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Passive vs. Aggressive Strategies: A Game Theoretic Analysis of Military Defense

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Abstract— **This paper describes a basic passive vs. aggressive defense model, and analyzes it in terms of defense strategies against an intelligent enemy. In response to varying combinations of passive and aggressive defense, we assume that the enemy can up- or down-regulate recruitment activity. This leads to a differential game formulation of battle scenarios that we analyze for a warfare situation. Specifically, we consider military counterterrorist activities in a civilian population. Simulation results, including uncertainty and sensitivity analyses, are provided to demonstrate the benefits and limitations of the proposed model in terms of understanding army defense plans.**

I. Introduction

FOR many different warfare scenarios, defense tactics may be classified into two general but distinct categories: passive strategies and aggressive strategies. Given that both of these strategies can be used simultaneously, overall defense operations may be further characterized by the particular balance of passive and aggressive strategies employed. This, of course, begs the question – 'what is the optimal combination of passive vs. aggressive defense?' Clearly, the answer to this question depends on how effective and how costly aggressive strategies are relative to their passive counterparts. Moreover, if the effectiveness or cost of a particular defense mode depends on actions taken by the opponent, the optimal combination of passive vs. aggressive defense will be inextricably linked with enemy operations as well. In general, we expect that aggressive strategies will be costly because they are necessarily associated with damage and destruction, while passive strategies, though less injurious, will also be less effective.

In this paper, we consider a differential game formulation that frames the trade-off between passive and aggressive defense tactics in mathematical terms. To the extent that our work is largely exploratory and aimed at defining general defense properties, we note that there are similarities between our approach and the model used in a recent paper by Caulkins and Grass [1]. In [1], the authors develop an optimal control formulation to consider the trade-off between 'fire' strategies (which eradicate insurgents in a comprehensive, nonselective manner) and 'water' strategies (which eradicate insurgents in an intelligence based selective manner). We focus on a different trade-off (passive strategies vs. aggressive strategies) that is applicable to a wider class of warfare scenarios including not only counterterrorism efforts, but also natural battlefield scenarios like immune defense. In addition, by including a fundamental aspect of enemy behavior, in this case recruitment strategy, we extend the optimal control framework used in [1] to a differential game framework. This allows us to consider optimal defense strategies in the face of an intelligent opponent. Finally, in light of the difficulties associated with parameterizing social models, we use a Latin Hypercube Sampling (LHS) scheme to perform an uncertainty/sensitivity analysis, thereby assessing the strengths of our general trend predictions, and also determining the parameters most likely to influence them.

While there are many different warfare scenarios that we might consider for model application, in this paper, we focus on military strategies against insurgents. We suggest that detailed mathematical modeling may help inspire new strategies for military operations.

Military scenario. In this paper, we consider defense against terrorist insurgent activity in a civilian population. Violent raids on the civilian population lie in direct opposition to military efforts aimed at maintaining the integrity of a country. As a result, counterterrorist activities must rely not only on aggressive eradication of enemy operatives, but also, on passive strategies which suppress growth of the insurgence movement. Passive strategies may include, but are not limited to, diplomacy efforts, open dialogue, infrastructure development, and negotiation. In other words, a passive strategy is any effort aimed at swaying civilian allegiance away from insurgent alignment and ideology without the use of violence or aggression. The time-dependent interplay between passive and aggressive tactics in military defense is paramount to the success or failure of any counterinsurgency military operation.

II. Dynamics Modeling

Our model assumes that the military has two broad categories of defense: passive strategies and aggressive strategies. Therefore we choose the effort devoted to passive strategies, denoted u_p, as the first military control variable and the effort devoted to aggressive strategies, denoted u_a, as the second. So far as the enemy is concerned, we assume that the

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insurgence movement attempts to counter military defense by regulating recruitment activities. Therefore the effort devoted to insurgence growth, denoted u_e, is the first and only enemy control variable in our differential game formulation. In order to define an objective function for our game, we assume that the military suffers a cost associated with the size of the enemy population, a cost associated with the effort devoted to passive tactics (risk, resources, etc.), a cost associated with the effort devoted to aggressive tactics (bystander damage, resources, etc.), and a benefit associated with forcing the enemy to increase growth efforts. We then assume that the game is zero-sum, thus the insurgency objective function takes on a benefit associated with insurgent population size, a benefit associated with increased military efforts (both passive and aggressive), and a cost associated with increased recruitment activities.

In terms of dynamics, we assume that the enemy grows at a per insurgent baseline rate r_0, and that this growth rate increases linearly with the enemy effort devoted to recruitment. The proportionality constant for the linear increase in growth rate with growth effort is denoted k_r. We additionally assume that military effort devoted to passive defense tactics causes a linear reduction in the insurgency growth rate. The proportionality constant for the decrease in growth rate with passive strategy effort is denoted k_p, and can be taken as reflective of the efficiency of passive defense. Finally, we assume that the military effort devoted to aggressive defense tactics causes a reduction in the enemy population that is proportional to both the military effort committed to aggression and the insurgent population size itself. The rate constant for insurgent reduction by aggressive means is denoted k_a, and can be taken as reflective of the efficiency of aggressive defense. Since the efficiency of violent attacks should improve significantly with increased intelligence, and since military intelligence is expected to depend on the level of insurgent recruitment[2, 3], we also assume that k_a is a linearly increasing function of enemy growth efforts, thus $k_a = k_{a0} + k_i u_e$, where k_{a0} is the rate of enemy eradication due to aggressive assault in the absence of military intelligence, while k_i is the increase in this rate per unit enemy effort devoted to growth. The above assumptions lead to the following model:

$$\max_{u_e} \min_{u_p, u_a} \int_0^{t_f} (x^2 + C_p u_p{}^2 + C_a u_a{}^2 - C_e u_e{}^2) dt \tag{1.a}$$

subject to

$$\dot{x} = [r_0 + k_r u_e - k_p u_p - (k_{a0} + k_i u_e) u_a] x$$
$$x(0) = x_0,\ x(t_f) = x_{end} \tag{1.b}$$

$$0 \le u_a \le u_{a,\max},\ 0 \le u_p \le u_{p,\max},$$
$$0 \le u_e \le u_{e,\max}, \quad C_p, C_a, C_e, x \ge 0 \tag{1.c}$$

where C_p is a weighting parameter which reflects the cost of passive defense strategies, C_a is a weighting parameter which reflects the cost of aggressive defense strategies and C_e is a weighting parameter which reflects the benefit (that is benefit to the military, but cost to the insurgency) of growth. All of these costs are taken relative the cost of an increased insurgent population, x. In equation (1), time dependence is suppressed for notational simplicity, however we point out that all of the control variables, u_p, u_a and u_e, as well as the state variable, x, are time-dependent.

In the above set of equations, (1.a) is the objective function for our game theoretic formulation, (1.b) defines the dynamics of the insurgent population from its initial value x_0 to its final value x_{end}, and (1.c) states additional social constraints on the effort devoted to passive or aggressive defense and the effort devoted to recruitment. In general, we will choose x_{end} such that the insurgent population is so low that it is unsustainable. We use this end point for our game since the dynamic equation (1.b) governing insurgent growth is such that $x(t) \to 0$ only as $t \to \infty$. Realistically, however, a insurgent population below a certain critical level will not be self-sustaining as a result of social factors not included in the model. Finally, we note that the problem, as formulated in equation (1), has a finite time horizon, thus the game ends at time $t = t_f$. For military applications, finite-time horizon models are appropriate since the military would, ideally, like to plan a defense strategy which guarantees effective pull-out on a particular, pre-determined date.

Before we continue, we must be more specific with respect to what is meant by a 'unit of passive effort', a 'unit of aggressive effort', and 'a unit of growth effort'. Let us begin by defining a unit of passive effort as the amount of effort which, if 100% effective, will reduce the per capita enemy growth rate by one enemy per military occupation period. Similarly, let us define a unit of aggressive effort as the amount of effort which, if 100% effective, will increase the per capita enemy death rate by one enemy per military occupation period. Finally, let us define a unit of growth effort as the amount of effort which, if 100% effective, will increase the enemy growth rate by one enemy per military occupation period.

To simplify the analysis of the game theoretic problem proposed in (1), we note that the various weighting parameters can be rescaled from costs per unit effort to costs per unit change in enemy growth, suppression or killing. To do this, we use the following substitutions

$$\begin{gathered} \tilde{u}_e = k_r u_e, \quad \tilde{u}_p = k_p u_p, \quad \tilde{u}_a = k_{a0} u_a, \quad k' = \frac{k_i}{k_{a0} k_r}, \\ \tilde{C}_p = \frac{C_p}{k_p^2}, \quad \tilde{C}_a = \frac{C_a}{k_{a0}^2}, \quad \tilde{C}_e = \frac{C_e}{k_r^2} \end{gathered} \tag{2}$$

The new differential game formulation is as follows

$$\max_{u_e} \min_{u_1, u_2} \int_0^{t_f} (x^2 + \tilde{C}_p \tilde{u}_p{}^2 + \tilde{C}_a \tilde{u}_a{}^2 - \tilde{C}_e \tilde{u}_e{}^2) dt \tag{3.a}$$

subject to

$$\dot{x} = [r_0 + \tilde{u}_e - \tilde{u}_p - (1 + k' \tilde{u}_e) \tilde{u}_a] x$$

$$x(0) = x_0,\, x(t_f) = x_{end} \tag{3.b}$$

$$0 \ge \tilde{u}_a \ge \tilde{u}_{a,\max},\; 0 \ge \tilde{u}_p \ge \tilde{u}_{p,\max}$$

$$0 \ge \tilde{u}_e \ge \tilde{u}_{e,\max},\quad \tilde{C}_p, \tilde{C}_a, \tilde{C}_e, x > 0 \tag{3.c}$$

We now have only two model parameters, r_0 and k', although there are still three weighting parameters, $\tilde{C}_p, \tilde{C}_a$ and $\tilde{C}_e$.

At this point, it is worthwhile to point out that our model predictions depend not only on the parameters r_0, k', $\tilde{C}_p, \tilde{C}_a$ and $\tilde{C}_e$, but also on the upper bounds that we choose in equation (3.c). In general, these bounds can be either fixed values, or else functions of the other control and/or state variables. For the current paper, we consider a scenario applicable to military contexts.

$$\tilde{u}_{a,\max} = \infty,\quad \tilde{u}_{e,\max} = \infty,\quad \tilde{u}_{p,\max} = \frac{M}{100}\tilde{u}_e \tag{4}$$

In other words, there is no extraneously imposed upper limit on aggressive military efforts or insurgent recruitment, however passive defense can, at most, slow recruitment activities by M%. For $M < 100$, the upper bound on $\tilde{u}_{p,\max}$ means that recruitment cannot be stopped entirely by passive tactics, nor can it be used to 'un-recruit' terrorists who have already committed to the insurgence movement. We believe that these assumptions are reflective of most counterterrorist operations.

It is impossible to solve the military differential game explicitly because of the highly nonlinear nature of equation (3.b). Therefore we resort to numerical methods. The basic technique used to predict the optimal military strategies u_a and u_p, the optimal insurgency strategy, u_e, and the insurgent population dynamics, x, is outlined in Section III.

Given that there are five unknown parameters in equations (3) and (4), and given that the exact characterization of these parameters is difficult as a result of limited social data and the approximate nature of our passive/aggressive strategy classification system, we analyze our model by considering its behavior over a wide region of the socially plausible parameter space. To do this, we use a stratified Monte Carlo sampling technique known as Latin Hypercube Sampling (LHS). In LHS, probability distributions are assigned to each of the parameters. The range of each parameter is then divided into N non-overalapping, equiprobable intervals, and a particular value from within each interval is chosen at random according to the probability distribution within that interval. The N randomly sampled values for the first parameter are then randomly paired with the N randomly sampled values for the second parameter. The N sampled pairs are then randomly paired with the N values for the third parameter, and so on, until an LHS table has been generated. An LHS table is an $N*K$ matrix, where K is the number of parameters. The details of the technique are described elsewhere[4]. Suffice it to say that the LHS scheme has proven to be an effective means by which to sample large parameter spaces efficiently [5, 6].

In addition to providing an uncertainty analysis, the LHS method can be used to calculate partial rank correlation coefficients (PRCC) for the different model parameters. This makes it possible to ascertain the potentially time-dependent impact of a particular parameter on a particular output measurement which, in this case, would be one of the various control variables, u_p, u_a and u_e or the state variable, x. Further discussion of LHS and PRCC calculation can be found in[5].

III. Control Strategies

We solve the optimal control problems in equations (3) and (4) using the Pontryagin Maximum Principal. We begin by setting up a Hamiltonian using equation (3)

$$\begin{aligned} H &= x^2 + \tilde{C}_p\tilde{u}_p{}^2 + \tilde{C}_a\tilde{u}_a{}^2 - \tilde{C}_e\tilde{u}_e{}^2 \\ &+ \lambda[r_0 + \tilde{u}_e - \tilde{u}_p - \left(1 + k'\tilde{u}_e\right)\tilde{u}_a]x \end{aligned} \tag{5}$$

where λ is the costate variable and is time dependent, although again, this has been suppressed for notational convenience. The condition that the various control variables maximize(minimize) the payoff function in (3.a) can be stated from the Hamiltonian as follows

$$\frac{\partial H}{\partial \tilde{u}_p} = 2\tilde{C}_p\tilde{u}_p - \lambda x = 0 \tag{6.a}$$

$$\frac{\partial H}{\partial \tilde{u}_a} = 2\tilde{C}_a\tilde{u}_a - \lambda\left(1 + k'\tilde{u}_e\right)x = 0 \tag{6.b}$$

$$\frac{\partial H}{\partial \tilde{u}_e} = -2\tilde{C}_e\tilde{u}_e + \lambda\left(1 - k'\tilde{u}_a\right)x = 0 \tag{6.c}$$

Minimization of the military control variables is guaranteed by $\partial^2 H/\partial\tilde{u}_p^2, \partial^2 H/\partial\tilde{u}_p^2 > 0$, while maximization of the insurgent control variable is guaranteed by $\partial^2 H/\partial\tilde{u}_e^2 < 0$, and all of these conditions hold for $\tilde{C}_p, \tilde{C}_a, \tilde{C}_e > 0$. Solving equation (6) for the three control variables, we find

$$\tilde{u}_p = \frac{\lambda x}{2\tilde{C}_p} \tag{7.a}$$

$$\tilde{u}_a = \frac{\lambda x\left(2\tilde{C}_e - \lambda x k'\right)}{4\tilde{C}_e\tilde{C}_a + \left(\lambda x k'\right)^2} \tag{7.b}$$

$$\tilde{u}_e = \frac{\lambda x\left(2\tilde{C}_a + \lambda x k'\right)}{4\tilde{C}_e\tilde{C}_a + \left(\lambda x k'\right)^2} \tag{7.c}$$

In addition, the adjoint equation can be found from (5) as

$$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial x} = -2x - \lambda\left(r_0 + \tilde{u}_e - \tilde{u}_p - \left(1 - k'\tilde{u}_e\right)\tilde{u}_a\right) \qquad (8)$$

Substituting equation (7) into equations (3.b) and (8), it is possible to define the dynamics of the system purely in terms of x and λ. A forward-backward sweep algorithm can then be used to iterate $x(t)$ forward from $x(0)$ and $\lambda(t)$ backwards from $\lambda(t_f) = \theta$, where θ is chosen in an iterative fashion in order to satisfy the condition $x(t_f) = x_{end}$.

IV. SIMULATION RESULTS

The proposed dynamics model and optimal control formulation have been implemented using Fortran code. In addition, we discuss uncertainty and sensitivity analyses of the passive-aggressive defense model. Simulation parameters and LHS parameter ranges are summarized in Tables 1. Figure 1 shows a sample simulation of the military model (equations (3) and (4)), for the parameter values listed in column 2 of Table 1.

Table 1 Parameters/parameter ranges used for the military model

Parameter	Value	Range	Description
r_0	0	n/a	Baseline recruitment rate
k'	0.3	$0.1-0.4$	Increase in aggressive killing per unit increase in intelligence (OP/insurgent)
$\tilde{C}_p$	5	$5-50$	cost of passive defense (impact·OP/insurgent)
$\tilde{C}_a$	200	$100-300$	cost of aggressive defense (impact·OP/insurgent)
$\tilde{C}_e$	150	$150-200$	cost of replication (impact·OP/insurgent)
$x(0)$	100	n/a	initial insurgent population
x_{end}	10	n/a	final insurgent population

*OP = occupation period

The values in Table 1 are chosen for illustration purposes, and application of the military model to a specific military scenario would require some additional effort in terms of quantifying the approximate ranges for the different weighting and model parameters. In general, we have assumed a low relative cost associated with passive defense, $\tilde{C}_p$, a significantly higher relative cost associated with aggressive defense, $\tilde{C}_a$, and an intermediate relative cost (to the insurgents) associated with recruitment activities. In addition, we have assumed that military intelligence can improve the efficiency of aggressive assaults significantly (an improvement of between ~10-40%). Certainly, many of these parameters can be estimated from resource costs, the risk of propaganda efforts, the rate of civilian casualty during offensive military strikes and estimates of the degree to which useful intelligence can be gleaned from terrorist recruiting activities.

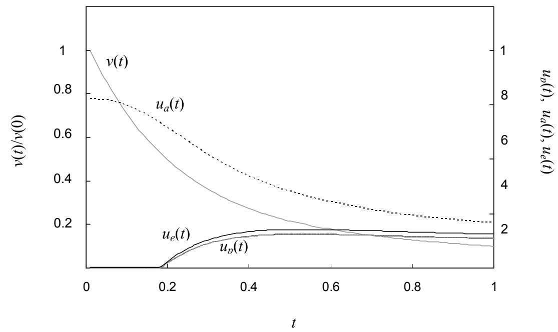

Fig. 1 Predicted control and state variables for the military model

From Figure 1 we see that the insurgent population is actually best to 'lie low' during the early stages of occupation, and should only start to recruit after the military has significantly decreased its aggressive efforts. In contrast, the military is best to attack aggressively at first, and ease off, supplementing aggressive defense with passive tactics, as time progresses through to the end of the occupation. In order to determine the degree to which these trends/strategy predictions are independent of model parameters, we perform an LHS uncertainty and sensitivity analysis.

Figure 2 a.-d. shows an LHS uncertainty analysis of the military model given the parameter ranges suggested in Table 1. In the absence of any additional information, we assume uniform probability distributions over all of our parameter ranges. Maximum and minimum predicted values for the state and control variables are shown as dotted lines, while the average values are shown as solid lines.

From Figure 2 it is clear that the predictions for the military model are strongly dependent on model and weighting parameters. Most significant, so far as the development of military strategy is concerned, is the wide variation in passive defense and insurgent recruitment strategies (Figure 2 b. and d.) during the early stages of the military occupation period. Optimal passive defense may, for instance, involve an initially strong response followed by a gradual decrease in passive efforts through time. It may instead, however, involve no initial response at all, with passive defense efforts rising to a relatively constant level only during the latter half of the occupation period. A similar trend follows for insurgent recruiting activities. In contrast, aggressive defense (Figure 2 c), does not show this same variation. In general an early aggressive defense operation is suggested over the entire range of parameters used to analyze our simple military model. In part to understand the different strategy variations that are apparent in Figure 2, we conducted a sensitivity analysis of the model in equations (3) and (4) for the parameter ranges listed in Table 1. By calculating PRCC values for the various state and control variables with respect to the various weighting and model parameters, it is possible to estimate the relative importance of the different parameters in terms of overall passive defense/recruitment schemes. Figure 3 a.-d. shows the PRCC values for x, u_p, u_a, and u_e

respectively.

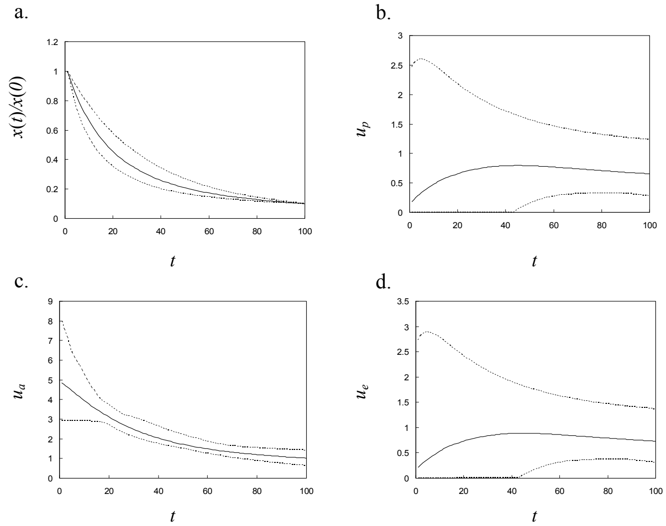

Fig. 2 Maximum (upper dotted line), minimum (lower dotted line) and average (solid line) predicted control and state variables (military model).

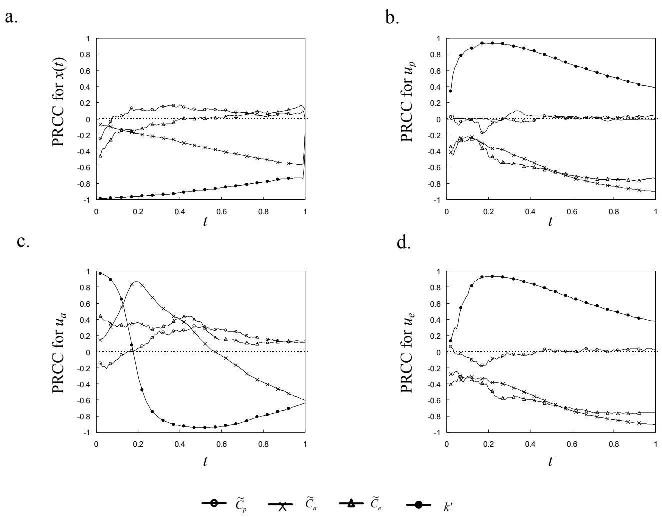

Fig. 3 PRCC analysis showing the dependence of the predicted control and state variables on the various model and weighting parameters

From Figure 3 b. and d., it is clear that both passive defense and insurgent recruitment are positively correlated with k' and negatively correlated with $\tilde{C}_a$ and $\tilde{C}_e$. Contrary to expectations, however, passive defense strategies exhibit only minimal dependence on $\tilde{C}_p$. In other words, the costs of aggressive defense and insurgent recruitment are far more significant in determining the passive defense strategy than is the cost of passive defense itself. In general, high values for $\tilde{C}_a$ and $\tilde{C}_e$ are associated with limited passive defense and insurgent recruitment, while high values for k' are associated with increased passive defense. These trends are somewhat intuitive, although the actual, time-dependent strategies predicted by the model are impossible to deduce without the use of a mathematical model. Several other interesting dependencies on the various model and weighting parameters are apparent in Figure 3, however these are beyond the scope of the current paper.

While Figure 3 illustrates the time-dependencies of the PRCC values for the control and state variables, unlike Figure 2, it is difficult to interpret Figure 3 in terms of overall strategy shape and, specifically, how the various parameters contribute to whether or not passive defense and recruitment strategies involve an initial suppression period. To answer this question, we calculated PRCC values for $t_{\mathrm{p,onset}}$ and $t_{\mathrm{e,onset}}$ - the initial detection times associated with passive defense and recruitment respectively. These values are summarized in Table 2.

Table 2 PRCC values for onset of passive defense and recruitment

Parameter	PRCC for $t_{\mathrm{p,onset}}$	PRCC for $t_{\mathrm{e,onset}}$
k'	-0.40	0.16
$\tilde{C}_p$	0.15	-0.04
$\tilde{C}_a$	0.49	0.14
$\tilde{C}_e$	0.29	-0.04

Interestingly, high costs associated with aggressive defense and a weaker ability to capitalize on intelligence gleaned from terrorist recruitment activities both lead to optimal defense strategies with delays in the onset of passive defense schemes. The same is true, though to a lesser extent, of high costs associated with passive defense and high costs associated with recruitment. In contrast to $t_{\mathrm{p,onset}}$, however, $t_{\mathrm{e,onset}}$ appears to exhibit very limited correlation with any of the model or weighting parameters, suggesting that insurgent behavior will be relatively difficult to predict, even based on evidence of insurgent behavior and resource potential.

V. Conclusions and Discussions

In this paper, we consider a very general model which interprets, mathematically, the trade-off between passive and aggressive defense in the face of a growing enemy population. Using this model, we attempt to predict optimal combinations of passive and aggressive defense for insurgent combat scenarios. Given the generality of the model, and the exploratory nature of the work, we analyse our predicted strategies in terms of LHS uncertainty and sensitivity analyses. This allows us to determine the degree to which strategies vary depending on model and weighting parameters, and also the parameters most likely to influence strategy decisions during the course of a military occupation. In general, we focus on scenarios where aggressive defense is significantly more costly than passive defense. This is in keeping with both military scenarios. Interestingly, it is also true of many other defense scenarios. Immune defense against hepatitis B virus, for instance, involves both cytolytic killing of infected cells (aggressive) and cytokine suppression of viral replication (passive), and thus could be modelled with a similar approach to the one used here. Overall, optimal aggressive strategies appear to follow a trend wherein there is significant aggression during the initial phase of the occupation period, but this aggression eases as time progresses. Optimal passive strategies are more varied. In the military model early suppression of passive strategy appears to be strongly dependent on high $\tilde{C}_a$ values and low k' values.

The model that we present in this paper is intentionally broad and general. As such, it can be applied, albeit with slight variation, to a wide range of different biological and social combat scenarios. We suggest that by analyzing a range of different warfare contexts using models that are, essentially, an extension of the basic model presented here, it may be possible to generalize basic features of defense strategy. This may, in turn, allow us to compare different forms of warfare on the same mathematical ground, helping both with tactical strategy development on the military front, and with other tangentially related forms of warfare like defense against disease both from the standpoint of medical intervention and epidemiology.

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