The 2009 IEEE/RSJ International Conference on
Intelligent Robots and Systems
October 11-15, 2009 St. Louis, USA

Extension of Colgate's Passivity Condition for Variable-Rate Haptics

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Abstract— **Colgate's passivity conditions for the virtual coupling and the virtual wall have been one of the most important results in haptics. However, due to its development being rooted in the frequency-domain, its theoretical justification has been limited to only uniform-rate haptics. In this paper, relying on analysis/derivation in the time-domain, we extend this Colgate's passivity condition to variable-rate haptics. We also present a similar passivity condition for the haptic rendering based on the recently proposed passive set-position modulation (PSPM).**

I. Introduction

The celebrated Colgate's passivity condition has been one of the most important results in haptics, which is often written as (with backward differentiation) [1], [2], [3]:

$$b \geq B + \frac{KT}{2} \tag{1}$$

where b is the device damping, B, K are the (discrete) damping/spring of the virtual coupling/wall, and T is the data update rate. This condition says that, when the sampling rate T is fixed, how large you can crank up the virtual coupling/wall's spring K and (even seemingly-benign) damping B are limited by the device's inherent (continuous-time) damping b.

On the other hand, the problem of variable-rate haptics (i.e. data update rate T_k is varying - see Fig. 1) is encountered/desired in many haptics applications, particularly when the complexity of the virtual world simulation is drastically changing from time to time (e.g. haptic interaction with a multi-dimensional deformable object with intermittent multi-point contacts [4]). Yet, perhaps, due to its relying on frequency domain concepts (e.g. [3]), with which nonlinear phenomena such as variable-rate data update are difficult to address, the Colgate's condition (1) has remained in the realm of uniform-rate haptics (i.e. $T_k = T$ for all $k \geq 0$). To this we may probably ascribe the fact that, in contrast to the uniform-rate haptics where the passivity condition (1) is readily applicable, most (or, to our knowledge, all, except [5], [6]) of the variable-rate haptics results (e.g. [7], [8]) do not enforce passivity, although this passivity is desirable in haptics in its capacity to ensure haptic interaction stability with a wide range of human-users, grips, etc.

In this paper, relying on time-domain analysis/derivation, we extend the Colgate's passivity condition (1) for the case of variable-rate haptics. More precisely, we will show that,

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Research supported in part by the National Science Foundation CAREER Award IIS-0844890.

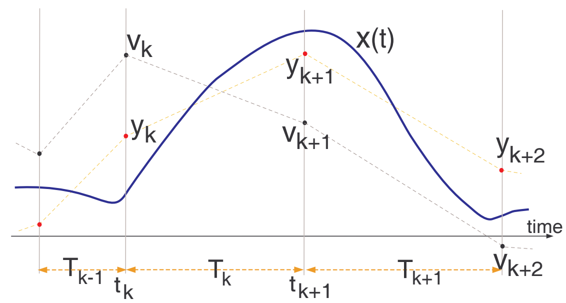

Fig. 1. Haptic device continuous position $x(t)$, and the virtual-world's discrete position y_k and velocity v_k

for the variable-rate virtual coupling and virtual wall, their passivity condition are given by:

$$b \geq \frac{B}{2}\left[1 + \left(\frac{T_k}{T_{k-1}}\right)\right] + \frac{KT_k}{2} \tag{2}$$

for all $k \geq 0$. A bit more practical version of this condition (2) can then be written as

$$b \geq \frac{B}{2}\left[1 + \lambda_{\max}\right] + \frac{KT_{\max}}{2}$$

where $\lambda_{\max} := \max_k(T_k/T_{k-1})$ and $T_{\max} := \max_k(T_k)$. Note that, if $T_k = T$ (i.e. uniform-rate), these conditions become exactly the same as the Colgate's passivity condition (1).

In this paper, we will also show that, under a condition similar to the new passivity condition (2), the recently proposed passive set-position modulation (PSPM) [5], [6], [9] can achieve passive haptic rendering while relaxing its assumption that the local device servo-loop should be much faster than the data update from the virtual environment. By doing so, this PSPM framework, together with non-iterative discrete-time passive integrator [10], can then achieve 1) passive haptic interaction even if communication is imperfect (e.g. varying-delay/packet-loss); and 2) complete separation between the device servo-loop and the virtual world simulation, although the B, K parameters of the PSPM-coupling still need to satisfy their passivity condition similar to (2).

The rest of this paper is organized as follows. We will first show respectively in Sec. II and Sec. III that the passivity conditions for the variable-rate virtual coupling and (unilateral) virtual wall are all given by (2). Passivity condition for the PSPM-based haptic rendering, which does not require the assumption that the device servo-rate is much faster than the data update rate from the virtual world, will be derived in Sec. IV. Concluding remarks with some comments on future research will be then given in Sec. V.

II. Passivity Condition for Variable-Rate Virtual Coupling

Following [1], with the sampling and zero-order-hold (ZOH), the virtual coupling law is given by: during $T_k := [t_k, t_{k+1})$,

$$\tau(t) := -b\dot{x}(t) - B\left(\frac{x_k - x_{k-1}}{T_{k-1}} - v_k\right) - K(x_k - y_k) \quad (3)$$

$$\tau_k := -B\left(v_k - \frac{x_k - x_{k-1}}{T_{k-1}}\right) - K(y_k - x_k) \quad (4)$$

where $\tau(t), \tau_k$ are respectively the virtual coupling control for the haptic device and the virtual proxy during the time-step $T_k \geq 0$, $x_k := x(t_k)$ (sampled position at t_k), $b \geq 0$ is (un-modeled) device damping, and y_k, v_k are respectively the discrete position/velocity of the virtual environment available at t_k, related by

$$v_k := \frac{y_k - y_{k-1}}{T_{k-1}}$$

i.e. backward differentiation [2]. See Fig. 1. Here, for simplicity, we assume zero initial condition with $x(t) = 0, \dot{x}(t) = 0$ for all $t \leq 0$ and $v_k = 0, y_k = 0$ for all $k \leq 0$.

Then, we want this variable-rate virtual coupling (3)-(4) to possess the following two-port hybrid (i.e. continuous-discrete) passivity [5]: $\forall N \geq 0$ and $\forall \bar{t} \in T_N := [t_N, t_{N+1})$,

$$\int_0^{\bar{t}} \tau(t)\dot{x}(t)dt + \sum_{k=0}^{N} \tau_k v_k T_{k-1} \leq c^2 \quad (5)$$

where $c \in \Re$ is a bounded constant. This hybrid passivity condition reflects the fact that, as depicted in Fig. 2, the virtual coupling (3)-(4) is a hybrid system, residing between the continuous-time haptic device and the discrete-time virtual environment and containing both discrete and continuous terms. Here, note that the term $\sum_{k=0}^{N} \tau_k v_k T_{k-1}$ in (5) is, in fact, the discrete-time energy-generation until the previous time-duration T_{N-1} (with its position increase given by $\Delta y_{k-1} = y_k - y_{k-1} = v_k T_{k-1}$). On the other hand, the term $\int_0^{\bar{t}} \tau(t)\dot{x}(t)dt$ in (5) is the continuous-time energy-generation until $\bar{t} \in T_N$. See Fig. 1.

This condition (5) implies that the maximum energy extractable from the virtual coupling by the human operator and the virtual world is bounded. If the (open-loop) haptic device is continuous-time passive (e.g. pure inertia) and also the virtual environment simulation is discrete-time passive (i.e. there is a bounded $d \in \Re$ s.t. $\sum_{k=0}^{N} f_k v_k T_{k-1} \leq d^2$ for all N, where f_k is interaction force of the virtual proxy), following the concept of controller passivity [11], [12], this condition (5) will then render the total haptic system as perceived by the human operator to be (one-port continuous-time) passive.

Now, let us consider that $\bar{t} = t_{k+1}$, i.e. reaching the end of the time-interval T_k. Then, for the passivity condition (5), we can decompose the energy generated by the virtual coupling (3)-(4) into three parts:

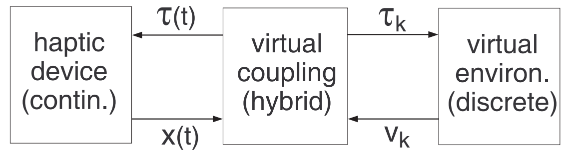

Fig. 2. Haptic systems consisting of: 1) continuous-time haptic device; 2) discrete-time virtual environment simulation; and 3) hybrid (i.e. discrete-continuous) virtual coupling

1) energy dissipation via device damping b during T_k: with $\Delta x_k := x_{k+1} - x_k$ and $x_k := x(t_k)$,

$$\text{diss}(k) := \int_{T_k} -b\dot{x}^2 dt \leq -\frac{b}{T_k}\Delta x_k^2 \quad (6)$$

which is derived from the Cauchy-Schwartz inequality (in integral form) [13], [14]:

$$\int_{T_k} f(t)g(t)dt \leq \sqrt{\int_{T_k} f^2(t)dt \times \int_{T_k} g^2(t)dt}$$

with $f(t) = 1$ and $g(t) = \dot{x}(t)$;

2) energy generation by the virtual coupling damping B:

$$\begin{aligned} E_B(k) &:= \int_{T_k} -B\left(\frac{\Delta x_{k-1}}{T_{k-1}} - v_k\right)\dot{x}dt - v_k T_{k-1} B\left(v_k - \frac{\Delta x_{k-1}}{T_{k-1}}\right) \\ &= -\frac{B}{T_{k-1}}\Delta x_{k-1}\Delta x_k + B\Delta x_k v_k - BT_{k-1}v_k^2 + Bv_k\Delta x_{k-1} \\ &\leq -\frac{B}{2T_{k-1}}(T_{k-1}v_k - \Delta x_{k-1} - \Delta x_k)^2 - \frac{B}{2}T_{k-1}v_k^2 \\ &\qquad + \frac{B}{2T_{k-1}}\Delta x_{k-1}^2 + \frac{B}{2T_{k-1}}\Delta x_k^2 \end{aligned} \quad (7)$$

using $\int_{T_k} \dot{x}(t)dt = \Delta x_k$ and $-\Delta x_k \Delta x_{k-1} \leq (\Delta x_k^2 + \Delta x_{k-1}^2)/2$; and

3) energy generation by the virtual coupling spring K: with $\Delta y_{k-1} := y_k - y_{k-1}$,

$$\begin{aligned} E_K(k) &:= \int_{T_k} -K(x_k - y_k)\dot{x}dt - v_k T_{k-1} K(y_k - x_k) \\ &= -K\Delta x_k(x_k - y_k) - K\Delta y_{k-1}(y_k - x_k) \\ &= K(-x_{k+1}x_k + x_{k+1}y_k + x_k^2 - y_k^2 + y_{k-1}y_k - y_{k-1}x_k) \\ &\leq K(-\frac{1}{2}y_k^2 + x_{k+1}y_k + \frac{1}{2}y_{k-1}^2 - y_{k-1}x_k + x_k^2 - x_{k+1}x_k) \\ &= -\frac{K}{2}(y_k - x_{k+1})^2 + \frac{K}{2}(y_{k-1} - x_k)^2 + \frac{K}{2}\Delta x_k^2 \end{aligned} \quad (8)$$

where we use $y_{k-1}y_k \leq (y_{k-1}^2 + y_k^2)/2$ to obtain the inequality.

Here, note that, similar to the passivity condition (5), the energy-generation for the device-side is during T_k (i.e. $\int_{T_k}$), while that for the virtual environment is for T_{k-1} (i.e. with $\Delta y_{k-1} = y_k - y_{k-1} = v_k T_{k-1}$).

Now, combining these (6)-(8) and adding them for $k =$

$0,1,\ldots,N$, we can then obtain: with $\bar{t} = T_{N+1}$ and $t_o = 0$

$$
\begin{aligned}
&\int_0^{\bar{t}} \tau(t)\dot{x}(t)dt + \sum_{k=0}^{N} \tau_k v_k T_{k-1} \\
&\le -\left[\frac{b}{T_o} - \frac{B}{2T_o} - \frac{K}{2}\right]\Delta x_o^2 - \sum_{k=1}^{N}\left[\frac{b}{T_k} - \frac{B}{2T_{k-1}} - \frac{B}{2T_k} - \frac{K}{2}\right]\Delta x_k^2 \\
&\quad - \sum_{k=1}^{N} \frac{B}{2}\left[\frac{(T_{k-1}v_k - \Delta x_{k-1} - \Delta x_k)^2}{T_{k-1}} + T_{k-1}v_k^2\right] \\
&\quad - \frac{K}{2}(y_N - x_{N+1})^2 - \frac{B}{2T_N}\Delta x_N^2 \qquad (9)
\end{aligned}
$$

where we also use the zero initial condition assumption. This inequality (9) then clearly shows that, with the new variable-rate passivity condition (2), the variable-rate virtual coupling (3)-(4) will be hybrid passive in the sense of (5) with $c = 0$. Here, we have $c = 0$, due to the zero initial condition assumption.

Of course, here, we have this inequality (9) with $\bar{t} = t_{N+1}$, rather than $\forall \bar{t} \in T_N = [t_N, t_{N+1})$ as required for (5). Yet, note that this inter-sample passivity (i.e. for all $\bar{t} \in T_N$) can still be deduced from the above inequality (9) (with (2)). This is because, in (9), similar to (5), the discrete-time energy generation is collected only up to T_{N-1}, which is completely independent from how we choose $\bar{t}$, as long as $\bar{t} \in T_N = [t_N, t_{N+1})$. This implies that, although $\bar{t} \neq t_{N+1}$, if $\bar{t} \in T_N$, we will still have (9) with $\Delta x_N, T_N$ and x_{N+1} in the RHS of (9) simply replaced by $x(\bar{t}) - x_{N-1}, \bar{t} - t_N$, and $x(\bar{t})$, respectively. Now, we summarize this result in the following theorem, which extends the Colgate's passivity condition [1], [2] to the case of variable-rate haptics.

Theorem 1: Variable-rate virtual coupling (3)-(4) is passive (i.e. two-port hybrid passivity (5)), if its parameters and update-rate satisfy the variable-rate passivity condition (2).

III. Passivity Condition for Variable-Rate Virtual Wall

The other well-known problem in haptics is the rendering of the virtual wall. Following [3], its rendering law can be written as: for $t \in T_k := [t_k, t_{k+1})$

$$
\tau(t) := \begin{cases} -b\dot{x}(t) - B\frac{x_k - x_{k-1}}{T_{k-1}} - K(x_k - y) & \text{if } x(t_k) \ge y \\ -b\dot{x}(t) & \text{if } x(t_k) < y \end{cases} \qquad (10)
$$

where b is the device damping, y is the boundary of the virtual wall (with the wall extending for $x \ge y$), $x_k = x(t_k)$, and B, K are the damping and spring constants of the virtual wall. Here, the key difference (and challenge) of this virtual wall rendering (10) from the virtual coupling (3)-(4) is its switching nature to realize the virtual wall's unilateral characteristics (i.e. wall only reacts when $x(t) \ge y$). When the data update rate is uniform, again, the Colgate's condition (1) ensures passivity of this virtual wall rendering [3], [14], which, here, we will extend for the case of variable-rate haptic rendering.

Since the virtual wall boundary is fixed at y, instead of the two-port passivity (5), here, we want to enforce one-port (continuous-time) passivity: $\forall N \ge 0$ and $\forall \bar{t} \in T_N := [t_N, T_{N+1})$,

$$
\int_0^{\bar{t}} \tau(t)\dot{x}(t)dt \le c^2 \qquad (11)
$$

where $c \in \Re$ is a bounded constant. This condition (11), together with the open-loop passive haptic device and the controller passivity concept [11], [12], implies then that the virtual wall, rendered by (10), will be perceived as being passive by its (continuous-time) human user, thereby, enhancing interaction stability/safety.

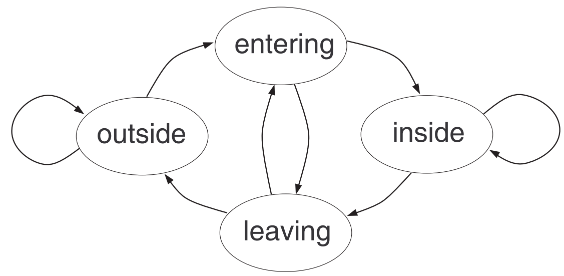

Fig. 3. Mode transition of the virtual wall rendering

Let us consider the four possible modes of the virtual wall haptic rendering: 1) **entering** the wall $x_k < y$, yet $x_{k+1} \ge y$; 2) staying **inside** the wall $x_k, \ldots, x_{k+p_1} \ge y$; 3) **leaving** the wall $x_k \ge y$, but $x_{k+1} < y$; and 4) staying **outside** the wall $x_k, \ldots, x_{k+p_2} < y$, where $p_1, p_2 \ge 1$ denote the duration of the motion inside/outside of the wall, and $x_k := x(t_k)$. Any interaction with the virtual wall can then be thought of as a combination of these four modes as shown by the directional mode transition graph in Fig. 3.

Let us first analyze the energy generated by the virtual wall rendering law (10) during T_k for each of the four modes (i.e. $\int_{T_k} \tau(t)\dot{x}(t)dt$):

1) energy dissipation for the outside mode is:

$$
E_{\text{out}}(k) := \int_{T_k} -b\dot{x}^2(t)dt \le -\frac{b}{T_k}\Delta x_k^2 \qquad (12)
$$

where we use the Cauchy-Schwartz inequality similar to (6);

2) energy generation during the entering mode $E_{\text{enter}}(k)$ is the same as $E_{\text{out}}(k)$ in (12), since this entering is not detected during this T_k with $x_k < y$ (i.e. the second line of (10) used);

3) energy generation for the inside mode is: using the first line of (10) and similar to (7),

$$
\begin{aligned}
E_{\text{in}}(k) :&= \int_{T_k} \left(-b\dot{x} - B\frac{\Delta x_{k-1}}{T_{k-1}} - K(x_k - y)\right)\dot{x}dt \\
&\le -\frac{b}{T_k}\Delta x_k^2 + \frac{B}{2T_{k-1}}\Delta x_k^2 + \frac{B}{2T_{k-1}}\Delta x_{k-1}^2 \\
&\quad - \frac{K}{2}(x_{k+1} - y)^2 + \frac{K}{2}(x_k - y)^2 + \frac{K}{2}\Delta x_k^2 \qquad (13)
\end{aligned}
$$

where we use the fact that

$$
K(x_k - y)\Delta x_k = \frac{K}{2}(x_{k+1} - y)^2 - \frac{K}{2}(x_k - y)^2 - \frac{K}{2}\Delta x_k^2
$$

and $\int_{T_k} \dot{x} dt = \Delta x_k$;

4) energy generation during the leaving mode $E_{\text{leave}}(k)$ is the same as this $E_{\text{inside}}(k)$ in (13), since this leaving is not detected with $x_k \geq y$, that is, the first line of (10) is used during T_k for this leaving mode.

Here, the fact that $E_{\text{out}}(k) = E_{\text{enter}}(k)$ (or $E_{\text{in}}(k) = E_{\text{leave}}(k)$, resp.) implies that, energetically, the outside and the entering modes (or the inside and the leaving modes, resp.) are indistinguishable.

Now, let us consider the energy generation by the virtual wall rendering (10) during a general scenario of interaction with it. For simplicity, let us assume that the device motion starts outside of the virtual wall (i.e. $x(0) < y$). We can then denote the time-interval of the initial outside modes by $T_o, T_1, ..., T_p$, $p \geq 0$. Then, during this time, the virtual wall is passive, since, from (12), we have

$$\int_0^{\bar{t}} \tau \dot{x} dt \leq -\sum_{k=0}^{p-1} \frac{b}{T_k} \Delta x_k^2 - \int_{t_{p-1}}^{\bar{t}} b \dot{x}^2 dt \leq 0 \qquad (14)$$

for all $\bar{t} \in T_p := [t_p, t_{p+1})$. This passivity is also true, even if the last time-interval T_p is switched to the entering mode, since $E_{\text{enter}}(k) = E_{\text{out}}(k)$.

Suppose then that, this segment of the outside mode motion with its last mode replaced by the entering mode (i.e. T_p is now the entering mode) is followed by a segment of the inside mode motion $T_{p+1}, ..., T_q$, $q \geq p+1$, Then, for all $\bar{t} \in T_q := [t_q, t_{q+1})$, combining the above inequality and using (13), we can show that:

$$\begin{aligned} \int_0^{\bar{t}} \tau \dot{x} dt \leq & -\sum_{k=0}^{p-1} \frac{b}{T_k} \Delta x_k^2 - \left[\frac{b}{T_p} - \frac{B}{2T_p} - \frac{K}{2} \right] \Delta x_p^2 \\ & - \sum_{k=p+1}^{q-1} \left[\frac{b}{T_k} - \frac{B}{2T_{k-1}} - \frac{B}{2T_k} - \frac{K}{2} \right] \Delta x_k^2 \\ & - \left[\frac{b}{\bar{T}_q} - \frac{B}{2T_{q-1}} - \frac{K}{2} \right] \Delta \bar{x}_q^2 - \frac{K}{2} (x(\bar{t}) - y)^2 \end{aligned} \qquad (15)$$

where for the first line, we use the fact that $(x_{p+1} - y)^2 \leq \Delta x_p^2$, and the last line is obtained by using (13) with its $T_q, \Delta x_q$ and x_{k+1} replaced by $\bar{T}_q := \bar{t} - t_q, \Delta \bar{x}_q := x(\bar{t}) - x_q$ and $x(\bar{t})$, respectively. Here, the first term of the last line will be zero, if $\bar{t} = t_q$.

This inequality (15) implies that, even with variable-rate data update, the virtual wall is passive, if the new passivity condition (2) is enforced, which will then make the RHS of this inequality (15) less than zero. Here, note that the first term of the last line of (15) is also non-positive with the condition (2). This virtual wall passivity will still be true, even if the last time-interval T_q of this outside motion segment $T_{p+1}, ..., T_q$ is replaced by the leaving mode, since $E_{\text{leave}}(k) = E_{\text{in}}(k)$. For instance, if $q = p+1$ (e.g. entering mode immediately followed by leaving mode), we then have: with (2),

$$\begin{aligned} \int_0^{\bar{t}} \tau \dot{x} dt \leq & -\sum_{k=0}^{p-1} \frac{b}{T_k} \Delta x_k^2 - \left[\frac{b}{T_p} - \frac{B}{2T_p} - \frac{K}{2} \right] \Delta x_p^2 \\ & - \left[\frac{b}{\bar{T}_{p+1}} - \frac{B}{2T_p} - \frac{K}{2} \right] \Delta \bar{x}_{p+1}^2 - \frac{K}{2} (x(\bar{t}) - y)^2 \leq 0 \end{aligned}$$

for all $\bar{t} \in [t_{p+1}, t_{p+2})$, where $\bar{T}_{p+1} = \bar{t} - t_{p+1}$, and $\Delta \bar{x}_{p+1} = x(\bar{t}) - x_{p+1}$. Here, again, the first term of the last line is zero, if $\bar{t} = t_{p+1}$.

Thus, so far, we have proved that the virtual wall (10) under the new passivity condition (2) is passive in the sense of (11) with $c = 0$, if we start from the outside/entering-modes and end up with the inside/leaving-modes. This, however, also implies that, whatever modes the device will enter in after this moment, the virtual wall will still be passive in the sense of (11) with $c = 0$. This is because: 1) if the device keeps staying in the outside mode, by concatenating (14) to (15), we will still have passivity (11) with $c = 0$; and 2) if the device again enters in the entering mode, and then, switches to either the leaving or inside mode, passivity is still be enforced, since, by concatenating (15) again to (15), the passivity condition (11) with $c = 0$ is still granted. Here, notice that we also allow immediate switching from the leaving mode to the entering mode, since we can have $p = 0$ and $q = p+1$ in (15). Although not presented here for brevity, a similar procedure can also be used for the case where we start either from the inside mode or the leaving mode. In this case, we may then have non-zero constant c instead of $c = 0$ for (11) due to non-zero initial condition. We now summarize this result in the following theorem.

Theorem 2: Variable-rate virtual wall rendering (10) is passive in the sense of (11), if its parameters and data update-rate satisfy the variable-rate passivity condition (2).

IV. Passivity Condition for PSPM-Coupling

Virtual coupling [1], and its extension to variable-rate haptics as proposed here in Sec. II, enable us to connect the haptic device and the virtual world (i.e. virtual proxy) with a very simple implementation architecture, and, now, even for variable-rate data-update. However, its usefulness (and its theoretical justification) starts being broken down, when the communication between the device and the virtual world is not perfect. One example for this is haptic interaction over the Internet [15], [16], [17], where the device and the virtual world are connected over the Internet with varying-delay and packet-loss (e.g. molecular simulation in a remote super-computer over the Internet). The other example is to have/allow some computation delay for the virtual environment simulation, particularly when it is complex (e.g. multi-dimensional deformable object with multi-point contacts [4]).

To address this communication issue, the passive set-position modulation (PSPM) has been recently proposed and demonstrated for the Internet teleoperation and variable-rate haptics [5], [6], [9], [10]. Here, we briefly review this PSPM and propose a passivity condition similar to (2), with which

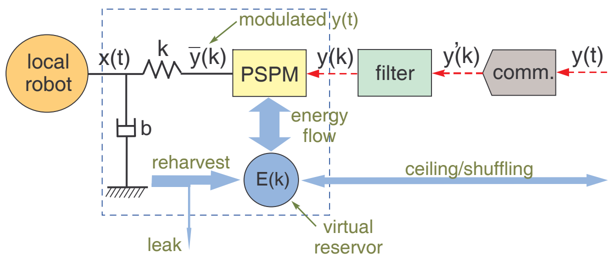

Fig. 4. Energetics of passive set-position modulation (PSPM)

a certain assumption of this PSPM can be relaxed. For more details on PSPM, see [9], [10], [5], [6].

Similar to the virtual coupling (3), the PSPM uses the following local spring control with damping injection for the haptic device:

$$\tau(t) := -b\dot{x}(t) - B\dot{x}(t) - K(x(t) - \bar{y}_k) \qquad (16)$$

where b is the (un-modeled) device damping, B, K are the PSPM damping and spring gains, $x(t)$ is the device position, and $\bar{y}_k$ is the **modulated** version of the virtual proxy's (raw) set-position y_k directly received from the virtual world via a possibly imperfect communication channel (e.g. Internet). See Fig. 1.

The key idea in the development of the PSPM is to recognize that, with both the open-loop haptic device and the continuous-part of the PSPM-coupling (16) being passive, if we use the raw set-position signal y_k received directly from the imperfect communication for (16), the only reason why the passivity may be compromised is the spring energy jump (stored in K) due to the switching of y_k (see also (20)). To avoid this possibly passivity-breaking spring energy jump, at each data reception time, the PSPM modulates this (raw) set-position signal y_k to $\bar{y}_k$ in such a way that $\bar{y}_k$ and y_k are as close with each other as possible (i.e. performance), yet, only to the extent permissible by the available energy in the system (i.e. passivity).

More precisely, at each time t_k of receiving y_k (see Fig. 1), the PSPM computes $\bar{y}_k$ by solving the following optimization problem:

$$\min_{\bar{y}_k} \quad ||y_k - \bar{y}_k|| \qquad (17)$$

$$E_k = E_{k-1} + \Delta E_k + D_{k-1}^{\min} - \Delta\bar{P}_k \geq 0 \qquad (18)$$

where

$$\Delta\bar{P}_k := \frac{K}{2}(x_k - \bar{y}_k)^2 - \frac{K}{2}(x_k - \bar{y}_{k-1})^2$$

is the spring-energy jump at t_k with the modulated set-position switches from $\bar{y}_{k-1}$ to $\bar{y}_k$; $E_k \geq 0$ is the virtual energy reservoir; $\Delta E_k > 0$ is the energy-shuffling term received from the virtual world; and $D_{k-1}^{\min} \geq 0$ is the estimation of the (passivity-enforcing) damping dissipation via B during the previous time-step T_{k-1} s.t.

$$\int_{T_{k-1}} -B\dot{x}^2(t)dt \leq -\frac{B}{T_{k-1}}\Delta x_{k-1}^2 =: -D_{k-1}^{\min}$$

from the Cauchy-Schwartz inequality (see also (6) and (12)).

From (17)-(18), we can now see that the PSPM pushes $\bar{y}_k$ as close to y_k as possible, yet, only to the extent that the (possibly passivity-breaking) spring energy jump for doing this (i.e. $\Delta\bar{P}_k$) is allowed (or can be absorbed) by the (passivity-enforcing) available energy in the system (i.e. $E_{k-1} + \Delta E_k + D_{k-1}^{\min}$ in (18)). Note also that the term $D_{k-1}^{\min}$ in (18) re-harvests portion of (otherwise-wasted) energy dissipation via (artificial) control damping B, thereby, improving energy-efficiency and, thus, performance (i.e. more energy to push $\bar{y}_k$ closer to y_k). We can also incorporate energy shuffling from the device to the virtual environment, similar to ΔE_k. See [9], [10], [5], [6] for more details. The energetics of this PSPM is illustrated in Fig. 4.

This PSPM-framework, unlike the virtual coupling technique, can be used even when the communication medium is imperfect (e.g. Internet), since it does not impose any condition on y_k other than its being a discrete sequence. A particularly interesting application of this PSPM is when it is combined with the recently proposed non-iterative discrete-time passive integrator [10] for haptic rendering [5]: from the passivity of this passive integrator, 1) exactly the same PSPM ((16), (17) and (18)) can be used for the virtual world (with a discrete-version of (16)) to achieve passive haptic interaction over Internet-like communication; and 2) as originally envisioned in [1], the virtual world simulation and the device servo-loop can be completely separated from each other.

Yet, one assumption for the PSPM is still lingering: as can be seen from (16), the device servo-loop should be much faster than the reception rate of y_k to consider $x(t), \dot{x}(t)$ as continuous-time signals. This assumption is a bit unsettling, particularly for some high-performance haptics applications where fast update of y_k is necessary (e.g. z-width [18]). Here, we aim to relax this assumption.

For this, instead of (16), let us consider the following (sampled-data) PSPM coupling: for $t \in T_k := [t_k, t_{k+1})$,

$$\tau(t) := -b\dot{x}(t) - B\frac{x_k - x_{k-1}}{T_{k-1}} - K(x_k - \bar{y}_k) \qquad (19)$$

where $x_k := x(t_k)$, $\bar{y}_k$ is the modulated set-position at t_k. Then, similar to (13), we can show that:

$$\int_{T_k} \tau(t)\dot{x}(t)dt \leq -\frac{b}{T_k}\Delta x_k^2 + \frac{B}{2T_{k-1}}\Delta x_k^2 + \frac{B}{2T_{k-1}}\Delta x_{k-1}^2 - \frac{K}{2}(x_{k+1} - \bar{y}_k)^2 + \frac{K}{2}(x_k - \bar{y}_k)^2 + \frac{K}{2}\Delta x_k^2$$

Using this, we can further compute the energy generated by the PSPM coupling (19) until $\bar{t} \in T_N := [T_N, T_{N+1})$ s.t.: with $t_o = 0$,

$$\int_0^{\bar{t}} \tau\dot{x}dt \leq \left[\frac{b}{T_o} - \frac{B}{2T_o} - \frac{B}{T_o} - \frac{K}{2}\right]\Delta x_o^2 - \sum_{k=1}^{N}\left[D_{k-1}^{\min} - \Delta\bar{P}_k\right] - \sum_{k=1}^{N-1}\left[\frac{b}{T_k} - \frac{B}{2T_{k-1}} - \frac{B}{2T_k} - \frac{B}{T_k} - \frac{K}{2}\right]\Delta x_k^2 - \left[\frac{b}{T_N} - \frac{B}{2T_{N-1}} - \frac{K}{2}\right](x(\bar{t}) - x_N)^2 - \frac{K}{2}(x(\bar{t}) - \bar{y}_N)^2 \qquad (20)$$

where we use the definitions of $D_{k-1}^{\min}$ and $\Delta\bar{P}_k$ as given above, $\bar{T}_N := \bar{t} - t_N$, and set the B-term in (19) is zero during T_o (i.e. $B(x_0 - x_{-1})/T_{-1} = 0$). Here, the second last term of this inequality is zero if $\bar{t} = t_N$. Also, note that B/T_o on the first line and B/T_k in the second line are due to the energy re-harvesting via the term $D_{k-1}^{\min}$ in (18).

Now, suppose that the following condition is satisfied

$$b \geq B + \frac{B}{2}\left[1 + \left(\frac{T_k}{T_{k-1}}\right)\right] + \frac{KT_k}{2} \tag{21}$$

where the first B in the RHS is to address the energy re-harvesting via $D_{k-1}^{\min}$ in (18). Then, we can show that: $\forall N \geq 0$ and $\forall \bar{t} \in [t_N, t_{N+1})$,

$$\int_0^{\bar{t}} \tau \dot{x} dt \leq E_o - E_N + \sum_{k=1}^{N} \Delta E_k \tag{22}$$

where we use

$$D_{k-1}^{\min} - \Delta\bar{P}_k = E_k - E_{k-1} - \Delta E_k$$

from the PSPM algorithm (18). Thus, if the sum of the the energy shuffling terms (i.e. $\sum \Delta E_k$ with $\Delta E_k > 0$) is bounded (e.g. passive virtual world with no data-duplication and bilateral energy shuffling [5]), the RHS of (22) will be bounded, implying that the PSPM coupling (19) with (17)-(18) be passive, without requiring the assumption that the data reception rate of y_k be much slower than the device servo-rate (for controlling $\dot{x}(t), x(t)$).

Note that, here, if we do not use the energy re-harvesting (i.e. no $D_{k-1}^{\min}$ in (18), thus, no B/T_o and B/T_k in (20)), the PSPM passive condition (22) will be the same as the previous variable-rate passivity condition (2), although performance may degrade without this re-harvesting (i.e. less energy to push $\bar{y}_k$ to y_k). In this sense, the first B in (21) may be thought of as the price paid for passifying the communication imperfectness, on the top of passifying the sampled-data spring-damper control B, K by (2). Now, the following theorem summarize our discussion so far on the passivity of the PSPM-coupling.

Theorem 3: The (sampled-data) PSPM-coupling (19) with (17)-(18) is passive in the sense of (22), if its parameters and data update-rate satisfy the variable-rate passivity condition (21). Moreover, if the energy re-harvesting is not used, its passivity condition is given by (2).

V. Summary and Future Works

In this paper, we extend the Colgate's passivity conditions for the virtual wall and the virtual coupling to the case of variable-rate data update. We also present a similar variable-rate passivity condition for the haptic rendering based on the recently proposed passive set-position modulation (PSPM).

Simulation and experimental study on the theoretical results proposed in this paper is under way in our laboratory. Since the presented results are based on time-domain analysis/derivation, we believe that they may be combined with the ideas of [14] to address sensor quantization. The concept of two-port hybrid passivity (originally proposed in [5], [6]), to our knowledge, is novel in haptics and we believe it may shed new insights on, or be useful for, some problems.

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