C\delta} \qquad \kappa = \frac{2\delta}{L} \qquad \alpha = \frac{\kappa}{L} \tag{14}$$

Equations (14) are the basis of a new, simple, and fast method to compute the sharpness and curvature of a clothoid segment of a continuous curvature path. Of course, the speed of this algorithm presupposes that you can quickly compute $\cos_C\delta$. This can be done, either using a table lookup technique, or using the technique presented in this next section.

IV. Computing the Clothoid Sine and Cosine

A common method of computing the Fresnel sine and cosine (4) is to use two auxiliary functions f and g [7 (page 301, 7.3.9 and 7.3.10)] shown as follows:

$$C(x) = \frac{1}{2} + f(x)\sin\left(\frac{\pi}{2}x^2\right) - g(x)\cos\left(\frac{\pi}{2}x^2\right)$$
$$S(x) = \frac{1}{2} - f(x)\cos\left(\frac{\pi}{2}x^2\right) - g(x)\sin\left(\frac{\pi}{2}x^2\right) \tag{15}$$

where auxiliary functions f and g [7 (page 300, 7.3.5 and 7.3.6)] are defined as:

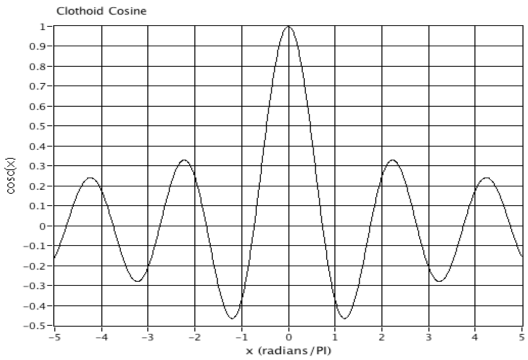


Fig 3a. Clothoid-sine from -5π to 5π

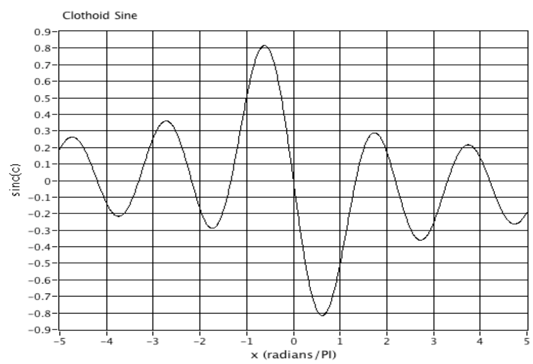


Fig 3b. Clothoid-cosine from -5π to 5π

$$f(x) = \left(\frac{1}{2} - S(x)\right)\cos\left(\frac{\pi}{2}x^2\right) - \left(\frac{1}{2} - C(x)\right)\sin\left(\frac{\pi}{2}x^2\right)$$
$$g(x) = \left(\frac{1}{2} - C(x)\right)\cos\left(\frac{\pi}{2}x^2\right) + \left(\frac{1}{2} - S(x)\right)\sin\left(\frac{\pi}{2}x^2\right) \quad (16)$$

and their derivatives [7 (page 301, 7.3.21)] are:

$f'(x) = -\pi\, x\, g(x) \qquad g'(x) = \pi\, x\, f(x) - 1$.

This approach is attractive because the auxiliary functions are well behaved and do not oscillate. There also exist several good approximations for them. For example, the rational approximations (17) compute *f* and *g* with less than 0.002 of absolute error [7 (page 302, 7.3.32 and 7.3.33)].

$$f(x) = \frac{1 + 0.926x}{2 + 1.792x + 3.104x^2}$$
$$g(x) = \frac{1}{2 + 4.142x + 3.492x^2 + 6.670x^3} \quad (17)$$

We can also use auxiliary functions *f* and *g* to compute our clothoid sine and cosine functions by substituting the Fresnel sine and cosine defined by (15) into (13). The resulting equations simplify nicely to:

$$\cos_C(\delta) = \frac{1}{\eta}\left(\frac{\cos\delta + \sin\delta}{2} - g(\eta)\right), \quad \delta > 0$$
$$\sin_C(\delta) = \frac{1}{\eta}\left(\frac{\cos\delta - \sin\delta}{2} - f(\eta)\right), \quad \delta > 0 \quad (18)$$

where $\eta = \sqrt{2|\delta|/\pi}$

Equations (18) are a fast, accurate, and convenient way of computing the clothoid sine and cosine in terms of auxiliary functions *f* and *g*. They are much faster than computing $\cos_C$ and $\sin_C$ using (13).

V. Computing a Clothoid Segment and Arc Segment

It is possible that the clothoid computed for a given forward distance x and deflection δ has a final curvature κ (computed by (14)) whose magnitude exceeds the maximum curvature κ_{MAX} the vehicle can drive (Fig. 4a,b). When this physical limitation is exceeded, the clothoid segment must be replaced with a clothoid and circular arc combination segment (Fig. 4c,d) that respects the κ_{MAX} limitation. The deflection of the clothoid part of the segment is δ_C and the deflection of the circular arc part is λ. The combined segment must satisfy the same input and continuity constraints as the original clothoid curve.

1. The total deflection angle must match the input (Fig 4d)
$$\delta = \delta_C + \lambda \quad (19)$$
2. The forward distance traveled must match the input (as shown in Fig 4c for right turns)
$$x = x_C \cos\lambda + (r + y_C)\sin\lambda \quad (20)$$
3. The curvature between the clothoid and circular arc must be continuous (Fig 4d for right turns).
$$\kappa_C = \frac{1}{r} = sign(\delta)\kappa_{MAX} \quad (21)$$

When solving a clothoid and circular arc combination segment, the arc deflection λ is the parameter that determines how much deflection is accomplished by the each of the two parts of the segment. Solving for λ is sufficient to determine both parts of the curve.

From (19) and (21), given the inputs δ and κ_{MAX}, and assuming λ has been found, the features of the clothoid part of the path can be easily determined as follows:

$$L_C = 2(\delta - \lambda)/\kappa_C \qquad \delta_C = \delta - \lambda \qquad \alpha_C = \kappa_C / L_C \quad (22)$$

To compute λ, we start by using (13) to write

$$x_C = L_C \cdot \cos_C(\delta - \lambda) \qquad y_C = L_C \cdot \sin_C(\delta - \lambda). \quad (23)$$

Then from (19), (20), (21) and (23), the following single relation can be derived. A residual function z is defined whose root is at λ:

$$2(\delta - \lambda)\big(\cos_C(\delta - \lambda)\cos\lambda + \sin_C(\delta - \lambda)\sin\lambda\big) + \sin\lambda - x\kappa_C = z(\lambda) = 0 \quad (24)$$

This equation for z has a single unknown variable λ for which to solve. Unfortunately, a closed form solution is not known and numerical methods must be used. This equation is well behaved and Newton's iterative solver works quickly to find λ. The Newton recurrence that converges to λ is:

$$\lambda_{k+1} = \lambda_k + \frac{z(\lambda_k)}{z'(\lambda_k)} = 3\lambda_k - 2\delta + \frac{x\kappa_C - \sin\lambda_k}{\cos_C(\delta - \lambda_k)\cos\lambda_k + \sin_C(\delta - \lambda_k)\sin\lambda_k} \quad (25)$$

The arc angle λ is between 0 (the case of all clothoid and no arc when $|\kappa_C| \le \kappa_{MAX}$) and δ (the case of no clothoid and all arc when $|\sin\delta| = x\kappa_{MAX}$). The bounds on λ are:

$$\delta > 0: \quad 0 \le \lambda \le \delta$$
$$\delta < 0: \quad \delta \le \lambda \le 0 \quad (26)$$

When $|\sin\delta| > x\kappa_{MAX}$ then the maneuver is not possible at all with a simple clothoid-arc path. Any λ in the range 0 to δ can be used to initialize the recurrence in Eqn. (25), for instance $\lambda_0 = \delta/2$ is a good choice. For all cases I have seen, z is nearly linear in the range 0 to δ and thus the recurrence converges quickly.

VI. Example Application: The Lane Change

To demonstrate how this method of computing a clothoid can be used to plan a more complex path, imagine planning a simple left lane change maneuver as shown in Fig. 5. Lets assume that the lanes are 4 meters wide, and you have a sufficient distance of 50 meters of straight road in which to accomplish the lane change. If possible, you don't want to slow down, so you want to perform the maneuver using a minimal amount of steering. The vehicle starts out heading down the right lane with zero curvature (the steering wheel is straight). The planned path shown in Fig. 5 contains four primitive clothoid (or clothoid-arc) segments. (It is not the purpose of this paper to discuss how to plan such paths.) Summing the individual deflections of the primitive segments along the path, the total deflection (change of heading) at the end of the path is $\delta + \delta - \delta - \delta = 0$, and thus the vehicle will be traveling in the same direction at the end of the maneuver as it was traveling at the start.

To solve for the clothoid parameters, we notice that the distance x (the forward distance for each primitive clothoid) is repeated four times along the diagonal, thus $x = \sqrt{50^2 + 4^2}/4 = 12.54$ meters. Also the deflection angle for all four primitive clothoids can be computed as $\delta = \arctan\left(\frac{4}{50}\right) = 0.07983$ radians. Assume that the maximum curvature for the vehicle is 0.2 radians/meter, which is typical for full size cars. Using equation (14) that was developed in this paper, the following parameters can be computed which completely characterize the primitive clothoid used in this maneuver:

$\cos_C \delta = \cos_C 0.07983 = 0.9983$,

$L = x/\cos_C \delta = 12.54/\cos_C 0{,}07983 = 12.5613$ meters,

$\kappa = 2\delta/L = 0.0127104$ radians per meter, and

$\alpha = \kappa/L = 0.00101187$ radians per meter^2.

Since $|\kappa| \le \kappa_{MAX} = 0.2$, a simple clothoid segment is sufficient, and a clothoid-arc is not needed. The fact that the maximum curvature κ is only 0.0127 is good news because it means that the steering wheel only turns very little to perform the maneuver.

This same clothoid is repeated four times to perform the entire maneuver. Clothoids are paired in such a way as to zero out the curvature (steering wheel in the straight position) anytime the vehicle crosses the diagonal: at the start, in the middle, and at the end. The final position of the vehicle relative to the start is forward 50 meters, lateral 4 meters to the left, no change of heading, and zero curvature.

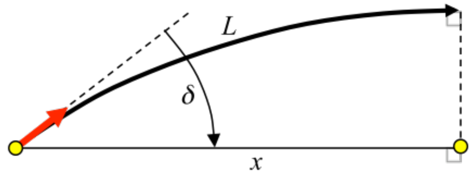


Fig 4a. A clothoid segment that exceeds κ_{max}

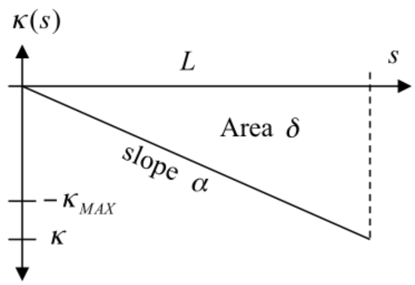


Fig 4b. The curvature diagram for 4a

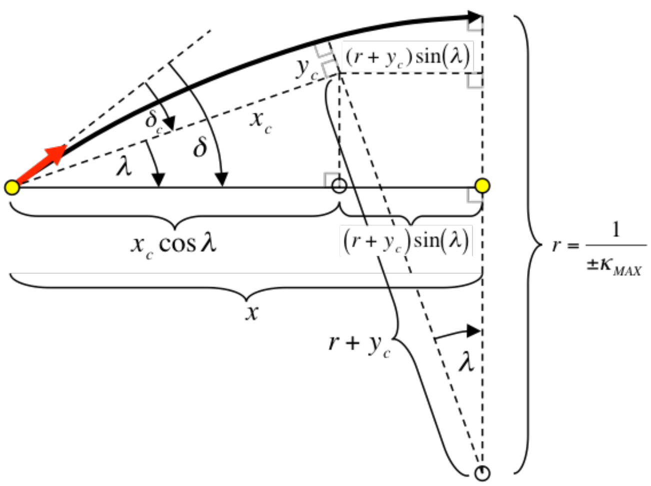


Fig 4c. A clothoid-arc combination segment that does not exceed κ_{max}

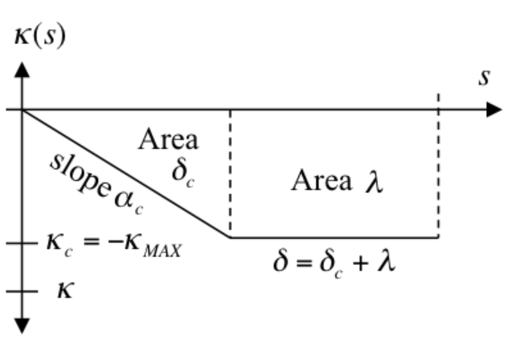


Fig 4d. The curvature diagram for 4c

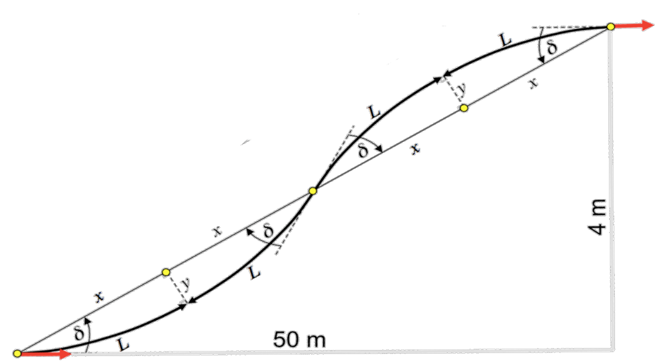


Fig. 5. A Lane Change Maneuver (not to scale)

VII. CONCLUSION

This paper has presented a new method for computing a clothoid with the least sharpness (and thus least maximum curvature) that deflects a given angle while traveling forward a given distance. The paper has also discussed how to compute a combination clothoid-arc segment if the maximum curvature is exceeded. The clothoid and clothoid-arc segments, along with the straight-line segment, form the set of primitive curves that can be combined in various combinations to form a general path. The computational methods discussed in this paper are an important part of a new motion planner [8] that is being developed for generating smooth and easily drivable trajectories for autonomous vehicles. These paths are an improvement over previously reported continuous curvature paths in that they use a minimal amount of steering (sharpness and curvature) that is needed to accomplish a specified maneuver, thus producing “smooth and easily drivable” paths. The author is continuing his work on how to best break down more general paths into these primitive segments.

VIII. ACKNOWLEDGEMENTS

This work is a continuation of work started with my friend and former graduate student Joshua Henrie. I am also grateful to the three reviewers who all gave me excellent feedback. I have tried to incorporate all of their suggestions. The paper was much improved because of their help.

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