The 2009 IEEE/RSJ International Conference on
Intelligent Robots and Systems
October 11-15, 2009 St. Louis, USA

Projection-Based Control of Parallel Manipulators

Farhad Aghili

Abstract— **This paper presents tracking and set point controllers for parallel mechanism based on the notion of a projection operator. The controller reported here works whether the system is overactuated or not; plus one does not need to derive the minimal-order dynamics model of the system. Since the dimension of projection matrix is fixed, the projection-based controller does not need to change its structure whenever the mechanical system changes its topology or number of degrees of freedom. Moreover, the derivation of the projection-based controller seems to be simpler than the inverse dynamics controller derived using Lagrange-D'Almbert formulation. This is because the structure of the former controller can be obtained from the Jacobian matrix of the constraints, which, in turn, can be deduced from the linkage geometry. The stability of the projection-based controllers are rigourously proved, while the condition for the controllability of parallel manipulators is also derived in detail. Finally, experimental results are appended.**

I. Introduction

The advantages of parallel manipulators such as their high stiffness, bandwidth and accuracy capacity make them more suitable than serial manipulators for certain industrial applications. Moreover, parallel manipulators have great payload capacity because not only their actuators are installed at the base (resulting in low effective inertia), but also much of the link gravity load appears in the form of constraint forces and hence actuation then takes less effort. However, in addition to their relatively low range of motion, parallel manipulators have a complex dynamics because of the existence of closed kinematic chains and passive joints that makes control design a challenging task.

Control algorithms reported in the literature for parallel manipulators can be classified in two categories: Those which are not based on the dynamics model of the system, and the ones that are, [1]–[7]. Parallel manipulators can be transformed into an open-chain tree-structure system by cutting all closed links at several points [1], [4], [8]. Therefore, in essence, control of parallel manipulators is tantamount to control of constrained multibody systems where the conventional method such as Lagrange multipliers [8], [9] can be applied. What makes the problem of controller design more challenging is that, if the number of independent constraints is larger than the number of passive joints, the system will become overactuated. On the other hand, when the numbers of independent constraints and passive joints are equal, the parallel manipulator system has as many controllable active joints as DOFs. Therefore, the controller must be able to handle both these cases.

F. Aghili is with the Spacecraft Engineering Division of the Space Technologies, Canadian Space Agency, Saint-Hubert, Quebec J3Y 8Y9, Canada farhad.aghili@space.gc.ca

A dynamic formulation of parallel manipulators with redundant actuators using the D'Alembert's principle is described by Nakamura *et al.* [1]. Derivation of the D'Alembert formulation was simplified in [4] for control purposes. Three controllers that took into account the overactuation were reported in [4] for redundantly actuated parallel manipulators. Two of these controllers are model-based, and the other is a simple PD controller. However, the stability of these controllers was not analyzed.

In this work, we propose tracking and set point controllers for parallel manipulator based on the notion of a projection operator [3], [10] that are proven to be exponentially stable. The controller reported here works whether the system is overactuated or not; plus one does not need to derive the minimal-order dynamics model of the system. Since the dimension of projection matrix is fixed, the projection-based controller does not need to change its structure whenever the mechanical system changes its topology or number of degrees of freedom. Moreover, the derivation of the projection-based controller seems to be simpler than the Lagrange-D'Almbert formulation [1], [4] because the structure of the former controller can be obtained from the Jacobian matrix of the constraints, which, in turn, can be deduced from the linkage geometry. The stability of the projection-based controllers are rigourously proved, while the condition for the controllability of parallel manipulators is also derived in detail.

This paper is organized as follows: Section II presents the optimal kinematic relation between the tangential component of entire torque vector and the vector of actuated joints using the notion of projection operator. The tangential component of the torque vector is determined in sections III-A and III-B for tracking control and set point control of parallel manipulators, respectively. A dynamic estimator to estimate the states of passive joints is described in Section IV. Then, the condition for controllability is derived in Section V. Finally, Section VI reports some experimental results.

II. Modelling of Parallel Manipulator Based on the Notion of Projection

In principle, parallel manipulators are constrained multibody systems. By cutting each loop of a parallel mechanism at one of the unactuated joints, the parallel manipulator can be transformed into a constrained mechanical system consisting of an open-loop tree-structure coupled by a series of algebraic constraint equations [1], [8]. A systematic method to create the dynamic model of a parallel manipulator in the form of an open-loop tree structure can be found in [1]. In this work, we assume that the model of the parallel

978-1-4244-3804-4/09/$25.00 ©2009 IEEE

mechanism in the form of a constrained multibody system has been already developed through a method similar to what is described in [1], [8].

The generalized coordinates of the open-loop tree-structure mechanism

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{q}_a \\ \boldsymbol{q}_p \end{bmatrix}$$

contain n *active joints* and m *passive, joints* denoted by $\boldsymbol{q}_a \in \mathbb{R}^n$ and $\boldsymbol{q}_p \in \mathbb{R}^m$, respectively. Unlike the active joints, the passive joints are not instrumented with any sensor or actuator. There are r constraint equations corresponding to the cut multi-loop closed-chain. The constraint equations at the velocity level are described by

$$\boldsymbol{A}_q \dot{\boldsymbol{q}}_a + \boldsymbol{A}_p \dot{\boldsymbol{q}}_p = \boldsymbol{0}, \tag{1}$$

where unless $n = r$ or $m = r$, neither of these two matrices in the above is square. The above equation can be written in a more compact form as

$$\boldsymbol{A}\dot{\boldsymbol{q}} = \boldsymbol{0} \ \in \mathbb{R}^r, \tag{2}$$

where $\boldsymbol{A} \equiv \begin{bmatrix} \boldsymbol{A}_a & \boldsymbol{A}_p \end{bmatrix}$. Note that (2) may or may not contain redundant constraint equations.

Now, let us assume the following holds:

Assumption 1: The Jacobian $\boldsymbol{A}_p$ remains full-rank.

Then, by making use of (1), one can *uniquely* compute the value of $\dot{\boldsymbol{q}}_p$ from the measured value of $\dot{\boldsymbol{q}}_a$ by

$$\dot{\boldsymbol{q}}_p = \boldsymbol{Q}\dot{\boldsymbol{q}}_a, \quad \text{where} \quad \boldsymbol{Q} \triangleq -\boldsymbol{A}_p^+ \boldsymbol{A}_a, \tag{3}$$

with $\boldsymbol{A}_p^+$ being a right inverse of $\boldsymbol{A}_p$, i.e., $\boldsymbol{A}_p \boldsymbol{A}_p^+ = \boldsymbol{1}$.

The full-order dynamics equations of a constrained mechanical system can then be derived as

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} - \boldsymbol{\tau}^c \tag{4}$$

which apparently is subject to the constraint equation (2), where vector $\boldsymbol{\tau} \in \mathbb{R}^{n+m}$ represents the generalized force in the active and passive joints; $\boldsymbol{\tau}^c$ represents the generalized constraint force associated to the Lagrange multipliers; $\boldsymbol{M}(\boldsymbol{q})$ is the $(n+m) \times (n+m)$ inertia matrix; $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$ is the $(n+m)$-dimensional vector containing the Coriolis and centrifugal terms; and $\boldsymbol{g}(\boldsymbol{q}) \in \mathbb{R}^{n+m}$ is the gravity torque.

Now, given the $r \times (n+m)$ constraint Jacobian matrix $\boldsymbol{A}$, we can uniquely define *symmetric* matrix $\boldsymbol{P} \in \mathbb{R}^{(n+m)\times(n+m)}$, the *null-space orthogonal projector* of $\boldsymbol{A}$, as [11]

$$\boldsymbol{P} \triangleq \boldsymbol{1}_{n+m} - \boldsymbol{A}^+ \boldsymbol{A} \tag{5}$$

where $\boldsymbol{1}_{n+m}$ is the $(n+m) \times (n+m)$ identity matrix. Because $\boldsymbol{P}$ is an orthogonal projection onto the null-space of the Jacobian—also known as the *tangent space* of the constraint manifold—any vector in $\mathcal{N}(\boldsymbol{A})$ is projected onto itself, whereas any vector perpendicular to the tangent space lies in the null-space of $\boldsymbol{P}$. The vector $\dot{\boldsymbol{q}}$ of generalized velocities belongs to the former group as $\boldsymbol{A}\dot{\boldsymbol{q}} = \boldsymbol{0}$, and the vector of constraint generalized forces $\boldsymbol{\tau}^c$ belongs to the latter because $\forall\, \dot{\boldsymbol{q}} \in \mathcal{R}(\boldsymbol{P}) \equiv \mathcal{N}(\boldsymbol{A}),\ \dot{\boldsymbol{q}}^T \boldsymbol{\tau}^c \equiv \boldsymbol{0}$. In other words, these two relations hold:

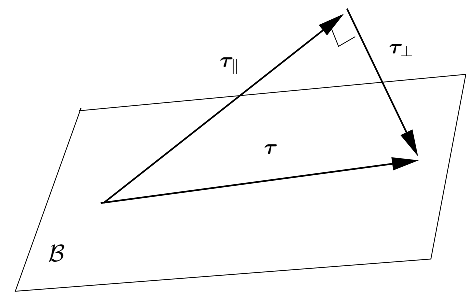

Fig. 1. The generalized torque is brought to subspace $\mathcal{B}$ by adding the impotent torque component $\boldsymbol{\tau}_{\perp}$.

$$\boldsymbol{P}\dot{\boldsymbol{q}} = \boldsymbol{P}^T \dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}, \quad \text{and} \quad \boldsymbol{P}\boldsymbol{\tau}^c = \boldsymbol{0}. \tag{6}$$

Hence, premultiplying (4) by $\boldsymbol{P}$, one can eliminate $\boldsymbol{\tau}^c$ from the set of equations:

$$\boldsymbol{P}\boldsymbol{M}\ddot{\boldsymbol{q}} = \boldsymbol{P}\big(\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q})\big). \tag{7}$$

Moreover, the vector of generalized forces $\boldsymbol{\tau}$ can be decomposed into two components denoted by subscripts $_{\|}$ and $_{\perp}$, lying in the orthogonal subspaces the tangent space $\mathcal{N}(\boldsymbol{A})$ and the null-space of $\boldsymbol{P}$, respectively:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_{\|} + \boldsymbol{\tau}_{\perp}. \tag{8}$$

Because $\boldsymbol{\tau}_{\perp} \in \mathcal{N}^{\perp}(\boldsymbol{A})$ and the constrained motion occurs in $\mathcal{N}(\boldsymbol{A})$, by definition, this component of the actuation generalized forces does not contribute to the motion of the system [10].

To keep the actuation torque to a minimum, the control law must specify a control torque $\boldsymbol{\tau}_{\|}$ that lies in the constraint tangent plane. However, the control torque thus computed may not actually be realizable in the system as it may need actuation in the unactuated joints, i.e., $\boldsymbol{\tau}_{\|}$ may have non-zero values among its last m entries. Because the passive joints are unactuated, the vector of the generalized forces should contain as many zeros as the number of the passive joints. There are n single-DOF active joints and m two-DOF passive joints. Thus, the generalized input force will be of this form:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_{\|} + \boldsymbol{\tau}_{\perp} = \begin{bmatrix} \boldsymbol{\tau}_a \\ \boldsymbol{0} \end{bmatrix} \tag{9}$$

where vector $\boldsymbol{\tau}_a \in \mathbb{R}^n$ represents the actuation torque, applied at the active joints. This implies that any admissible generalized force satisfies

$$\boldsymbol{B}\boldsymbol{\tau} = \boldsymbol{\tau}, \quad \text{and} \quad \boldsymbol{\tau} \in \mathcal{B} \triangleq \mathcal{R}(\boldsymbol{B}), \tag{10}$$

in which orthogonal projection $\boldsymbol{B}$ onto the *actuator space* $\mathcal{B}$ is defined as

$$\boldsymbol{B} \triangleq \begin{bmatrix} \boldsymbol{1}_n & \boldsymbol{0}_{n\times m} \\ \boldsymbol{0}_{m\times n} & \boldsymbol{0}_{m\times m} \end{bmatrix}. \tag{11}$$

The control law (23) produces control torques that are completely in the tangent space. However, these torques may not necessarily be admissible, meaning that they may not lie in the range of $\boldsymbol{B}$. Therefore, we need to modify the motion control law (23) so as to fulfill the condition in (10). If $\mathcal{N}(\boldsymbol{A}) \subseteq \mathcal{R}(\boldsymbol{B})$, then (10) is automatically satisfied by choosing $\boldsymbol{\tau} = \boldsymbol{\tau}_{\|}$, where $\boldsymbol{\tau}_{\|}$ is obtained from (23). Otherwise, we need to add an $\mathcal{N}^{\perp}$ component, say $\boldsymbol{\tau}_{\perp}$, to $\boldsymbol{\tau}_{\|}$ so that $\boldsymbol{\tau} = \boldsymbol{\tau}_{\|} + \boldsymbol{\tau}_{\perp}$ lie in $\mathcal{B}$; see Fig. 1. Note that, since $\boldsymbol{\tau}_{\perp}$ does not affect the system motion at all, the motion tracking performance of the controller is preserved.

Pre-multiplying both sides of (9) by $\boldsymbol{P}$, we arrive at

$$\boldsymbol{\tau}_{\|} = \begin{bmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\tau}_a \\ \boldsymbol{0}_m \end{bmatrix} = \boldsymbol{P}_1 \boldsymbol{\tau}_a, \tag{12}$$

in which the projection matrix has been partitioned into submatrices $\boldsymbol{P}_1 \in \mathbb{R}^{(n+m)\times n}$ and $\boldsymbol{P}_2 \in \mathbb{R}^{(n+m)\times m}$.

Remark 1: The two submatrices $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ can be used to select the $\dot{\boldsymbol{q}}_a$ and $\dot{\boldsymbol{q}}_p$ parts of $\dot{\boldsymbol{q}}$:

$$\dot{\boldsymbol{q}}_a = \boldsymbol{P}_1^T \dot{\boldsymbol{q}} \quad \text{and} \quad \dot{\boldsymbol{q}}_p = \boldsymbol{P}_2^T \dot{\boldsymbol{q}} \tag{13}$$

Now given $\boldsymbol{\tau}_{\|}$, equation (12) will have at least one solution for $\boldsymbol{\tau}_a$ if

$$\mathcal{N}(\boldsymbol{A}) \subseteq \mathcal{R}(\boldsymbol{P}_1). \tag{14}$$

In that case, there is a $\boldsymbol{\tau}_a$ that can produce the generalized torque control $\boldsymbol{\tau}_{\|}$. The minimum-norm solution can be obtained using the pseudo-inverse of $\boldsymbol{P}_1$:

$$\boldsymbol{\tau}_a = \boldsymbol{P}_1^+ \boldsymbol{\tau}_{\|} \qquad \Rightarrow \qquad \|\boldsymbol{\tau}_a\| \;\rightarrow\; \min. \tag{15}$$

One can readily verify that the generalized torque generated by (15),

$$\boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{P}_1^+ \boldsymbol{\tau}_{\|} \\ \boldsymbol{0} \end{bmatrix},$$

satisfies $\boldsymbol{P}\boldsymbol{\tau} = \boldsymbol{\tau}_{\|}$ meaning that the tangential component of the admissible torque vector has not changed.

A. Actuation Saturation

If the actuator torque limitation is taken into account, then the problem of finding optimal actuator torque can be formulated by

$$\min \quad \|\boldsymbol{\tau}_a\| \tag{16a}$$

$$\text{subject to:} \quad \boldsymbol{h} = \boldsymbol{P}_1 \boldsymbol{\tau}_a - \boldsymbol{\tau}_{\|} = \boldsymbol{0} \tag{16b}$$

$$\boldsymbol{c} = |\boldsymbol{\tau}_a| - \boldsymbol{\tau}_{\max} \leq \boldsymbol{0} \tag{16c}$$

where $|\boldsymbol{\tau}_a| \triangleq \text{col}(|\tau_{a_1}|, \cdots, |\tau_{a_n}|)$ and $\boldsymbol{\tau}_{\max} \triangleq \text{col}(\tau_{\max_1}, \cdots, \tau_{\max_n})$. The above *quadratic optimization program* with r equality constrains (16b) and n inequality constraints (16c) can be solved by minimizing the quadratic function over a polyhedron [12]. Clearly, in the absence of the inequality constraints (16c), the optimal solution coincides with the pseudo-inverse solution (15).

III. Projection-Based Controllers

The following sections present tracking control and set-point control of parallel manipulator based on the projection approach.

A. Tracking Control Using Inverse Dynamics

The number of independent generalized coordinates of the system is the DOF of the system $d = n + m - r'$, where $r' \leq r$ is the number of independent constraints. This means that one can control the constrained mechanical system by only controlling an independent set $\boldsymbol{x}(\boldsymbol{q}) \in \mathbb{R}^k$ of the generalized coordinates, where

$$k \leq d.$$

Now, differentiation of the given function $\boldsymbol{x}(\boldsymbol{q})$ with respect to time yields

$$\dot{\boldsymbol{x}} = \boldsymbol{\Lambda}\dot{\boldsymbol{q}}, \quad \text{and} \quad \ddot{\boldsymbol{x}} = \dot{\boldsymbol{\Lambda}}\dot{\boldsymbol{q}} + \boldsymbol{\Lambda}\ddot{\boldsymbol{q}} \tag{17}$$

where $\boldsymbol{\Lambda} = \partial \boldsymbol{x}(\boldsymbol{q})/\partial \boldsymbol{q} \in \mathbb{R}^{n\times k}$. Since vector $\boldsymbol{x}(\boldsymbol{q})$ constitutes a set of independent functions, the Jacobian matrix $\boldsymbol{\Lambda}$ must be of full rank.

Now, we propose the *projected inverse-dynamics control* law as follow

$$\boldsymbol{\tau}_{\|} = \boldsymbol{P}\big(\boldsymbol{C}\dot{\boldsymbol{q}} + \boldsymbol{g}\big) + \boldsymbol{P}\boldsymbol{M}\boldsymbol{\Lambda}^+\big(-\dot{\boldsymbol{\Lambda}}\dot{\boldsymbol{q}} + \ddot{\boldsymbol{x}}_d + \boldsymbol{K}_D\dot{\boldsymbol{e}} + \boldsymbol{K}_P\boldsymbol{e}\big), \tag{18}$$

where $\boldsymbol{e} = \boldsymbol{x}_d - \boldsymbol{x}$ is the position tracking error, and $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$ are $k \times k$ gain matrices.

Theorem 1: Tracking error of system (4) under the control law (18) exponentially converges to zero.

PROOF Substituting (18) into (7) yields dynamics of the closed loop system as

$$\boldsymbol{P}\boldsymbol{M}\boldsymbol{\Lambda}^+\big(\ddot{\boldsymbol{e}} + \boldsymbol{K}_D\dot{\boldsymbol{e}} + \boldsymbol{K}_P\boldsymbol{e}\big) = \boldsymbol{0} \tag{19}$$

The stability proof rests on showing that matrix $\boldsymbol{P}\boldsymbol{M}\boldsymbol{\Lambda}^+$ is full rank. In a proof by contradiction, we will show that the latter matrix is indeed full rank. If the matrix is rank deficient, then there must exist a non-zero vector $\boldsymbol{\zeta}$ such that

$$\boldsymbol{P}\boldsymbol{M}\boldsymbol{\Lambda}^+\boldsymbol{\zeta} = \boldsymbol{0} \qquad \text{where} \quad \boldsymbol{\zeta} \neq \boldsymbol{0} \tag{20}$$

Denote $\boldsymbol{\xi} \triangleq \boldsymbol{\Lambda}^+\boldsymbol{\zeta}$. Then, since $\mathcal{R}(\boldsymbol{\Lambda}^+) \subseteq \mathcal{R}(\boldsymbol{P})$, we can say $\boldsymbol{\xi} \in \mathcal{R}(\boldsymbol{P})$ meaning that $\boldsymbol{P}\boldsymbol{\xi} = \boldsymbol{\xi}$. Moreover, $\boldsymbol{\xi} \neq \boldsymbol{0}$ because matrix $\boldsymbol{\Lambda}^+$ is full rank and $\boldsymbol{\zeta} \neq \boldsymbol{0}$. Therefore, pre-multiplying both sides of (20) by $\boldsymbol{\xi}^T$ yields

$$\boldsymbol{\xi}^T\boldsymbol{M}\boldsymbol{\xi} = \boldsymbol{0} \qquad \text{where} \quad \boldsymbol{\xi} \neq \boldsymbol{0}, \tag{21}$$

which is a contradiction because $\boldsymbol{M}$ is a positive-definite matrix. Consequently, matrix $\boldsymbol{P}\boldsymbol{M}\boldsymbol{\Lambda}^+$ can not be rank deficient and the only possibility for (19) to happen is that the expression inside the parenthesis of (19) is identically zero. This completes the proof by noting that the gains are positive definite. Thus $\boldsymbol{x} \rightarrow \boldsymbol{x}_d$, $\dot{\boldsymbol{x}} \rightarrow \dot{\boldsymbol{x}}_d$ and $\ddot{\boldsymbol{x}} \rightarrow \ddot{\boldsymbol{x}}_d$ as $t \rightarrow \infty$. □

Finally, a solution to the optimal actuation torque can be obtained by substituting (18) into either (15) or (16a).

B. Regulation Using Lyapunov Control

In this section, we assume that the control objective is to regulate the independent set $\boldsymbol{x}(\boldsymbol{q}) \in \mathbb{R}^d$ to their desired values $\boldsymbol{x}_d$, i.e., $\dot{\boldsymbol{x}}_d = \ddot{\boldsymbol{x}}_d = \boldsymbol{0}$. In view of the property of the projection and (17), we can say

$$\dot{\boldsymbol{x}} = \boldsymbol{\Lambda}\boldsymbol{P}\dot{\boldsymbol{q}}. \tag{22}$$

Now, let us consider the following control law:

$$\boldsymbol{\tau}_{\|} = -\boldsymbol{P}\boldsymbol{\Lambda}^T\big(\boldsymbol{K}_D\dot{\boldsymbol{x}} + \boldsymbol{K}_P(\boldsymbol{x}-\boldsymbol{x}_d)\big) + \boldsymbol{P}\boldsymbol{g}(\boldsymbol{q}) \tag{23}$$

where $\boldsymbol{K}_D$ and $\boldsymbol{K}_P$ are $d \times d$, positive-definite feedback gains.

Theorem 2: The constrained mechanical system (4) under the control law (23) asymptotically converges to the desired position $\boldsymbol{x}_d$.

PROOF: Substituting control law (23) in the dynamics equation (7), we obtain

$$\boldsymbol{P}\boldsymbol{M}\ddot{\boldsymbol{q}} = -\boldsymbol{P}\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{P}\boldsymbol{\Lambda}^T\boldsymbol{K}_D\dot{\boldsymbol{x}} - \boldsymbol{P}\boldsymbol{\Lambda}^T\boldsymbol{K}_P\boldsymbol{e}, \tag{24}$$

where $\boldsymbol{e} = \boldsymbol{x} - \boldsymbol{x}_d$ is the set point error. Now, consider the following candidate Lyapunov function:

$$V = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}\dot{\boldsymbol{q}} + \frac{1}{2}\boldsymbol{e}^T\boldsymbol{K}_P\boldsymbol{e}, \quad \forall\,\dot{\boldsymbol{q}} \in \mathcal{N}(\boldsymbol{A}). \tag{25}$$

Then, using (22) and the first of (6) and knowing that $\dot{\boldsymbol{M}} - 2\boldsymbol{C}$ is a skew-symmetric matrix [13], one can compute the time-derivative of the above function along the solution of (24):

$$\begin{aligned}\dot{V} =& \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{M}}\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^T\boldsymbol{M}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{x}}^T\boldsymbol{K}_P\boldsymbol{e}\\ =& \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{M}}\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^T\boldsymbol{P}\boldsymbol{M}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{x}}^T\boldsymbol{K}_P\boldsymbol{e}\\ =& -\dot{\boldsymbol{x}}^T\boldsymbol{K}_D\dot{\boldsymbol{x}} \leq 0\end{aligned}$$

which is negative-semidefinite. Clearly, we have $\dot{V} = 0$ only if $\dot{\boldsymbol{x}} = \boldsymbol{0}$, or if $\dot{\boldsymbol{q}} = \boldsymbol{0}$ because there is a one-to-one relationship between $\dot{\boldsymbol{x}}$ and $\dot{\boldsymbol{q}}$—recall that here we take $\boldsymbol{x} \in \mathbb{R}^d$. Now, substituting $\dot{\boldsymbol{x}} = \boldsymbol{0}$ and $\ddot{\boldsymbol{q}} = \dot{\boldsymbol{q}} = \boldsymbol{0}$ in (24), we can find the largest invariant set with respect to system (24) as the following

$$\Omega = \{\boldsymbol{x}, \dot{\boldsymbol{x}} : \dot{\boldsymbol{x}} = \boldsymbol{0}, \quad \boldsymbol{P}\boldsymbol{\Lambda}^T\boldsymbol{K}_p(\boldsymbol{x}-\boldsymbol{x}_d) = 0\} \tag{26}$$

On the other hand, from (22), one can see that $\boldsymbol{\Lambda}\boldsymbol{P}$—and thus its transpose $\boldsymbol{P}\boldsymbol{\Lambda}^T$—must be a full-rank matrix as $\dot{\boldsymbol{x}}$ are selected to be a complete set of independent generalized velocities. Therefore, the vector equation inside (26) can only hold if $\boldsymbol{x} - \boldsymbol{x}_d$ vanishes. Then, $\Omega = \{\boldsymbol{x} = \boldsymbol{x}_d, \dot{\boldsymbol{x}} = \boldsymbol{0}\}$ is the largest invariant set which satisfies $V = 0$. Therefore, according to LaSalle's Global Invariant Set Theorem [14], [15, p. 115], the solution of system (24) asymptotically converges to the invariant set Ω. Consequently, as the time progresses, $\boldsymbol{x}$ asymptotically approaches its desired value $\boldsymbol{x}_d$. □

Remark 2: The control torque proposed above is of minimal norm in the sense that it does not contribute to the constraint forces.

Finally, substituting $\boldsymbol{\tau}_{\|}$ from (23) into (15), we can derive the motor-torque control law as

$$\boldsymbol{\tau}_a = -\boldsymbol{P}_1^{+}\boldsymbol{P}\boldsymbol{\Lambda}^T\big(\boldsymbol{K}_D\dot{a} + \boldsymbol{K}_P(\boldsymbol{x}-\boldsymbol{x}_d)\big) + \boldsymbol{P}_1^{+}\boldsymbol{P}\boldsymbol{g}(\boldsymbol{q}). \tag{27}$$

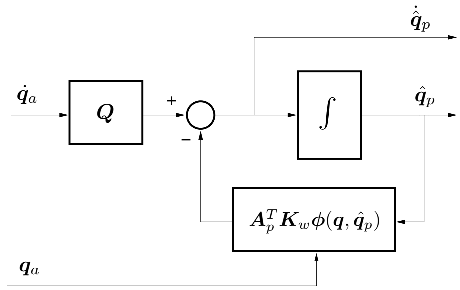

Fig. 2. Estimating the states of the passive joints

IV. DYNAMIC ESTIMATOR

Implementation of the controllers described in sections III-A and III-B require the values of the states of both active and passive joints. Since only active joints are instrumented with position (and velocity) sensors, the values of the states of passive joints should be estimated from those of the active joints. Having computed $\dot{\boldsymbol{q}}_p$ from (3), one can then integrate it to obtain the time-history of the passive-joint variables. The integration, however, will inevitably lead to a drift in the position error. In order to suppress this drift, we use a dynamic estimator which employs the constraint equations $\boldsymbol{\phi}(\boldsymbol{q}_a, \boldsymbol{q}_p) = \boldsymbol{0} \in \mathbb{R}^r$ as a measure of the estimation error; note that $\boldsymbol{A} = \partial\boldsymbol{\phi}/\partial\boldsymbol{q}$. For convenience, it would be easier to use the same number of "measurements" as the number of variables to be estimated. Therefore, we choose a combination $\boldsymbol{\phi}' \in \mathbb{R}^m$ of the constraint equations defined by

$$\boldsymbol{\phi}' \triangleq \boldsymbol{W}\boldsymbol{\phi},$$

where $\boldsymbol{W}$ is a $m \times r$ full-rank matrix; in its simplest form, $\boldsymbol{W}$ can be chosen as a selection matrix that picks m independent equations of the r constraints.

As shown in Fig. 2, to obtain the estimate of the value of the passive joints $\hat{\boldsymbol{q}}_p$, we realize the dynamic estimator by closing the loop using an $\boldsymbol{A}_p^T\boldsymbol{K}_w\boldsymbol{\phi}$ feedback. This feedback and the feedforward given by (3) result in

$$\dot{\hat{\boldsymbol{q}}}_p = \boldsymbol{Q}(\boldsymbol{q}_a, \hat{\boldsymbol{q}}_p)\dot{\boldsymbol{q}}_a - \boldsymbol{A}_p^T(\boldsymbol{q}_a, \hat{\boldsymbol{q}}_p)\boldsymbol{K}_w\,\boldsymbol{\phi}(\boldsymbol{q}_a, \hat{\boldsymbol{q}}_p), \tag{28}$$

where estimator gain matrix $\boldsymbol{K}_w \in \mathbb{R}^{r\times r}$ is constructed from a positive-definite matrix $\boldsymbol{K} \in \mathbb{R}^{m\times m}$ and the weighting matrix $\boldsymbol{W}$ as

$$\boldsymbol{K}_w = \boldsymbol{W}^T\boldsymbol{K}\boldsymbol{W}$$

Note that the inputs and outputs of the estimator loop are $\{\boldsymbol{q}_a, \dot{\boldsymbol{q}}_a\}$ and $\{\hat{\boldsymbol{q}}_p, \dot{\hat{\boldsymbol{q}}}_p\}$, respectively.

Proposition 1: Let us assume that $\boldsymbol{A}_p$ remains a full-rank matrix during the estimation precess, and that an estimate of $\boldsymbol{q}_p$ is obtained from system (28). Then, the constraint equation $\boldsymbol{\phi}'(\boldsymbol{q}_a, \boldsymbol{q}_p)$ as a function of the estimated values of the passive joints globally uniformly asymptotically converges to zero. The estimator will also be globally exponentially stable.

PROOF: Consider the positive-definite Lyapunov function candidate

$$V = \frac{1}{2}\boldsymbol{\phi}'^T(\boldsymbol{q}_a, \hat{\boldsymbol{q}}_p)\boldsymbol{K}\boldsymbol{\phi}'(\boldsymbol{q}_a, \hat{\boldsymbol{q}}_p),$$

which satisfies the following bounds

$$\lambda_{\min}(\boldsymbol{K})\left\|\boldsymbol{\phi}'\right\|^2 \le V \le \lambda_{\max}(\boldsymbol{K})\left\|\boldsymbol{\phi}'\right\|^2. \tag{29}$$

Differentiating V with respect to time along the trajectories of (28) yields

$$\begin{aligned}\dot{V} &= \boldsymbol{\phi}'^T\boldsymbol{K}\boldsymbol{W}\big(\boldsymbol{A}_a\dot{\boldsymbol{q}}_a + \boldsymbol{A}_p\dot{\hat{\boldsymbol{q}}}_p\big)\\ &= \boldsymbol{\phi}'^T\boldsymbol{K}\boldsymbol{W}\big(\boldsymbol{A}_a\dot{\boldsymbol{q}}_a + \boldsymbol{A}_p\boldsymbol{Q}\dot{\boldsymbol{q}}_a - \boldsymbol{A}_p\boldsymbol{A}_p^T\boldsymbol{W}^T\boldsymbol{K}\boldsymbol{\phi}'\big)\\ &= -\boldsymbol{\phi}'^T\boldsymbol{K}(\boldsymbol{W}\boldsymbol{A}_p)(\boldsymbol{W}\boldsymbol{A}_p)^T\boldsymbol{K}\boldsymbol{\phi}'\\ &\le -\lambda^2_{\min}(\boldsymbol{W}\boldsymbol{A}_p)\lambda^2_{\min}(\boldsymbol{K})\left\|\boldsymbol{\phi}'\right\|^2. \end{aligned} \tag{30}$$

Therefore, based on the Lyapunov stability theory for non-autonomous system [15, p. 138], it can be inferred from (29) and (30) that $\boldsymbol{\phi}'(\boldsymbol{q}_a, \hat{\boldsymbol{q}}_p) = \boldsymbol{0}$ must be a globally uniformly asymptotically stable equilibrium point of the system (28). This means that, as t becomes large, $\boldsymbol{\phi}'(\boldsymbol{q}_a, \hat{\boldsymbol{q}}_p)$ approaches zero. Consequently, $\hat{\boldsymbol{q}}_p$ will approach the actual value $\boldsymbol{q}_p$, and $\boldsymbol{\phi}'(\boldsymbol{q}_a, \hat{\boldsymbol{q}}_p)$ will asymptotically vanish. Furthermore, because the bounding functions of V and $\dot{V}$ are of the form $a\left\|\boldsymbol{\phi}'\right\|^b$ where a and b are strictly positive constants, the system is also globally exponentially stable [15, p. 140]. □

Apparently, the estimator (28) is similar to the closed-loop inverse kinematics (CLIK) scheme [16], [17], where the inverse kinematics problem is solved by reformulating it in terms of the convergence of an equivalent feedback control system. It should be noted that, whereas the conventional CLIK algorithm [16], [17] only confines the constraint error inside a small ball, our estimator asymptotically eliminates it.

V. KINEMATIC CONDITIONS FOR CONTROLLABILITY

Condition (14) may seem too restrictive or difficult to satisfy, especially that one cannot easily manipulate either of the two subspaces involved to satisfy the condition. However, this concern is a nonissue. In fact, we can show that, if Assumption 1 holds, (14) is automatically satisfied. In other words, if the values of the passive joints can be uniquely determined from those of the active joints, then they can also be changed to their desired values.

Proposition 2: If the Jacobian matrix $\boldsymbol{A}_p$ is full-rank, then

i) there is no nonzero vector that lies in both $\mathcal{N}(\boldsymbol{A})$ and $\mathcal{N}(\boldsymbol{P}_1^T)$:

$$\mathcal{N}(\boldsymbol{P}_1^T) \cap \mathcal{N}(\boldsymbol{A}) = \emptyset \tag{31}$$

ii) the range of $\boldsymbol{P}_1$ is the same as the null-space of the constraint Jacobian:

$$\mathcal{R}(\boldsymbol{P}_1) = \mathcal{N}(\boldsymbol{A}). \tag{32}$$

PROOF: We prove the first part of the proposition by contradiction. To this end, let us consider a vector $\boldsymbol{\xi} \neq \boldsymbol{0}$ that lies in both $\mathcal{N}(\boldsymbol{A})$ and $\mathcal{N}(\boldsymbol{P}_1^T)$. Then, by definition, $\boldsymbol{\xi}$ must satisfy both

$$\boldsymbol{A}\boldsymbol{\xi} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{P}_1^T\boldsymbol{\xi} = \boldsymbol{0}. \tag{33}$$

TABLE I

MANIPULATOR'S LINK PARAMETERS (INCLUDED ARE MASS AND INERTIA OF THE MOTORS).

Properties	Link 1	Link 2	Link 3	Link 4
Link mass (kg)	0.773	0.407	0.327	0.15
Link length (m)	0.30	0.20	0.12	0.05
Link inertia (kgm^2)	0.058	0.015	2.4×10^{-3}	3×10^{-4}
Center of mass (m)	0.25	0.18	0.05	0.04

The first relation is the same as (1); as such, one can divide $\boldsymbol{\xi}$ into two sub-arrays $\boldsymbol{u} \in \mathbb{R}^n$ and $\boldsymbol{v} \in \mathbb{R}^m$—corresponding to $\dot{\boldsymbol{q}}_a$ and $\dot{\boldsymbol{q}}_p$, respectively—such that $\boldsymbol{\xi}^T = \mathrm{col}(\boldsymbol{u}, \boldsymbol{v})$. Moreover, if $\boldsymbol{A}_p$ is full-rank, one can compute $\boldsymbol{v}$ from $\boldsymbol{u}$ using a relation similar to (3): $\boldsymbol{v} = \boldsymbol{Q}\boldsymbol{u}$. Then, comparing the second of (33) with (13), we can see that

$$\boldsymbol{u} = \boldsymbol{P}_1^T\boldsymbol{\xi} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{v} = \boldsymbol{Q}\boldsymbol{u} = \boldsymbol{0}$$
$$\Longrightarrow \quad \boldsymbol{\xi} \equiv \begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{bmatrix} = \boldsymbol{0},$$

which is a contradiction, i.e., the only vector $\boldsymbol{\xi}$ that satisfies (33) is the zero vector, the trivial solution. That completes the proof of the first part of the proposition.

For the second part, we notice that (31) amounts to

$$\mathcal{N}(\boldsymbol{A}) \subseteq \mathcal{N}^{\perp}(\boldsymbol{P}_1^T). \tag{34}$$

To relate the above relation to the range of $\boldsymbol{P}_1$, we resort to the fundamental theorem of linear operator transformation, which states that the range of a linear operator is the same as the null-space orthogonal of its transpose.[1] For $\boldsymbol{P}_1$, this means

$$\mathcal{R}(\boldsymbol{P}_1) = \mathcal{N}^{\perp}(\boldsymbol{P}_1^T) \tag{35}$$

which combined with (34) results in $\mathcal{N}(\boldsymbol{A}) \subseteq \mathcal{R}(\boldsymbol{P}_1)$. However, $\mathcal{R}(\boldsymbol{P}_1)$ is evidently a subset of $\mathcal{R}(\boldsymbol{P}) \equiv \mathcal{N}(\boldsymbol{A})$. Hence, we must have

$$\mathcal{N}(\boldsymbol{A}) \subseteq \mathcal{R}(\boldsymbol{P}_1) \subseteq \mathcal{N}(\boldsymbol{A}) \quad \Rightarrow \quad \mathcal{R}(\boldsymbol{P}_1) = \mathcal{N}(\boldsymbol{A}),$$

which completes the proof. □

The results of the above development can be briefly stated as follows:

Corollary 1: Let Assumption 1 holds. Then,

i) the states of the passive joints can be uniquely obtained from those of the active joints, e.g., equations (3) or (28), and
ii) the torque-control laws (18) and (23) applied to the active joints can achieve the tracking and set-point control, respectively, while demanding minimum actuation force.

VI. EXPERIMENT

Fig.3 shows a four-link parallel mechanism with three active joints and two passive joints. The active joints are driven by geared motors RH-8-6006, RH-11-3001, and RH-14-6002 from Hi-T Drive that are equipped with their own

[1] The fundamental relationship between the null-space and the range space of a linear transformation S is $\mathcal{R}(S^T) = \mathcal{N}^{\perp}(S)$ [11], [18].

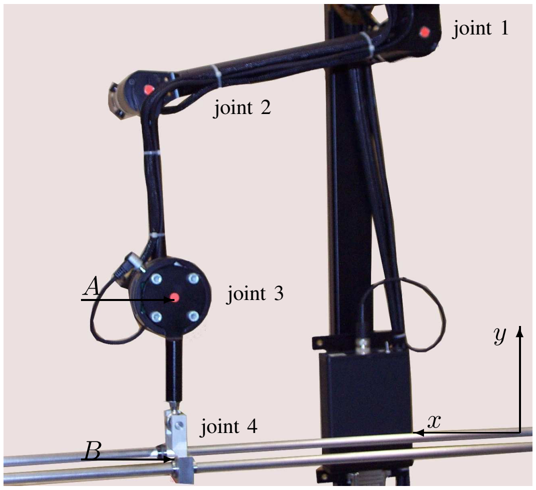

Fig. 3. The experimental setup.

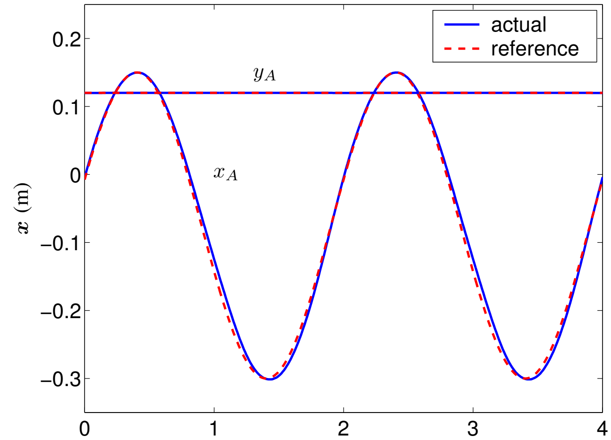

Fig. 4. Trajectories of the task space variables.

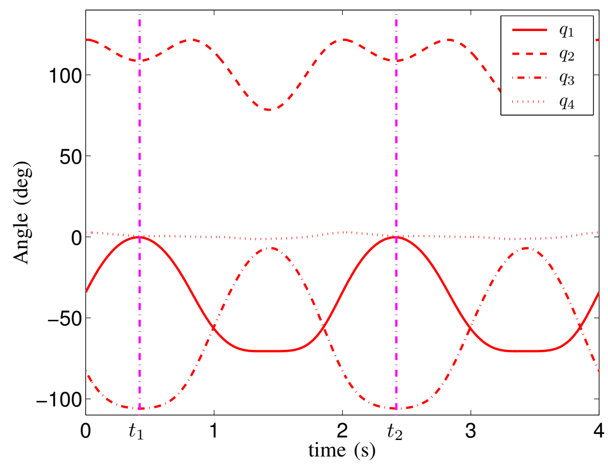

Fig. 5. Trajectories of the joint angles.

built-in optical encoders. The passive joints are comprised of a hinge and a slider. The hinge connects the third link to the last one, while the slider constrains the vertical motion and orientation of the last link. The slider uses linear bearings to minimize friction along x-axis motion. Therefore, by virtually cutting the last link at point B, we can specify the constraint equations as $\boldsymbol{\phi} = \text{col}\,(y_B(\boldsymbol{q}),\ \theta_B(\boldsymbol{q})) = \boldsymbol{0}$, where y_B and θ_B represent the position and orientation. Here, vector $\boldsymbol{q} = \text{col}\,(q_1, \cdots q_4)$ includes the angles of the active joints, q_1-q_3, and the angle of the hinge, q_4. Therefore, cutting the parallel mechanism at point B virtually creates a 4-dof planar manipulator with three active joints and one passive joint. The inertia properties of the planar manipulator are given in Table I.

The control objective is to control the position of point A, i.e., $\{x_A,\ \ y_A\}$ which are expressed in the inertial frame. Therefore, the task space variable is defined as: $\boldsymbol{x}(\boldsymbol{q}) = \text{col}\big(x_A(\boldsymbol{q}), y_A(\boldsymbol{q})\big)$. The desired position trajectories are specified as

$$\boldsymbol{x}_d(t) = \begin{bmatrix} -0.075 + 0.225\sin(\pi t + \frac{\pi}{9}) \\ 0.12 \end{bmatrix} \quad \text{(m)}$$

The parallel mechanism is redundant in most kinematics configurations because there are three actuators to control only two degrees-of-freedoms. However, the mechanism may lose some of its dofs at particular configurations wherein it becomes non-redundant. For instance, by inspection, one can show that the parallel mechanism becomes non-redundant when

$$q_1 = q_4 = 0. \tag{36}$$

In our experiment, the controller gains are selected as $\boldsymbol{K}_P = 480\boldsymbol{1}_2$ and $\boldsymbol{K}_D = 45\boldsymbol{1}_2$. Fig. 4 shows the tracking performance of the projection-based controller (15) and (18). Trajectories of the joint angles and joint torques are plotted in Figs. 5 and 6, respectively. Despite of the fact that the manipulator changed its topology from redundant to non-redundant and vice versa around $t_1 = 0.4$ s and $t_2 = 2.4$ s., the projection-based controller has achieved good tracking performance with a smooth control action; note that (36) occurs at the specified times; see Fig. 5. For a comparison, trajectories of the joint torques obtained by an inverse control law without filtering the requested torque control by the projection operator $\boldsymbol{P}$ is depicted in the Fig. 7, while trajectories of the Euclidean norm of the joint torques are plotted in Fig.8. It is apparent from the latter figure that the projection operator significantly reduces the magnitude of the joint torques by filtering out the normal component of the joint torque vector. It is interesting to note that the two trajectories converge to one other at $t_1 = 0.4$ s and $t_2 = 2.4$ s wherein the manipulator becomes non-redundant leading to a unique solution.

VII. Conclusion

Tracking and set point controllers for parallel manipulator based on the notion of a projection operator has been presented. The main advantage of the projection-based tracking controller is that it works whether the system is overactuated or not; the controller does not need to change its structure whenever the mechanical system changes its topology or number of degrees of freedom. Moreover, the derivation of the projection-based controller seems to be simpler than the

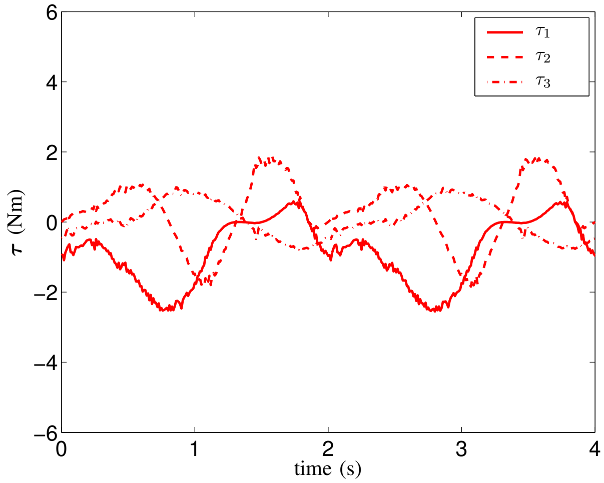

Fig. 6. Trajectories of the joint torque with applying projection filer.

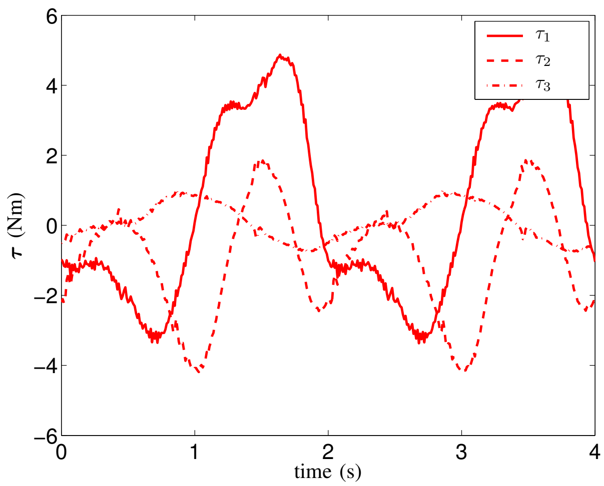

Fig. 7. Trajectories of the joint torque without applying projection filer.

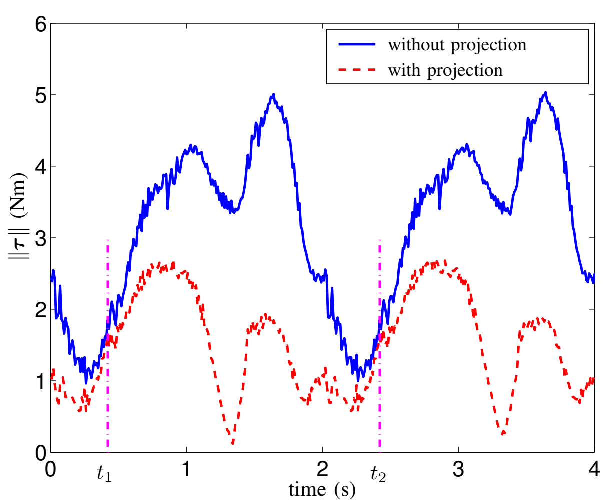

Fig. 8. Euclidean norm of joint torques.

inverse dynamics derived using D'Almbert principal. The stability of the closed-loop system under the projection-based controllers have been rigourously proved. Moreover, the condition for the controllability of parallel manipulators has been also derived. The analysis results showed that if the constraint Jacobian with respect to the passive joints remains full-rank, then not only the states of the passive joints can be uniquely obtained from those of the active joints, but also the projection-based control law applied to the active joints can achieve the tracking and set-point control, respectively, while demanding minimum actuation force. Experiments conducted on a parallel mechanism has demonstrated tracking performance.

References

[1] Y. Nakamura and M. Ghodoussi, "Dynamics computation of closed-link robot mechanism with nonredundant actuators," *IEEE Trans. Robotics and Automation*, vol. 5, pp. 294–302, 1989.

[2] F. H. Ghorbel, O. Chételat, R. Gunawardana, and R. Longchamp, "Modeling and set point control of closed-chain mechanisms: Theory and experiments," *IEEE Transactions on Control Systems Technology*, vol. 8, no. 5, pp. 801–815, 2000.

[3] F. Aghili, "Inverse and direct dynamics of constrained multibody systems based on orthogonal decomposition of generalized force," in *Proceedings of the IEEE Conference on Robotics and Automation*, Taipei, Taiwan, 2003, pp. 4035–4041.

[4] H. Cheng, Y.-K. Yiu, and Z. Li, "Dynamics and control of redundantly actuated parallel manipulator," *IEEE Trans. on Mechatronics*, vol. 8, no. 4, pp. 483–491, Dec. 2003.

[5] F. Aghili, "Control of constrained mechanical systems with passive joints," in *IEEE/RSJ Int. Conf. On Intelligent Robots and Systems*, Sendai, Japan, 2004, pp. 3288–3294.

[6] Z. Wang and F. H. Ghorbel, "Control of closed kinematic chains using a singularly perturbed dynamics model," *J. Dynamic Systems, Measurement, and Control*, vol. 128, no. 1, pp. 142–151, 2006.

[7] Y. Su, D. Sun, L. Ren, and J. K. Mills, "Integration of saturated synchronous control and PD feedback for control of parallel manipulators," *IEEE Trans. on Robotics*, vol. 22, no. 1, pp. 202–207, Feb. 2006.

[8] J. Y. S. Luh and Y. Zheng, "Computation of input generalized forces for robots with closed kineamtic chain mechanism," *IEEE J. Robotics Automat.*, vol. RA-1, pp. 95–103, 1985.

[9] B. Paul, "Analytical dynamics of mechanisms–a computer oriented overview," *Mech. Math. Theory*, vol. 10, pp. 481–507, 1975.

[10] F. Aghili, "A unified approach for inverse and direct dynamics of constrained multibody systems based on linear projection operator: Applications to control and simulation," *IEEE Trans. on Robotics*, vol. 21, no. 5, pp. 834–849, October 2005.

[11] G. H. Golub and C. F. V. Loan, *Matrix Computations*. Baltimore and London: The Johns Hopkins University Press, 1996.

[12] S. Boyd and L. Vandenberghe, *Convex Optimization*. New York: Cambridge University Press, 2004, p. 152.

[13] J.-J. Slotine and W. Li, *Applied Nonlinear Control*. Englewood Cliffs, New Jersey: Prentice Hall, 1991.

[14] J. P. LaSalle, "Some extensions of Lyapunov's second method," *IRE Trans. Circuit Theory*, vol. 7, no. 4, pp. 520–527, 1960.

[15] H. K. Khalil, *Nonlinear Systems*. New-York: Macmillan Publishing Company, 1992.

[16] P. Chiacchio, S. Chiaverini, L. Sciavicco, and B. Siciliano, "Closed-loop inverse kinematics schemes for constrained redundant manipulators with task space augmentation and task priority strategy," *The Int. Journal of Robotics Research*, vol. 10, no. 4, pp. 410–425, August 1991.

[17] W. A. Wolovich and H. Elliott, "A computational technique for inverse kinematics," in *IEEE Conf. On Decision and Control*, New York, 1984, pp. 1359–1363.

[18] C. Desoer, *Notes for a Second Course on Linear Systems*. Van Nostrand Reinhold, 1970.