Multi-robot Active Target Tracking
with Distance and Bearing Observations

Ke X. Zhou* and Stergios I. Roumeliotis†
*Dept. of Electrical and Computer Engineering, †Dept. of Computer Science and Engineering
University of Minnesota, Minneapolis, MN 55455 Email: {kezhou|stergios}@cs.umn.edu

Abstract—In this paper, we study the problem of optimal trajectory generation for a team of mobile robots tracking a moving target using distance and bearing measurements. Contrary to previous approaches, we explicitly consider limits on the robots’ speed and impose constraints on the minimum distance at which the robots are allowed to approach the target. We first address the case of a single sensor and show that although this problem is non-convex with non-convex constraints, in general, its optimal solution can be determined analytically. Moreover, we extend this approach to the case of multiple sensors and propose an iterative algorithm, Gauss-Seidel-relaxation (GSR), for determining the set of feasible locations that each sensor should move to in order to minimize the uncertainty about the position of the target. Extensive simulation results are presented demonstrating that the performance of the GSR algorithm, whose computational complexity is linear in the number of sensors, is indistinguishable of that of a grid-based exhaustive search, with cost exponential in the number of sensors, and significantly better than that of a random, towards the target, motion strategy.

I. INTRODUCTION

Optimally tracking a moving target under motion and processing constraints is necessary in a number of applications such as environmental monitoring, surveillance, human-robot interaction, as well as defense applications. In most cases in practice, multiple wireless static sensors are employed in order to improve the tracking accuracy and increase the size of the surveillance area. Contrary to static sensors, whose density and sensing range are fixed, mobile sensors (robots) can cover larger areas over time without the need to increase their number. Additionally, their spatial distribution can change dynamically so as to adapt to the motion of the target, and hence provide informative measurements about its position. Selecting the best sensing locations is of particular importance especially when considering time-critical applications (e.g., when tracking a hostile target), as well as limitations on the robots’ processing and communication resources.

In this paper, our objective is to determine optimal trajectories for a team of heterogeneous robots that track a moving target using distance and bearing measurements. Since accurately predicting the motion of the target over multiple time steps is impossible, we focus our attention to the case where the robots must determine their optimal sensing locations for one time step at a time. Specifically, we seek to minimize the uncertainty about the position of the target, expressed as the trace of the posterior covariance matrix for the target’s position estimates, while considering maximum-speed limitations on the robots’ motion. Additionally, and in order to avoid collisions, we impose constraints on the minimum distance between any of the robots and the target (cf. Section III). Due to the non-linearities of the robots’ motion and measurement models, this formulation results in a non-convex objective function with non-convex constraints on the optimization variables (i.e., the robots’ sensing locations).

The main contributions of this work are the following:

• We first investigate the case of a single sensor and for the first time we prove that the optimal solution to the active target tracking problem, when using distance and bearing measurements, can be determined analytically (cf. Section IV-A). In particular, we show that depending on the distance between the robot and the target, three distinct cases must be considered, each corresponding to different pairs of polynomial equations in two variables. The solutions of these bivariate polynomial systems are computed by first employing the Sylvester resultant to eliminate one of the variables and subsequently determining the roots of the resulting univariate polynomial through the companion matrix [1].

• We extend this approach to the case of multiple sensors by employing the non-linear Gauss-Seidel-relaxation (GSR) algorithm whose computational complexity is linear in the number of sensors (cf. Section IV-B). Additionally, we compare the performance of the GSR algorithm to that of a grid-based exhaustive search (GBES), whose cost is exponential in the number of sensors, and show that GSR achieves comparable tracking accuracy at a significantly lower computational cost (cf. Section V).

Before describing the formulation and solution of the active target tracking problem, in Sections III and IV, respectively, we first review related work in Section II. Simulation results are presented in Section V, while the conclusions of this work and directions of future research are discussed in Section VI.

II. LITERATURE REVIEW

Although target tracking has received considerable attention, in most cases the sensors involved are static and the emphasis is on the optimal processing of the available
information (e.g., given communication constraints [2], [3]). Significant work on the problem of single-robot active target tracking using bearing-only measurements, has been presented in [4], [5], [6]. Additionally, the case of multi-robot active target tracking using distance-only measurements has been addressed in [7], [8], [9]. Due to the key differences in the observation models when distance-only or bearing-only, instead of distance and bearing, measurements are considered and their impact on selecting the next best sensing location, we hereafter limit our discussion to multi-robot tracking approaches that use both distance and bearing measurements.

Stroupe and Balch [10] propose an approximate tracking behavior, where the mobile sensors attempt to minimize the target’s location uncertainty using distance and bearing measurements. The objective function is the determinant of the target position estimates’ covariance matrix. The optimization process in this case does not consider the set of all possible trajectories. Instead, a greedy search is performed over the discretized set of candidate headings, separately for each sensor. Additionally, the expected information gain from the teammates’ actions is approximated by assuming that the other sensors’ measurements in the next time step will be the same as these recorded at their current locations.

Chung et al. [11] present a decentralized motion planning algorithm for solving the multi-sensor target tracking problem using both distance and bearing measurements. The authors employ the determinant of the target’s position covariance matrix as the cost function. The decentralized control law in this case is based on the gradient of the cost function with respect to each of the sensor’s coordinates, multiplied by constant step-size of 1. The authors, however, do not account for the speed constraints on the motion of sensors. In addition, the convergence rate of the gradient-based method and the existence of local minima are not considered.

The main drawback of the previous approaches is that no constraints on the speed of the sensors are considered. Furthermore, their impact on the computational complexity of the optimization algorithm used is not examined. The only exception is the work presented in [10]. In that case, however, these constraints are used only to define the discretized region over which the heading of each sensor is optimized independently (i.e., each sensor determines its next sensing location without considering the constraints on the motion of its teammates). Contrary to previous approaches, in our formulation, we account for the existence of prior information and explicitly consider constraints on the motion of the sensors (maximum speed and minimum allowed distance to the target).

III. PROBLEM FORMULATION

Consider a group of mobile sensors (or robots) moving in a plane and tracking the position of a moving target by processing distance and bearing measurements. In this paper, we study the case of global tracking, i.e., the position of the target is described with respect to a fixed (global) frame of reference, instead of a relative group-centered one. Hence, we hereafter employ the assumption that the position and orientation (pose) of each tracking sensor are known with high accuracy within the global frame of reference (e.g., from precise GPS and compass measurements).

Furthermore, we consider the case where each sensor can move in 2D with speed $v_i$, which is upper bounded by $v_{\text{max}}$. From time-step $k$ to $k+1$, the sensor can only move within a circular region centered at its position at time-step $k$ with radius $r_i = v_{\text{max}} \delta t$. Furthermore, to avoid collision with the target, sensor-$i$ is prohibited to move inside a circular region centered at the target’s position estimate at time-step $k + 1$ with radius $\rho_i$. $S_i$ is the sensor’s position with respect to sensor-$i$. The distance measurement of sensor-$i$ is the norm of $r_i [x_T(k+1) - y_T]$ plus noise, and the bearing measurement of sensor-$i$ is $\theta_i(k + 1)$ plus noise.

Fig. 1. Illustration of the $i$-th sensor’s and target’s motion: Sensor-$i$ moves in 2D with speed $v_i$, which is bounded by $v_{\text{max}}$. From time-step $k$ to $k+1$, the sensor can only move within a circular region centered at its position at time-step $k$ with radius $r_i = v_{\text{max}} \delta t$. Furthermore, to avoid collision with the target, sensor-$i$ is prohibited to move inside a circular region centered at the target’s position estimate at time-step $k + 1$ with radius $\rho_i$. $S_i$ is the sensor’s position with respect to sensor-$i$. The distance measurement of sensor-$i$ is the norm of $r_i [x_T(k+1) - y_T]$ plus noise, and the bearing measurement of sensor-$i$ is $\theta_i(k + 1)$ plus noise.

In the next two sections, we present the target’s state propagation equations and the sensors’ measurement model.

A. State Propagation

In this work, we employ the Extended Kalman Filter (EKF) for recursively estimating the target’s state, $x_T(k)$. This is defined as a vector of dimension $2N$, where $N - 1$ is the highest-order time derivative of the position, $(x_T(k), y_T(k))$, described by the motion model, and can include components such as position, velocity, and acceleration:

$$x_T(k) = [x_T(k), y_T(k), \dot{x}_T(k), \dot{y}_T(k), \ddot{x}_T(k), \ddot{y}_T(k), ...]^T$$ (1)

We consider the case where the target moves randomly and assume that we know the stochastic model describing the motion of the target (e.g., constant-acceleration or constant-velocity, etc. [12]). However, as it will become evident later
on, our sensing strategy does not depend on the particular selection of the target’s motion model.

The discrete-time state propagation equation is:

\[ x_T(k+1) = \Phi_k x_T(k) + G_k w_d(k) \]  

(2)
where \( w_d \) is a zero-mean white Gaussian noise process with covariance \( Q_d = E[w_d(k)w_d^T(k)] \).

The estimate of the target’s state is propagated as:

\[ \hat{x}_T(k+1|k) = \Phi_k \hat{x}_T(k|k) \]  

(3)
where \( \hat{x}_T(\ell|j) \) is the state estimate at time-step \( \ell \), after measurements up to time-step \( j \) have been processed.

The error-state covariance matrix is propagated as:

\[ P_{k+1|k} = \Phi_k P_{k|k} \Phi_k^T + G_k Q_d G_k^T \]

where \( P_{\ell|j} \) is the covariance of the error, \( \hat{x}_T(\ell|j) \), in the state estimate. The state transition matrix, \( \Phi_k \), and the process noise Jacobian, \( G_k \), appear in the preceding expressions depend on the motion model used [12]. In our work, these can be arbitrary, but known, matrices, since no assumptions on their properties are imposed.

B. Measurement Model

At time-step \( k+1 \), sensor-\( i \) measures its distance \( d_i(k+1) \) and bearing \( \theta_i(k+1) \) to the target \( (i = 1, \ldots, M) \), as shown in Fig. 1. Assume the orientation of sensor-\( i \) is \( \phi_i(k+1) \), therefore the measurement equation is:

\[ z(k+1) = \begin{bmatrix} d_1(k+1) \\ \theta_1(k+1) \\ \vdots \\ d_M(k+1) \\ \theta_M(k+1) \end{bmatrix} + \begin{bmatrix} n_{d_1}(k+1) \\ n_{\theta_1}(k+1) \\ \vdots \\ n_{d_M}(k+1) \\ n_{\theta_M}(k+1) \end{bmatrix} \]  

(4)
with \( (i = 1, \ldots, M) \)

\[ d_i(k+1) = \sqrt{\Delta x_T^2(k+1) + \Delta y_T^2(k+1)} \]

\[ \theta_i(k+1) = \arctan \left( \frac{\Delta y_T(k+1)}{\Delta x_T(k+1)} \right) - \phi_i(k+1) \]

where

\[ \Delta x_T(k+1) = x_T(k+1) - x_S(k+1) \]
\[ \Delta y_T(k+1) = y_T(k+1) - y_S(k+1) \]

and \( p_T(k+1) = [x_T(k+1) \ y_T(k+1)]^T \). \( p_S(k+1) = [x_S(k+1) \ y_S(k+1)]^T \) are the positions of the target and the \( i \)-th sensor, respectively, expressed in the global frame of reference. Note also that \( n_i(k+1) = [n_{d_i}(k+1) \ n_{\theta_i}(k+1)]^T \) is the noise in the \( i \)-th sensor’s measurements, which is a zero-mean white Gaussian process with covariance \( R_i = E[n_i(k+1)n_i^T(k+1)] = \text{diag}(\sigma_{d_i}^2, \sigma_{\theta_i}^2) \), and independent of the noise in other sensors, i.e., \( E[n_i(k+1)n_j^T(k+1)] = R_i \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker delta. Thus, the covariance corresponding to the measurement noise \( n(k+1) = [n_1^T(k+1), \ldots, n_M^T(k+1)]^T \) is \( R = E[n(k+1)n^T(k+1)] = \text{diag}(R_i) \).

The measurement equation (4) is a nonlinear function of the state variable \( x_T \). The measurement-error equation, obtained by linearizing (4) is:

\[ \hat{z}(k+1|k) = z(k+1) - \tilde{z}(k+1|k) \]
\[ \simeq H_{k+1} x_T(k+1|k) + n(k+1) \]  

(5)
where

\[ \hat{z}(k+1|k) = [\hat{z}_1^T(k+1|k), \ldots, \hat{z}_M^T(k+1|k)]^T \]
\[ \hat{z}_i(k+1|k) = [\hat{d}_i(k+1|k) \ \hat{\theta}_i(k+1|k)]^T \]

\[ \hat{d}_i(k+1|k) = \sqrt{\Delta x_T^2(k+1|k) + \Delta y_T^2(k+1|k)} \]

\[ \hat{\theta}_i(k+1|k) = \arctan \left( \frac{\Delta y_T(k+1|k)}{\Delta x_T(k+1|k)} \right) - \phi_i(k+1) \]

\[ \Delta x_T(k+1|k) = \hat{x}_T(k+1|k) - x_S(k+1) \]
\[ \Delta y_T(k+1|k) = \hat{y}_T(k+1|k) - y_S(k+1) \]
\[ \hat{x}_T(k+1|k) = x_T(k+1) - \hat{x}_T(k+1|k) \]

Note that the measurement matrix in (5) has a block column structure, which is given by the following expression:

\[ H_{k+1} = [H_{e,k+1} \ 0_{2M \times (2N-2)}] \]  

(6)
where \( 2N \) is the dimension of the state vector and

\[ H_{e,k+1} = [h_1(k+1) \ , \ldots, \ h_M(k+1)] \]  

(7)
\[ h_i(k+1) = [h_{d_i}(k+1) \ h_{\theta_i}(k+1)] \]  

(8)
\[ h_{d_i}(k+1) = \frac{-1}{\sqrt{p_i^T p_i}} p_i, \quad h_{\theta_i}(k+1) = \frac{1}{p_i^T p_i} J p_i \]  

(9)
\[ p_i = p_S(k+1) - \hat{p}_T(k+1|k) \]  

(10)
where \( J = C \left( -\frac{1}{2} \right) \) and \( C(\cdot) \) is the \( 2 \times 2 \) rotational matrix.

C. State and Covariance Update

Once the distance and bearing measurements, \( z(k+1) \), from all the sensors are available, the target’s state estimate and its covariance are updated as:

\[ \bar{x}_T(k+1|k) = \hat{x}_T(k+1|k) + K_{k+1} \hat{z}(k+1|k) \]

\[ P_{k+1|k+1} = P_{k+1|k} - K_{k+1} S_{k+1} K_{k+1}^T \]  

(11)
where \( K_{k+1} = P_{k+1|k} H_{k+1}^T S_{k+1}^{-1} \) is the Kalman gain, \( S_{k+1} = H_{k+1} P_{k+1|k} H_{k+1}^T + R \) is the measurement residual covariance.

Our objective in this work is to determine the active sensing strategy that minimizes the uncertainty for the position estimate of the target. In order to account for the impact of the prior state estimates on the motion of the sensors, we first present the following lemma.

Lemma 1: The posterior (updated) covariance for the target’s position estimate depends on (i) the prior (propagated) covariance sub-matrix of the target’s position (i.e., it is independent of the uncertainty in the estimates of higher-order time derivatives of the position such as velocity, acceleration, etc., and hence it is independent of the target’s motion model) and (ii) the measurement information matrix corresponding to the target’s position, i.e.,

\[ \text{...} \]  

(12)
\[ P_{k+1|k+1,11} = \left( (P_{k+1|k,11})^{-1} + H_{k+1}^T R_{k}^{-1} H_{k+1} \right)^{-1} \] (12)

where \( P_{l|j,11} \) denotes the 2 × 2 upper diagonal sub-matrix of \( P_{l|j} \) corresponding to the covariance in the position estimates.

**Proof:** The proof is shown in [13].

The importance of this lemma is that the optimization algorithm presented in Section IV can be derived based on (12) for the position covariance update – instead of (11) for the entire state covariance update – regardless of the stochastic process model employed for describing the target’s motion.

In the next section, we formulate the sensors’ one-step-ahead optimal motion strategy as a constrained optimization problem, and discuss its properties.

**D. Problem Statement and Reformulation**

As evident from (7)-(10) and (12), after each update step the target’s position covariance matrix will depend on all the next sensors’ positions \( P_{S_i}(k+1) = [x_{S_i}(k+1) \ y_{S_i}(k+1)]^T, \ i = 1, \ldots, M. \) Assume that at time-step \( k \), sensor-\( i \) is at location \( p_{S_i}(k) = [x_{S_i}(k) \ y_{S_i}(k)]^T \). At time-step \( k+1 \) its position \( p_{S_i}(k+1) \) is confined in a circular region centered at \( p_{S_i}(k) \), due to the maximum speed constraint, and outside a circular region centered at \( p_T(k+1) \) to avoid collisions (cf. Fig. 1):

\[ \|p_{S_i}(k+1) - p_{S_i}(k)\| \leq r_i = v_{\text{max}} \delta t \] (13)

\[ \|p_{S_i}(k+1) - \hat{p}_T(k+1)\| \geq \rho_i \] (14)

Substituting \( p_i \) [cf. (10)] in the above two equations, yields:

\[ \|p_i - [p_{S_i}(k) - \hat{p}_T(k+1)]\| \leq r_i \] (15)

\[ ||p_i|| \geq \rho_i \] (16)

where the feasible region of \( p_i \) is inside a circle of radius \( r_i \) centered at \( p_{S_i}(k) - \hat{p}_T(k+1) \), and outside a circle of radius \( \rho_i \) centered at the origin \([0,0]^T\). Note that the estimate \( \hat{p}_T(k+1) \) [cf. (3)] is shared among all sensors, and can be treated as a constant at time-step \( k+1 \). Hence, once \( p_{S_i}, \ i = 1, \ldots, M, \) is determined, the location of sensor-\( i \) at time-step \( k+1 \), \( p_{S_i}(k+1), \ i = 1, \ldots, M, \) can be obtained through (10).

The problem we address in this work is that of determining the sensors’ optimal motion strategy, i.e., the set \( C(k+1) = \{ p_i, \ i = 1, \ldots, M \} \), that minimizes the trace of the target’s position estimate covariance matrix, under the constraints specified in (15)-(16). To proceed, we exploit the fact that \( R \) is a diagonal matrix, and substitute (7)-(9) in (12) to obtain the following optimization problem:

**Optimization Problem 1 (Π₁)**

\[
\min_{p_1, \ldots, p_M} \ \text{tr}\left( (P_{k+1|k,11})^{-1} + \sum_{i=1}^{M} \frac{1}{\sigma_d^2} p_i^T P_i \right) \] (17)

s.t. \[ ||p_i - [p_{S_i}(k) - \hat{p}_T(k+1)]|| \leq r_i, \ i = 1, \ldots, M \]

\[ ||p_i|| \geq \rho_i, \ i = 1, \ldots, M \]

In what follows, we will consider, without loss of generality, the case \( P_{k+1|k,11} \) being a diagonal matrix. If \( P_{k+1|k,11} \) is non-diagonal, we can always apply a coordinate transformation (cf. Lemma 2), to transform it into a diagonal one.

**Lemma 2:** Assume \( P_{k+1|k,11} \geq 0_{2 \times 2} \) is non-diagonal. The eigen-decomposition of \( P_{k+1|k,11} = C(\varphi_0) \Lambda C(-\varphi_0) \), and \( \Lambda = \text{diag}(\lambda_1, \lambda_2) \), \( \lambda_1 \geq \lambda_2 > 0 \). Then,

\[ \text{tr}(P_{k+1|k+1,11}) = \text{tr}\left( \Lambda + \sum_{i=1}^{M} \frac{1}{\sigma_d^2} s_i^T s_i \right)^{-1} \] (18)

where \( s_i = C(\varphi_0)p_i, \ i = 1, \ldots, M \).

**Proof:** Substituting \( P_{k+1|k+1,11} = C(\varphi_0) \Lambda C(-\varphi_0) \) and \( p_i = C(\varphi_0)s_i \) in (17), employing the equality \( C(\varphi_0)J = JC(-\varphi_0) \) which holds since both are \( 2 \times 2 \) rotational matrices, and noting that the trace operation is invariant to similarity transformations results in (18).

Note also that the similarity transformation does not change the norm of a vector; thus, constraint (15) is equivalent to \( ||s_i - c_i|| \leq r_i \), with \( c_i = C(\varphi_0)p_i - \hat{p}_T(k+1) \), and constraint (16) is equivalent to \( ||s_i|| \geq \rho_i \). Therefore, from now on, we will mainly focus on the following equivalent optimization problem:

**Optimization Problem 2 (Π₂)**

\[
\min_{s_1, \ldots, s_M} \ \text{tr}\left( \Lambda + \sum_{i=1}^{M} \frac{1}{\sigma_d^2} s_i^T s_i \right) \] (19)

s.t. \[ ||s_i - c_i||^2 \leq r_i^2, \ i = 1, \ldots, M, \] (20)

\[ ||s_i||^2 \geq \rho_i^2, \ i = 1, \ldots, M, \] (21)

Once the optimal solution \( \{ s_i, \ i = 1, \ldots, M \} \) is obtained, the best position of the \( i \)-th sensor at time-step \( k+1 \), \( p_{S_i}(k+1) \), can be calculated through \( p_i = C(\varphi_0)s_i \) and (10).

**Remark 1:** The optimization problem \( \Pi_2 \) is a nonlinear programming problem since both the objective function [cf. (19)] and constraints [cf. (20)-(21)] are nonlinear functions with respect to the optimization variable \( s = [s_1^T, \ldots, s_M^T]^T \). Moreover, \( \Pi_2 \) (and equivalently, \( \Pi_1 \)) is not a convex programming since the feasible set defined by constraint (21) is not convex.

**IV. Problem Solution**

As mentioned in the previous section, the problem of optimal trajectory generation for multiple sensors with mobility constraints that track a moving target using distance and bearing measurements is not convex in general. Hence, finding the global optimal solution for the original optimization problem, or for its equivalent formulation (cf. \( \Pi_1 \Leftrightarrow \Pi_2 \)), becomes challenging. Ideally, the optimal solution can be determined if one discretizes the feasible set of all sensors [cf. (20)-(21)] and performs an exhaustive search. This approach, however, has computational complexity exponential in the number of sensors, which is of limited practical use given realistic processing constraints.

In order to design algorithms that can operate in real time, appropriate relaxations of the original optimization problem become necessary. In what follows, we first present an analytic solution for the single-sensor case (cf. Section IV-A) and based on that we propose a Gauss-Seidel relaxation.
Lemma 3: Assume $\Omega = \Omega \cup \partial \Omega$ is a compact and connected set in 2D, and the origin $[0, 0]^T \not\in \Omega$. For any $s \in \Omega$, the line segment connecting $s$ and the origin will inevitably intersect $\partial \Omega$ at one or multiple points, and $s^\dagger \in \partial \Omega$ denotes the one closest to $[0, 0]^T$ (cf. Fig. 2). Then $f_0(s^\dagger) \leq f_0(s)$.

Proof: Based on the construction of $s^\dagger$, we have: $s^\dagger = \epsilon s$, with $\epsilon \in (0, 1)$, hence we conclude:

$$
\frac{(s^\dagger)^T(s^\dagger)}{s^T s} J(s^\dagger)(s^\dagger)^T (s^\dagger)^T = \frac{1}{c^T (s^\dagger)^2 s^T J^T}{(s^\dagger)^T(s^\dagger)} \Rightarrow \left(\Lambda + \frac{1}{\sigma_2^T(s^\dagger)^2} \frac{J(s^\dagger)(s^\dagger)^T}{J^T} \right) \leq \left(\Lambda + \frac{1}{\sigma_2^T(s^\dagger)^2} \frac{J(s^\dagger)(s^\dagger)^T}{J^T} \right)
$$

Therefore, $f_0(s^\dagger) \leq f_0(s)$.

Lemma 3 establishes the fact that the global optimal solution for $\Pi_3$, when optimizing over the feasible set $\Omega$ (cf. Fig. 2), is always attained on $\partial \Omega$. Moreover, by applying the same argument as before (cf. Fig. 2), it can be easily shown that $f_0(s^\dagger) \leq f_0(s^\dagger)$. Therefore, the global optimal solution $s^*$ resides only in the portion (the curve $ADB$) of $\partial \Omega$ facing the origin, denoted as $\Theta$ (cf. Fig. 2).

As shown in Figs. 3(a)-(c), depending on the parameters $c$, $r$, and $\rho$, there exist three cases we need to consider for the feasible set $\Omega$ of $\Pi_3$, described by the constraints (23)-(24). In what follows, we present the corresponding solutions satisfying the KKT conditions (25)-(27) for each of these three cases analytically. For clarity, we employ the following notation: $s^* = [x y]^T$, and $c = [c_1, c_2]^T$. Furthermore, we set $r = \min(v_{\max}, \|s\|)$. 

1) Case-I, $\rho \leq \|c\| - r$: As shown in Fig. 3(a), the only active constraint for Case-I is the maximum speed constraint [cf. (23)]. Based on Lemma 3 and setting $v = v_{\max}$, the optimal solution $s^*$ must reside on the red-colored arc $ADB$. Since the collision avoidance constraint (24) is inactive, its corresponding Lagrange multiplier is $\mu^* = 0$, and (25)-(27) are simplified to:

\[ \nabla f_0(s^*) + \mu^*(s^* - c) = 0_{2 \times 1} \quad (28) \]

\[ \|s^* - c\|^2 - r^2 = 0 \quad (29) \]

Clearly, (29) is a 2nd order polynomial equation in the variables $x$ and $y$, i.e.,

\[ 0 = f_2(x, y) = (x - c_1)^2 + (y - c_2)^2 - r^2 \quad (30) \]

Since we aim at transforming (28) into a polynomial equation only containing $x$ and $y$, we eliminate $\mu^*$ by multiplying both sides of (28) with $(s^* - c)^T C (\Sigma) C^{-1}$:

\[ (s^* - c)^T C (\Sigma) C^{-1} \nabla f_0(s^*) = 0 \quad (31) \]

After rearranging terms in (31), we obtain the following 8th order polynomial equation in $x$ and $y$:

\[ 0 = f_3(x, y) = \beta_3 x y \Delta + (\alpha_3 x + \alpha_7 y + \beta_3) x y \Delta^2 + (\alpha_5 x^2 + \alpha_4 x y + \alpha_6 y^2 + \alpha_8 y^3 + \beta_1) y \Delta + (\alpha_2 x + \alpha_1 y) x y \]

where $\Delta := x^2 + y^2$, and the parameters $\alpha_i$, $i = 1, \ldots, 8$, are defined in (23)-(24).
and $\beta_i$, $i = 1, 2, 3$, are known coefficients expressed in terms of $\lambda_1, \lambda_2, c_1, c_2, \sigma_2, \sigma_2^\prime$, and $r$. Due to space limitations, the interested reader can refer to [13] for more details.

In order to obtain all the critical points of $\Pi_1$ in Case-I, we need to solve the two polynomial equations $f_2(x, y) = 0$ and $f_1(x, y) = 0$ analytically [cf. (30) and (32)]. By employing the Sylvester resultant [15, Ch. 3], we are able to eliminate variable $y$ from (30) and (32), and obtain the following 10th order univariate polynomial in variable $x$:

$$0 = f_3(x) = \sum_{j=0}^{10} \gamma_j x^j$$

(33)

where $\gamma_j$, $j = 0, \ldots, 10$, are known coefficients expressed in terms of $\lambda_1, \lambda_2, c_1, c_2, \sigma_2, \sigma_2^\prime$, and $r$ [13].

The roots of $f_3$ are the eigenvalues of the corresponding $10 \times 10$ companion matrix [1], [13]. Although there are 10 solutions for $x$, we only need to consider the real eigenvalues of the companion matrix. Once $x$ is determined, $y$ is computed from (30), and the critical point $[x, y]^T$ is added into the set $\Xi_1$, which has at most 20 elements.

The final step is to evaluate the objective function $f_0(s)$ [cf. (22)] at all the critical points in $\Xi_1$ and select the one with the smallest objective value as the global optimal solution for $\Pi_1$.

2) Case-II, $\rho \geq \sqrt{|c|^2 - r^2}$: As shown in Fig. 3(b) and based on Lemma 3, the only active constraint for Case-II is the collision avoidance constraint [cf. (24)]. Since the maximum speed constraint (23) is inactive, its corresponding Lagrange multiplier $\nu^* = 0$, and (25)-(27) are simplified to:

$$\nabla f_0(s^*) - \nu^* s^* = 0_{2 \times 1}$$

(34)

$$||s^*||^2 - \rho^2 = 0$$

(35)

Clearly, (35) is a 2nd order bivariate polynomial equation, i.e.,

$$0 = f_3(x, y) = x^2 + y^2 - \rho^2$$

(36)

Applying the same technique as in Case-I to eliminate $\nu^*$ in (34), yields:

$$0 = f_4(x, y) = xy$$

(37)

It is easy to verify that the four real solutions satisfying (36) and (37) are $\{[\pm \rho, 0]^T, [0 \pm \rho]^T\}$. However, not all these critical points necessarily belong to the feasible region $\bar{\Omega}$, or equivalently, reside in the arc $ADB$, which is defined by (35) and inside the disk characterized by (23) [cf. Fig. 3(b)]. Hence, the set $\Xi_2$ containing all the feasible critical points can possibly have zero, and up to four elements. In particular, for the plot shown in Fig. 3(b), $\Xi_2 = \emptyset$.

Note, however, that since the curve $ADB$ is an arc of the circle defined by (35), it is also necessary to consider the objective value attained at the two boundary points $A$ and $B$ [cf. Fig. 3(b)]. Thus, the set $\Xi_2$ is augmented into $\Xi_2 = \Xi_2 \cup \{A, B\}$, which contains at least two, and up to six elements. The global optimal solution for $\Pi_2$ is selected as $s^* \in \Xi_2$ the one with the smallest objective value $f_0(s^*)$.

3) Case-III, $\rho \in (|c| - r, \sqrt{|c|^2 - r^2})$: As shown in Fig. 3(c) and based on Lemma 3, the optimal solution $s^* \in \bar{\Omega}$ must reside in the red-colored curve $ADB$, which is composed of three segments, i.e., $A = \Theta_1 \cup \Theta_2 \cup \Theta_3$. Here, $\Theta_1$ and $\Theta_2$ are due to the maximum speed constraint (23), and $\Theta_3$ is because of the collision avoidance constraint (24).

To obtain the critical points in Case-III, we proceed as follows: We first ignore the collision avoidance constraint (24) and calculate the set $\Xi_1$ of all critical points of $\Pi_3$ under the maximum speed constraint (23), following the process outlined in Case-I. Then, we apply the method outlined in Case-II to compute the optimal solution $s^\dagger$ of $\Pi_3$, while confining the feasible set to $\Theta_3$ only. Since $\Theta_1$ and $\Theta_2$ are two arcs of the circle defined by (29), it is also necessary to consider the objective value attained at the two boundary points $A$ and $B$ [cf. (36)], where $\Xi_1 = \Xi_1 \cup \{s^\dagger, A, B\}$, which is a finite set with cardinality at most 23.

The final step is to evaluate the objective function $f_0(s)$ at all the critical points in $\Xi_3$, and select the $s^*$ with the smallest objective value $f_0(s^*)$ as the global optimal solution for $\Pi_3$.

B. Multiple-sensor Target Tracking: Gauss-Seidel Relaxation

Motivated by the simplicity of the analytic-form solution for the case of one sensor (cf. Section IV-A), a straightforward approach to finding a minimum of the optimization problem $\Pi_2$ is to iteratively minimize its objective function
[cf. (19)] for each optimization variable separately, i.e., [16, Ch. 3]:

**Optimization Problem 4 (Π₄):**

\[
\min_{\mathbf{s}} \; \text{tr} \left( \left( \mathbf{P}_i^{(\ell+1)} \right)^{-1} + \frac{1}{\sigma_{d_i}^2} \left( \mathbf{s}_i^{(\ell+1)} \right)^T \left( \mathbf{s}_i^{(\ell+1)} \right) \right) \\
+ \frac{1}{\sigma_{d_j}^2} \left( \mathbf{s}_j^{(\ell+1)} \right)^T \mathbf{J} \left( \mathbf{s}_j^{(\ell+1)} \right) \right)^{-1} \\
\text{s.t.} \; \left\| \mathbf{s}_i^{(\ell+1)} - \mathbf{c}_i \right\| \leq r_i \; \text{and} \; \left\| \mathbf{s}_i^{(\ell+1)} \right\| \geq \rho_i
\]

where \( \mathbf{s}_i^{(\ell+1)} \) is the sought new optimal value of \( s_i \) at iteration \( \ell + 1 \), and \( P_i^{(\ell+1)} \) is defined as:

\[
\left( P_i^{(\ell+1)} \right)^{-1} = \Lambda + \\
\sum_{j=1}^{M} \left( \frac{1}{\sigma_{d_j}^2} \left( \mathbf{s}_j^{(\ell)} \right)^T \right)^2 + \frac{1}{\sigma_{d_j}^2} \left( \mathbf{s}_j^{(\ell)} \right)^T \mathbf{J} \left( \mathbf{s}_j^{(\ell)} \right) \\
+ \sum_{j=1}^{i-1} \left( \frac{1}{\sigma_{d_j}^2} \left( \mathbf{s}_j^{(\ell+1)} \right)^T \right)^2 + \frac{1}{\sigma_{d_j}^2} \left( \mathbf{s}_j^{(\ell+1)} \right)^T \mathbf{J} \left( \mathbf{s}_j^{(\ell+1)} \right)
\]

where \( \mathbf{s}_j^{(\ell+1)}, j = 1, \ldots, i-1 \), and \( \mathbf{s}_j^{(\ell)}, j = i+1, \ldots, M \), are the remaining optimization variables, considered fixed during this step, computed sequentially during the previous iterations.

Note that the matrix \( P_i^{(\ell+1)} \) is positive definite, and in general, non-diagonal. However, based on Lemma 2, through a similarity transformation, the optimization method employed in the single-sensor case (cf. Section IV-A) can be readily applied to solve \( \Pi_4 \). Moreover, the optimization process in the Gauss-Seidel relaxation (GSR) algorithm is carried out only for one variable (i.e., \( s_i \)) at every step. Thus, the GSR process has computational complexity only linear in the number of sensors. Furthermore, it is easily implemented, has low memory requirements and, as demonstrated in Section V, it achieves the same level of positioning accuracy as the exhaustive search approach.

**V. Simulation Results**

In order to evaluate the presented constrained optimal motion strategy, Gauss-Seidel Relaxation (GSR), we have conducted extensive simulation experiments and compared the performance of GSR to the following methods:

- **Grid-Based Exhaustive Search (GBES).** In this case, we discretize the feasible set of all sensors and perform an exhaustive search over all possible combinations of these to find the one that minimizes the trace of the covariance matrix for the target’s position estimates. Ideally, the GBES should return the global optimal solution and it could be used as a benchmark for evaluating the GSR, if the grid size is sufficiently small. However, this is difficult to guarantee in practice since its computational complexity and memory requirements are exponential in the number of sensors. Hence implementing the GBES becomes prohibitive when the number of sensors, \( M \), increases and/or when the size of the grid cells decreases.

- **Random Motion (RM).** This is a modification of an intuitive strategy that would require the sensors to move towards the target. In this case, however, and in order to ensure that the sensors do not converge to the same point, we require that at every time step sensor-\( i \) selects its heading direction with uniform probability towards points within the curve \( ADB \) shown in Figs. 3(a)–3(c).

1) **Simulation Setup:** For the purposes of this simulation, we adopt a zero-acceleration target motion model

\[
\dot{\mathbf{x}}_T(t) = F \cdot \mathbf{x}_T(t) + G \cdot \mathbf{w}(t)
\]

where

\[
F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & -x_T(t) \\ 0 & 0 & y_T(t) \\ 0 & 1 & \dot{x}_T(t) \\ 0 & 1 & \dot{y}_T(t) \end{bmatrix}
\]

and \( \mathbf{w}(t) = [w_x(t), w_y(t)]^T \) is a zero-mean white Gaussian noise vector with covariance \( E[\mathbf{w}(t)\mathbf{w}^T(\tau)] = qI_2\delta(t-\tau) \), \( q = 10 \), and \( \delta(t-\tau) \) is the Dirac delta. In our implementation, we discretize the continuous-time system model [cf. (39)] with time step \( \delta t = 0.1 \) sec.

The initial true state of the target is \( \mathbf{x}_T(0) = [0, 0, -8, 6]^T \). The initial estimate for the target’s state is \( \hat{\mathbf{x}}_T(0) = [2, -2, 0, 0]^T \). This can be obtained by processing the first measurements from the sensors at time-step 0. At the beginning of the experiment, the sensors are randomly distributed within a circle of radius 5 m, which is at a distance of about 20 m from the target’s initial position. The maximum speed for each sensor is set to 10 m/s, i.e., the largest distance that a sensor can travel during any time step is 1 m. The minimum distance between the target and sensors is set to \( \rho = 2 \) m.

The duration of the simulations is 5 sec (i.e., 50 time steps). At every time step, we employ the methods described (i.e., GBES, GSR, and RM) to calculate the next sensing location of each sensor. Throughout the simulations, we set the GBES cell size to \( \pi/200 \); i.e., we discretize the curve \( ADB \) [cf. Figs. 3(a)–3(c)] in segments of length \( \pi/200 \) m.

Finally, the measurement noise covariance matrix for the three sensors considered is set to \( \mathbf{R}_i = \text{diag}(\sigma_{d_i}^2, \sigma_{d_i}^2) \), with \( \sigma_{d_1}^2 = 2 \text{ m}^2, \sigma_{d_2}^2 = 4 \text{ m}^2, \sigma_{d_3}^2 = 0.25 \text{ rad}^2, \) and \( \sigma_{d_4}^2 = 0.5 \text{ rad}^2 \).

2) **Target Tracking with 3 Sensors (Heterogeneous team):** Figs 4(a)–4(c) depict the actual and estimated trajectories of the target, along with the trajectories of the three sensors, when employing as motion strategy GBES, GSR, and RM, respectively. As evident, the accuracy of the target’s position estimates for GSR is significantly better than that of RM and almost identical to that of GBES. Furthermore, the EKF estimates from the GSR and GBES are consistent.

Interestingly, in this case the heterogeneous sensor team splits into two groups. Sensor-1 (the most accurate one with

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4Due to space limitations we only describe here the results for the case of a team with 3 heterogeneous sensors. Further studies are presented in [13].
measurement noise variances $\sigma_{d_1}^2 = 2\text{ m}^2$, $\sigma_{\theta_1}^2 = 0.25\text{ rad}^2$ follows the target from the right, while sensors 2 and 3 form a separate cluster approaching the target from the left while moving very close to each other. The reason for this is the following: As sensors 2 and 3 measure the distance and bearing to the target from approximately the same location at every time step, their four independent measurements become equivalent, in terms of accuracy, to two with variances

$$\frac{1}{\sigma_{d_2,3}^2} \simeq \frac{1}{\sigma_{d_2}^2} + \frac{1}{\sigma_{d_3}^2} = \frac{1}{2}, \text{ or } \sigma_{d_2,3}^2 \simeq 2$$

$$\frac{1}{\sigma_{\theta_2,3}^2} \simeq \frac{1}{\sigma_{\theta_2}^2} + \frac{1}{\sigma_{\theta_3}^2} = 4, \text{ or } \sigma_{\theta_2,3}^2 \simeq 0.25$$

Hence, this problem becomes equivalent to that of 2 sensors with equal distance and bearing noise variances [13], with the difference that in this case the “second” sensor is realized by requiring sensors 2 and 3 to move close to each other.

VI. CONCLUSIONS

In this paper, we address the problem of determining optimal trajectories for a team of heterogeneous sensors (robots) that track a moving target using distance and bearing measurements. The optimality criterion used is the minimization of the trace of the target’s position covariance matrix. In our formulation, we have accounted for the existence of prior information and considered motion constraints on the robots (maximum speed and minimum distance to the target). We have shown that this non-linear constrained optimization problem is non-convex, in general, and have derived the optimal solution for the single-robot case in analytical form. Moreover, and in order to provide real-time solutions, we have introduced an iterative algorithm, Gauss-Seidel relaxation (GSR), for generating optimal trajectories for a team of robots whose computational and memory requirements are significantly lower compared to those of a grid-based exhaustive search (GBES) method (linear vs. exponential in the number of robots). Simulation studies demonstrate that the GSR achieves the same level of tracking accuracy as GBES, and significantly better when compared to the case when the robots move randomly towards the target.

In our future work, we plan to extend our current approach and address the cases when the robots’ poses are uncertain and when multiple targets are present. Finally, we intend to consider additional constraints both on the visibility and the motion of the robots due to obstacles in their surroundings.

REFERENCES