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On Stochastically Observable Directions of the Estimation Theoretic SLAM State Space

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Abstract—**The theory of stochastic observability is vital in describing the performance of Simultaneous Localization and Mapping (SLAM) as a nonlinear stochastic state estimation problem quantifying effects of random noise on its observability. We show that the eigen space corresponding to the stochastically unobservable states of the state error covariance matrix of the SLAM problem initialized with unknown initial conditions are in the null space of the information matrix associated with observations of the SLAM problem. We establish by using theoretical analysis and examples that the stochastically unobservable directions of the SLAM state space can be changed by modifying the observation model of the SLAM problem. We then use simulations and experiments to show that stochastically observable directions of state space and their degree of stochastic observability can be modified as required in a particular application (such as surveying, mapping and surveillance) by changing the vehicle path with respect to the landmarks in the environment by selecting which landmarks to observe and by modifying the observation model.**

Index Terms—**SLAM, stochastic observability**

I. Introduction

SIMULTANEOUS Localization and Mapping (SLAM) ([1] and [2]) is still considered one of the major challenges in autonomous localization faced by the mobile robotics research community. An effective, scalable and efficient solution to the SLAM problem is a key to many applications such as exploration, surveying, surveillance, transportation, mining etc where deployment of autonomous vehicles is promising. The SLAM problem is a highly nonlinear, stochastic and dynamic state estimation problem encompassing a state space which widely varies over the time. In particular process and observation models used in SLAM are of highly stochastic nature involving random process and measurement noise.

Observability of SLAM was first addressed in [3] using Linear system techniques. Piecewise constant theory and linear system techniques are used in addressing the observability of SLAM in [4]. Nonlinear observability theory is used to analyze SLAM in [5] and [6]. However, all existing literature in particular [3]-[6] does not consider the effects of process and measurement noise on the observability of SLAM.

For deterministic systems, the knowledge of initial conditions is adequate to determine the state of the systems at any given time. However, for stochastic state estimation problems such as SLAM knowledge of the initial state alone without observability is not sufficient to recover the system state at any time. In such stochastic systems, the important concept is the relationship of measurements and inputs to the initial state, which can then be used to derive the state at any time from the measurements and the inputs. Therefore stochastic state estimation problems must fulfil observability even if the initial conditions of the system are known. Although, SLAM is usually initialized using known vehicle initial conditions, initial conditions of the landmark states added to the SLAM state are usually completely or partially unknown. Hence, we can't completely treat SLAM as a problem initialized with known initial conditions.

SLAM problem in general has noisy process and observation models. Since, the observability of the deterministic system corresponding to the stochastic system (when noise injection is zero) is a prerequisite for the observability of the stochastic system, study of the observability of the SLAM problem ([3]-[6]) is essential. However, it is also important that the observability of the SLAM problem be addressed in a stochastic context to have a clear picture of the effects of random noise on its observability properties. Hence, in our investigation we address the stochastic observability of the SLAM problem and its effects on initial conditions.

There are several motivations of understanding stochastic observability of the SLAM problem. Knowledge of stochastic observability is essential in designing efficient stochastic observers and designing sensor configurations for the SLAM problem to have desired target (or landmark) observation capabilities. Study of stochastic observability is also essential in understanding the effect of initial state uncertainty of the SLAM problem on its estimator

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performance. Understanding of the stochastic observability also provides means of quantifying how process and measurement noises affect the observability and estimability of the SLAM problem.

The paper is organized as follows. Section II describes the SLAM problem. Section III investigates the SLAM problem in information form, its stochastic observability, effects of initial conditions on SLAM and their interrelations. In Section IV we show how we can completely avoid stochastically unobservable directions in the SLAM state space by modifying the observation model. Section V provides simulations and experiments to substantiate the theoretical results established. Section VI concludes the work.

II. The Simultaneous Localization and Mapping Problem

A vehicle is said to be implementing a SLAM algorithm if it is building a map of its surrounding environment and localizing at the same time with respect to the constructed map. In a nutshell, the discrete time feature based SLAM problem [9] comprises the following process and measurement models. Suppose a vehicle is moving on a two dimensional (2D) flat surface while estimating its pose $\mathbf{x}_v(k)$ and location states of n point landmarks in the surroundings. The estimated states are;

$$\mathbf{x}_v(k) = \begin{bmatrix} x_v(k) & y_v(k) & \theta_v(k) \end{bmatrix}^T \tag{1}$$

$$\mathbf{m}(k) = \begin{bmatrix} x_1(k) & y_1(k) & \ldots & x_n(k) & y_n(k) \end{bmatrix}^T \tag{2}$$

$$\mathbf{x}_n(k) = \begin{bmatrix} \mathbf{x}_v^T(k) & \mathbf{m}^T(k) \end{bmatrix}^T \tag{3}$$

where $x_v(k)$ is the vehicle longitudinal coordinate $y_v(k)$ is the vehicle lateral coordinate, $\theta_v(k)$ is the vehicle heading, and $x_i(k)$ and $y_i(k)$ $\forall i$ are the longitudinal and lateral coordinates of the i^{th} estimated landmark all at time step k. The process model assuming a car-like (or bicycle) vehicle model is;

$$\mathbf{x}_n(k) = \mathbf{f}(\mathbf{x}_n(k-1),\ \mathbf{u}(k-1)) + \boldsymbol{\eta}_1(k-1) \tag{4}$$

$$\mathbf{f}(n) = \begin{bmatrix} u(k)\cos(\theta(k)) & u(k)\sin(\theta(k)) & \omega(k) & \mathbf{0} \end{bmatrix}^T \tag{5}$$

$$\omega(k) = u(k)\tan\gamma(k)/L \tag{6}$$

where $\boldsymbol{\eta}_1(k)$ is a zero mean uncorrelated noise term representing the process noise with the covariance matrix $\mathbf{Q}(k)$, $\mathbf{u}(k) = \begin{bmatrix} u(k) & \gamma(k) \end{bmatrix}^T$, $u(k)$ is the speed input, $\gamma(k)$ is the steering angle input all at time step k and L is the vehicle wheel base. The transition function is denoted by $\mathbf{f}(.)$. The measurement model assuming a range and bearing sensor is;

$$\mathbf{z}(k) = \mathbf{h}(\mathbf{x}_n(k)) + \boldsymbol{\eta}_2(k) \tag{7}$$

$$\mathbf{h}(.) = [(\mathbf{h}_1)^T \quad (\mathbf{h}_2)^T \quad \ldots \quad (\mathbf{h}_n)^T]^T \tag{8}$$

$$\mathbf{h}_i = \begin{bmatrix} \sqrt{(x_i(k)-x_v(k))^2 + (y_i(k)-y_v(k))^2} \\ \tan^{-1}\{(y_i(k)-y_v(k))/(x_i(k)-x_v(k))\} - \theta_v(k) \end{bmatrix} \tag{9}$$

where $\boldsymbol{\eta}_2(k)$ is a zero mean uncorrelated noise term representing the measurement noise with the covariance matrix $\mathbf{R}(k)$. Measurement function for observing n landmarks is denoted as $\mathbf{h}(.)$. Once the process and measurement models are defined, we can use any estimation algorithm to recursively update and estimate the SLAM state vector.

III. Information Filter and Stochastic Observability

A. Information Matrix Associated with Observations

Information filter [7] is a widely used state estimation filter in practice. We use the information filter equations here to investigate the stochastic observability of SLAM. The state update equation of the Fisher Information form is;

$$\mathbf{P}^{-1}(k\,|\,k) = \mathbf{P}^{-1}(k\,|\,k-1) + \mathbf{I}(k) \tag{10}$$

$$\mathbf{I}(k) = \mathbf{H}^T(k)\mathbf{R}^{-1}(k)\mathbf{H}(k) \tag{11}$$

where $\mathbf{P}(k\,|\,k)$ is the updated error covariance matrix of the estimated state vector $\mathbf{x}_n(k\,|\,k)$ at time step k, $\mathbf{P}(k\,|\,k-1)$ is the predicted error covariance matrix at time step k given $\mathbf{x}_n(k-1\,|\,k-1)$, $\mathbf{H}(k)$ is the Jacobian of the measurement model and $\mathbf{R}(k)$ is the measurement noise covariance matrix both at time step k. We can obtain $\mathbf{P}(k\,|\,k-1)$ using the following equation of the Kalman filter prediction.

$$\mathbf{P}(k\,|\,k-1) = \mathbf{F}(k-1)\mathbf{P}(k-1\,|\,k-1)\mathbf{F}^T(k-1) + \mathbf{Q}(k-1) \tag{12}$$

where $\mathbf{P}(k-1\,|\,k-1)$ is the error covariance matrix of $\mathbf{x}_n(k-1\,|\,k-1)$, $\mathbf{F}(k-1)$ is the Jacobean of the process model and $\mathbf{Q}(k-1)$ is the process noise covariance matrix at time step k-1. The information state vector is $\mathbf{y}(k\,|\,k) = \mathbf{P}^{-1}(k\,|\,k)\mathbf{x}_n(k\,|\,k)$ and $\mathbf{P}^{-1}(k\,|\,k)$ is the information matrix of the state vector $\mathbf{x}_n(k\,|\,k)$. $\mathbf{I}(k) = \mathbf{H}^T(k)\mathbf{R}^{-1}(k)\mathbf{H}(k)$ in (11) is known as the Information Matrix Associated with the Observations (IMAO). Since both the process and observation noises of the SLAM problem are assumed Gaussian, the inverse of the state error covariance matrix is equal to the Fisher information matrix.

The prediction (12) of the covariance matrix always increases uncertainty. Therefore, if we initialize SLAM with an infinitely uncertain initial conditions $\mathbf{P}^{-1}(k\,|\,k-1)$ will also be infinite. Thus, $\mathbf{P}^{-1}(k\,|\,k-1)$ in (10) should be zero. Thus, in order to have a finite covariance matrix $\mathbf{P}^{-1}(k\,|\,k)$ from the measurement updating we should be able to invert $\mathbf{I}(k)$ suggesting that when $\mathbf{I}(k)$ is singular it cannot give any information about at least some of the states and thus, the uncertainty of these states cannot be reduced.

Since $\mathbf{R}(k)$ is a positive definite matrix we find using

Property 1 of the Appendix that $\mathbf{R}^{-1}(k)$ also is a positive definite matrix. Since $\mathbf{H}^T(k)\mathbf{R}^{-1}(k)\mathbf{H}(k)$ is Hermitian, it follows from Property 2 of the Appendix that rank and null space of $\mathbf{H}^T(k)\mathbf{R}^{-1}(k)\mathbf{H}(k)$ is similar to the rank and the null space of $\mathbf{H}(k)$. In the ensuing discussion we therefore for simplicity analyze $\mathbf{H}(k)$ to determine stochastic observability properties of $\mathbf{H}^T(k)\mathbf{R}^{-1}(k)\mathbf{H}(k)$.

B. Stochastic observability of the SLAM problem

Since SLAM is a stochastic estimation problem, it is often required to investigate how the observability conditions affect the SLAM problem in the presence of random noise. In a deterministic problem observability or providing known initial conditions alone is adequate to determine all the states from time zero to any finite value. However, in the presence of random noise, the observability conditions can be violated resulting in unobservability. Hence, in the ensuing discussion we try to compare and contrast the effectiveness of the SLAM problem with a stochastic observability or estimability ([11] and [12]) measure on systems. One important measure on observability of stochastic nonlinear systems is the eigen values (designating the maximum variance of the linear combination of error states [12] of the error covariance matrix). The smallest eigen values of the error covariance matrix therefore correspond to the linear combination of the most observable states and the largest eigen values of the error covariance matrix correspond to the linear combination of the least observable states. The appropriate linear combinations of states having the provided degree of observability are given by the respective eigen vectors.

When the initial state covariance matrix $\mathbf{P}(0\,|\,0)$ is finite, it is extremely difficult to symbolically determine the eigen values and vectors of unobservable states. We therefore, numerically evaluate such conditions for SLAM in simulations and experiments. When the initial conditions are completely unknown it follows from (10) that the IMAO denoted by $\mathbf{I}(k)$ is the inverse of the error covariance matrix. Hence, the unobservable states in the error covariance matrix must result in infinite eigen values in the error covariance matrix, thus resulting zero eigen values in the inverse of the error covariance matrix. Hence we have to look for zero eigen values and corresponding eigen vectors in determining unobservable states. However, the eigen space S_E comprising all the eigen vectors of $\mathbf{I}(k)$ for an eigen value λ by definition is;

$$S_E = null(\mathbf{I}(k) - \lambda \mathbf{I}) \tag{13}$$

where $\mathbf{I}$ is an identity matrix with the same dimension as $\mathbf{I}(k)$. Hence for zero eigen values, the eigen space $S_{E,0}$ is given by the null space of $\mathbf{I}(k)$. However, since the null space of $\mathbf{I}(k)$ is similar to that of $\mathbf{H}(k)$ we have;

$$S_{E,0} = null\left(\mathbf{H}(k)\right) \tag{14}$$

Therefore, the eigen space of the linear combinations of unobservable states is given by the null space of the measurement Jacobian.

IV. Stochastically Unobservable State Space of SLAM Initialized with Unknown Initial Conditions

It is important to know the stochastically unobservable directions in the state space, in designing efficient and effective observers for SLAM so that landmarks located at a known configuration to the vehicle path are observed well. If we know such directions, we can decide on the path we can direct the vehicle to observe certain important landmarks depending on the configuration of the landmarks in the environment, availability of traversable paths and other constraints. In this Section we investigate the stochastically unobservable directions of the SLAM problem initialized with unknown initial conditions based on the theory developed in Section III and find closed form solutions for some important scenarios. Herein after we refer $x_v(k\,|\,k-1)$, $y_v(k\,|\,k-1)$, $x_i(k\,|\,k-1)$, and $y_i(k\,|\,k-1)$ as x_v, y_v, x_i, and y_i respectively for notational simplicity. It follows from (7)-(9) that if all the landmarks are observed the n landmark SLAM problem has a measurement Jacobian given by;

$$\mathbf{H} = [(\mathbf{H}_1)^T \quad (\mathbf{H}_2)^T \quad \ldots \quad (\mathbf{H}_n)^T\,]^T \tag{15}$$

$$\mathbf{H}_i = \begin{bmatrix} \Delta x_i/r_i & \Delta y_i/r_i & 0 & \mathbf{0} & -\Delta x_i/r_i & -\Delta y_i/r_i & \mathbf{0} \\ -\Delta y_i/r_i^2 & \Delta x_i/r_i^2 & -1 & \mathbf{0} & \Delta y_i/r_i^2 & -\Delta x_i/r_i^2 & \mathbf{0} \end{bmatrix} \tag{16}$$

where $\Delta x_i = x_v - x_i$, $\Delta y_i = y_v - y_i$, and $r_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$. Suppose now that the j^{th} column of $\mathbf{H}_i$ and $\mathbf{H}$ are $\mathbf{C}_i^j$ and $\mathbf{C}^j$ respectively. It follows from (16) that;

$$\mathbf{C}_i^1 + \mathbf{C}_i^{2+2i} = \mathbf{0} \tag{17}$$

$$\mathbf{C}_i^2 + \mathbf{C}_i^{3+2i} = \mathbf{0} \tag{18}$$

$$-\Delta y_1 \mathbf{C}_i^1 + \Delta x_1 \mathbf{C}_i^2 + \mathbf{C}_i^3 + (y_1 - y_i)\mathbf{C}_i^{2+2i} - (x_1 - x_i)\mathbf{C}_i^{3+2i} = \mathbf{0} \tag{19}$$

Hence, from (15) and (17)-(19) it follows that;

$$\mathbf{C}^1 + \sum_{i=1}^{n} \mathbf{C}^{2+2i} = \mathbf{0} \tag{20}$$

$$\mathbf{C}^2 + \sum_{i=1}^{n} \mathbf{C}^{3+2i} = \mathbf{0} \tag{21}$$

$$\begin{aligned} &-(y_v - y_1)\mathbf{C}^1 + (x_v - x_1)\mathbf{C}^2 + \mathbf{C}^3 + \\ &\qquad \sum_{i=1}^{n} \{(y_1 - y_i)\mathbf{C}^{2+2i} + (-x_1 + x_i)\mathbf{C}^{3+3i}\} = \mathbf{0} \end{aligned} \tag{22}$$

Therefore, it follows that (20), (21) and (22) represent null vectors of $\mathbf{H}$ and hence eigenvectors of $\mathbf{I}(k)$ corresponding to zero eigen values where $\mathbf{C}^i$ now corresponds to the column vectors of $\mathbf{I}(k)$. Hence, the vectors (20), (21) and (22) are along the stochastically unobservable state space in the n landmark SLAM problem initialized with unknown initial conditions if only the estimated landmarks are observed.

Since, when only all the estimated landmarks are observed SLAM has three stochastically unobservable directions in the state space, we investigate in the ensuing discussion whether we can get rid of these stochastically unobservable directions in the state space by modifying the SLAM observation model. When all the estimated landmarks and the vehicle heading are observed; the new measurement Jacobian ${}^1\mathbf{H}$ is given by ${}^1\mathbf{H} = \begin{bmatrix} \mathbf{H}^T & \bar{\mathbf{H}}^T \end{bmatrix}^T$ where $\bar{\mathbf{H}} = \begin{bmatrix} 0 & 0 & 1 & \mathbf{0} \end{bmatrix}$. Hence, it follows that (17), (18) and (20), (21) are still true for ${}^1\mathbf{H}$ when j^{th} column of $\mathbf{H}_i$ and ${}^1\mathbf{H}$ are $\mathbf{C}_i^j$ and $\mathbf{C}^j$ respectively. Therefore, it follows that (20) and (21) represent null vectors of ${}^1\mathbf{H}$ and hence eigenvectors of $\mathbf{I}(k)$ corresponding to zero eigen values where $\mathbf{C}^i$ now corresponds to the column vectors of $\mathbf{I}(k) = {}^1\mathbf{H}^T(k)\mathbf{R}^{-1}(k){}^1\mathbf{H}(k)$. Hence, the vectors (20) and (21) are along the stochastically unobservable state space in the n landmark SLAM problem initialized with unknown initial conditions if only the estimated landmarks and the vehicle's heading are observed.

When all the estimated landmarks and the vehicle's lateral and longitudinal coordinates are observed; the new measurement Jacobian ${}^1\mathbf{H}$ is now given by ${}^1\mathbf{H} = \begin{bmatrix} \mathbf{H}^T & \bar{\mathbf{H}}^T \end{bmatrix}^T$ where

$$\bar{\mathbf{H}} = \begin{bmatrix} 1 & 0 & 0 & \mathbf{0} \\ 0 & 1 & 0 & \mathbf{0} \end{bmatrix} \tag{23}$$

It follows from (16) and (23) that;

$$\mathbf{C}_i^3 + (y - y_i)\mathbf{C}_i^{2+2i} - (x - x_i)\mathbf{C}_i^{3+2i} = \mathbf{0} \tag{24}$$

where j^{th} column of $\mathbf{H}_i$ and ${}^1\mathbf{H}$ are $\mathbf{C}_i^j$ and $\mathbf{C}^j$ respectively.

Hence, using (15), (16), (23) and (24) we obtain;

$$\mathbf{C}^3 + \sum_{i=1}^{n}\left((y - y_i)\mathbf{C}^{2+2i} - (x - x_i)\mathbf{C}^{3+2i}\right) = \mathbf{0} \tag{25}$$

Therefore, it follows that (25) is a null vector of ${}^1\mathbf{H}$ and hence eigenvectors of $\mathbf{I}(k)$ corresponding to zero eigen values where $\mathbf{C}^i$ now corresponds to the column vectors of $\mathbf{I}(k) = {}^1\mathbf{H}^T(k)\mathbf{R}^{-1}(k){}^1\mathbf{H}(k)$. Hence, the vector (25) is along the stochastically unobservable state space in the n landmark SLAM problem initialized with unknown initial conditions if only the estimated landmarks and the vehicle's longitudinal and lateral coordinates are observed.

When all the estimated landmarks and one *a priori* known landmark are observed; the new measurement Jacobian ${}^1\mathbf{H}$ is given by ${}^1\mathbf{H} = \begin{bmatrix} \mathbf{H}^T & \bar{\mathbf{H}}_1^T \end{bmatrix}^T$ and

$$\bar{\mathbf{H}}_i = \begin{bmatrix} \Delta\bar{x}_i/\bar{r}_i & \Delta\bar{y}_i/\bar{r}_i & 0 & \mathbf{0} & 0 & 0 & \mathbf{0} \\ -\Delta\bar{y}_i/\bar{r}_i^2 & \Delta\bar{x}_i/\bar{r}_i^2 & -1 & \mathbf{0} & 0 & 0 & \mathbf{0} \end{bmatrix} \tag{26}$$

where $\Delta\bar{x}_i = x_v - \bar{x}_i$, $\Delta\bar{y}_i = y_v - \bar{y}_i$, and $\bar{r}_i = \sqrt{\Delta\bar{x}_i^2 + \Delta\bar{y}_i^2}$. $\bar{\mathbf{H}}_i$ is obtained from (26) when $i = 1$. Let the j^{th} column of $\mathbf{H}_i$, ${}^1\mathbf{H}$ and $\bar{\mathbf{H}}_1$ are $\mathbf{C}_i^j$, $\mathbf{C}^j$ and $\bar{\mathbf{C}}_1^j$ respectively. It follows from (26) that;

$$(\bar{y}_1 - y)\bar{\mathbf{C}}_1^1 - (\bar{x}_1 - x)\bar{\mathbf{C}}_1^2 + \bar{\mathbf{C}}_1^3 + = \mathbf{0} \tag{27}$$

If we now consider $\mathbf{H}_i$ it follows that;

$$\begin{aligned}(\bar{y}_1 - y)\mathbf{C}_i^1 - (\bar{x}_1 - x)\mathbf{C}_i^2 + \mathbf{C}_i^3 \\ + (\bar{y}_1 - y_i)\mathbf{C}_i^{2+2i} - (\bar{x}_1 - x_i)\mathbf{C}_i^{3+2i} = \mathbf{0}\end{aligned} \tag{28}$$

Therefore, using (27) and (28) we have;

$$\begin{aligned}(\bar{y}_1 - y)\mathbf{C}^1 - (\bar{x}_1 - x)\mathbf{C}^2 + \mathbf{C}^3 \\ + \sum_{i=1}^{n}\left((\bar{y}_1 - y_i)\mathbf{C}^{2+2i} - (\bar{x}_1 - x_i)\mathbf{C}^{3+3i}\right) = \mathbf{0}\end{aligned} \tag{29}$$

Therefore, it follows that (29) is a null vector of ${}^1\mathbf{H}$ and hence eigenvectors of $\mathbf{I}(k)$ corresponding to zero eigen values where $\mathbf{C}^i$ now corresponds to the column vectors of $\mathbf{I}(k) = {}^1\mathbf{H}^T(k)\mathbf{R}^{-1}(k){}^1\mathbf{H}(k)$. Hence, the vector (29) is along the stochastically unobservable state space in the n landmark SLAM problem initialized with unknown initial conditions if only the estimated landmarks and *a priori* known landmark are observed.

When all the estimated landmarks and two *a priori* known landmarks are observed; the new measurement Jacobian ${}^1\mathbf{H}$ is given by ${}^1\mathbf{H} = \begin{bmatrix} \mathbf{H}^T & \bar{\mathbf{H}}_1^T & \bar{\mathbf{H}}_2^T \end{bmatrix}^T$ where $\bar{\mathbf{H}}_i$ for $i = 1$ and 2 are given by (26). When the two *a priori* known landmarks and the vehicle are collinear it follows that

$$\bar{\mathbf{H}}_2 = \begin{bmatrix} k\Delta\bar{x}_1/\bar{r}_1 & k\Delta\bar{y}_1/\bar{r}_1 & 0 & \mathbf{0} & 0 & 0 & \mathbf{0} \\ -\Delta\bar{y}_1/\bar{r}_1^2 & \Delta\bar{x}_1/\bar{r}_1^2 & -1 & \mathbf{0} & 0 & 0 & \mathbf{0} \end{bmatrix} \tag{30}$$

where k is a scalar parameter. Hence, it follows that any column operation done on first three columns of $\bar{\mathbf{H}}_1$ to make a null column (see (29)) will result in a null column in $\bar{\mathbf{H}}_2$ as well. When one of the two *a priori* known landmarks and the vehicle are collinear with the i^{th} estimated landmark, it follows that

$$\bar{\mathbf{H}}_2 = \begin{bmatrix} k\Delta x_i/r_i & k\Delta y_i/r_i & 0 & \mathbf{0} & 0 & 0 & \mathbf{0} \\ -\Delta y_i/r_i^2 & \Delta x_i/r_i^2 & -1 & \mathbf{0} & 0 & 0 & \mathbf{0} \end{bmatrix} \tag{31}$$

where k is a scalar parameter. Hence, it follows that any column operation done on first three columns of $\mathbf{H}_i$ to make a null column (see (29)) will result in a null column in $\bar{\mathbf{H}}_2$ as well. Therefore, when all the estimated landmarks and two *a priori* known landmarks which are collinear with the vehicle are observed or when one *a priori* landmarks and an estimated landmark are collinear with the vehicle are observed the n landmark SLAM problem initialized with unknown initial conditions has a stochastically unobservable direction in state space along the vector given by (29). Assume now that (1) the two *a priori* landmarks and the vehicle are not collinear and (2) any one of the two *a priori* known landmarks is not collinear with an estimated landmark and the vehicle. It now follows from the structure of the ${}^1\mathbf{H}$ that all the columns are not null vectors on their own. Since the first three columns of ${}^1\mathbf{H}$ has four rows corresponding to $\bar{\mathbf{H}}_1$ and $\bar{\mathbf{H}}_2$ both with zero elements in columns other than 1-3, the non zero terms in $\bar{\mathbf{H}}_1$ and $\bar{\mathbf{H}}_2$ can be made null only by the column operations on first three columns of ${}^1\mathbf{H}$. Assume that there are three column operations $T_i \neq 0$ for $i = 1$, 2, 3 on first three columns of

$^1\mathbf{H}$ that result in null columns.

$$\left(\Delta\bar{x}_1/\bar{r}_1\right)T_1 + \left(\Delta\bar{y}_1/\bar{r}_1\right)T_2 = 0 \tag{32}$$

$$\left(-\Delta\bar{y}_1/\bar{r}_1^2\right)T_1 + \left(\Delta\bar{x}_1/\bar{r}_1^2\right)T_2 - T_3 = 0 \tag{33}$$

$$\left(\Delta\bar{x}_2/\bar{r}_2\right)T_1 + \left(\Delta\bar{y}_2/\bar{r}_2\right)T_2 = 0 \tag{34}$$

$$\left(-\Delta\bar{y}_2/\bar{r}_2^2\right)T_1 + \left(\Delta\bar{x}_2/\bar{r}_2^2\right)T_2 - T_3 = 0 \tag{35}$$

When one of T_i for $i=1$, 2, 3 is zero it results in inconsistent equations. Similarly, if all T_i for $i=1$, 2, 3 are considered together it follows that only solution we have is $T_i = 0$ for all $i=1$, 2, 3. Hence, the assumption is a contradiction. There are no column operations on columns 1,2 and 3, either all together or in pares which result in a null column. Hence, only possibility of having a null column is using column operations on any column pare corresponding to a landmark. Let there be two column operations $T_{2+2i} \neq 0$ and $T_{3+2i} \neq 0$ which result in a null column by operating on $(2+2i)^{th}$ and $(3+2i)^{th}$ columns of $^1\mathbf{H}$. It then follows that;

$$\left(-\Delta x_i/r_i\right)T_{2+2i} + \left(-\Delta y_i/r_i\right)T_{3+2i} = 0 \tag{36}$$

$$\left(\Delta y_i/r_i^2\right)T_{2+2i} + \left(-\Delta x_i/r_i^2\right)T_{3+2i} = 0 \tag{37}$$

Since, the determinant of the system of equations (36) and (37) on variables T_{2+2i} and T_{3+2i} are not equal to zero, the only solution we have is $T_{2+2i} = T_{3+2i} = 0$. This is a contradiction to the assumption that $T_{2+2i} \neq 0$ and $T_{3+2i} \neq 0$. Hence, it is not possible to have null columns in $^1\mathbf{H}$ by any column operation. Therefore, when all the estimated landmarks and two *a priori* known landmarks which are nether collinear with the vehicle nor any one of two *a priori* known landmarks are collinear with any estimated landmark and the vehicle are observed; the n landmark SLAM problem initialized with unknown initial conditions has no completely stochastically unobservable direction in the state space.

It is very important to note from (25) and (29) that we can select $(\bar{x}_i, \bar{y}_i)$ or (x, y) so that certain directions along state space are much more observable than the others. That means by selecting observed landmarks (known) and taking certain vehicle paths one can improve the stochastic observability of the SLAM problem.

V. Simulations and Experiments

This section comprises of simulations and experiments to substantiate the theoretical results we have established in the previous sections.

A. Simulations

We have used an Extended Kalman Filter [9] based estimation theoretic algorithm in simulations to evaluate the theory of stochastic observability of SLAM. We assumed a car like mobile robot moving in a 2D simulation environment (Fig. 1) according to a specified trajectory while observing point landmarks in the environment using a range and bearing sensor. We also use a nearest neighbor data association method [2] and a map management method [11] in the SLAM algorithm.

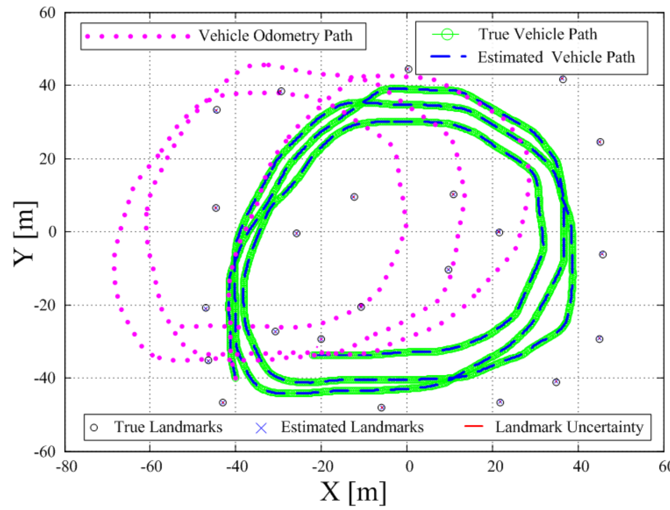


Fig. 1 Simulation of SLAM

We evaluate the effect of the initial conditions on the stochastic observability of the SLAM problem by first assuming finite initial state error covariance and then by assuming infinitely large initial state error covariance. In evaluating the eigen values and eigen vectors of the error covariance matrices of SLAM we normalize (see [12]) the error covariance matrix prior to calculating singular values and vectors so that, singular values are dimensionally homogeneous and within the same range.

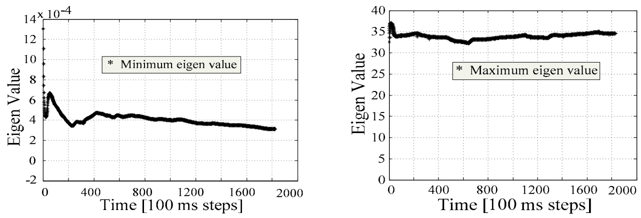
$$\mathbf{P}'(k\,|\,k) = \left(\sqrt{\mathbf{P}(k\,|\,K)}\right)^{-1}\mathbf{P}(k\,|\,k)\left(\sqrt{\mathbf{P}(k\,|\,K)}\right)^{-1} \tag{38}$$

$$\mathbf{P}_N(k\,|\,k) = \left(n/Trace\left(\mathbf{P}'(k\,|\,k)\right)\right)\mathbf{P}'(k\,|\,k) \tag{39}$$

We then use the normalized error covariance matrix $\mathbf{P}_N(k\,|\,k)$ to evaluate eigen values and eigen vectors.

Fig. 2, Fig. 6 and Fig. 7 show the variation of minimum and maximum eigen values of the SLAM error covariance matrix when SLAM is initialized with zero initial state uncertainty of the vehicle location and (1) only all the estimated landmarks are observed, (2) all the estimated landmarks and one *a priori* known landmark are observed and (3) all the estimated landmarks and two *a priori* known landmarks which are not collinear with either the vehicle or any other estimated landmarks are observed respectively.

Fig. 2, Fig. 6 and Fig. 7 establish that the maximum eigen value or in other words the maximum variance along the least observable direction is lowest when the SLAM problem satisfies the full rank conditions of $\mathbf{I}(k) = {}^1\mathbf{H}^T(k)\mathbf{R}^{-1}(k)\,{}^1\mathbf{H}(k)$. This result further extends the result of Section IV that SLAM initialized with unknown initial conditions has no completely stochastically unobservable directions when $\mathbf{I}(k)$ is full rank to the case of finite initial uncertainties. Further, the maximum eigen value increases beyond initial value in Fig. 2 and 6 when $\mathbf{I}(k)$ is not full rank thus, resulting in stochastically unobservable SLAM. Recall from [11] that a state estimator is estimable only if there is increasing amount of information in the form of Fisher.



(a) (b)

Fig. 2 Eigen values of the SLAM state error covariance matrix when only all the estimated landmarks are observed and starting with finite initial uncertainty.

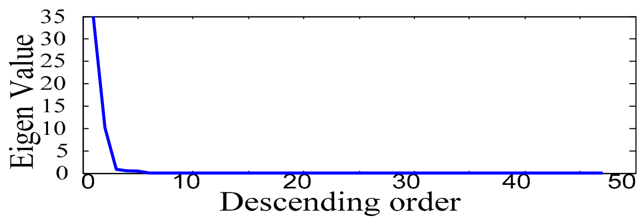


Fig. 3 Eigen value variation of the SLAM state error covariance matrix at time step 1800 when only all the estimated landmarks are observed.

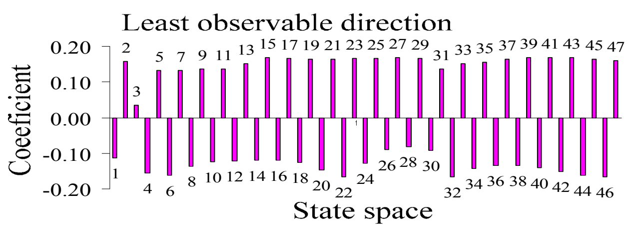


Fig. 4 Eigen vector in along the least observable direction in the state space of the SLAM problem at time step 1800 when only all the estimated landmarks are observed.

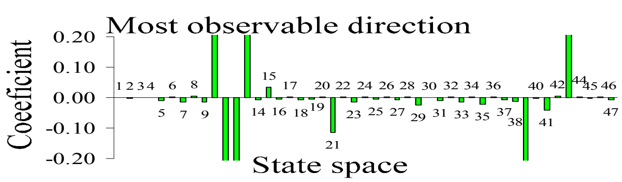
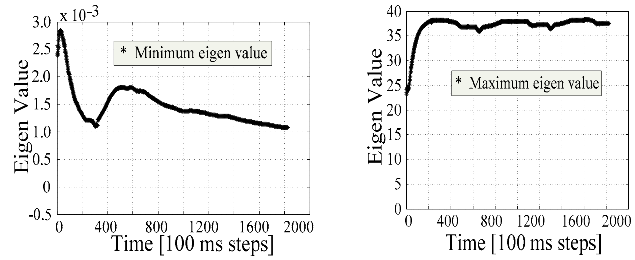


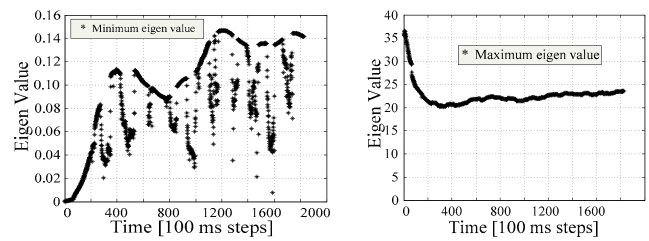
Fig. 5 Eigen vector in along the most observable direction in the state space of the SLAM problem at time step 1800 when only all the estimated landmarks are observed.

Fig. 3 shows the variation of eigen values of SLAM at time step 1800. It shows SLAM has directions of state space which are stochastically observable in varying degrees. Fig. 4 and 5 illustrate the least and most stochastically observable directions of SLAM state space at time step 1800. Fig. 8-10 shows the variation of eigen values and stochastically observable directions of the SLAM problem initialized with very large uncertainty (in this case it is 10^{10}). It verifies that SLAM still has a stochastically observable solution when initialized with very large uncertaities when $\mathbf{I}(k)$ is full rank as shown in Section IV. Fig. 9, Fig. 10 and Fig. 11 show the variation of eigen value and the least and most stochastically observable directions of SLAM state space when initialized with very large uncertainty.



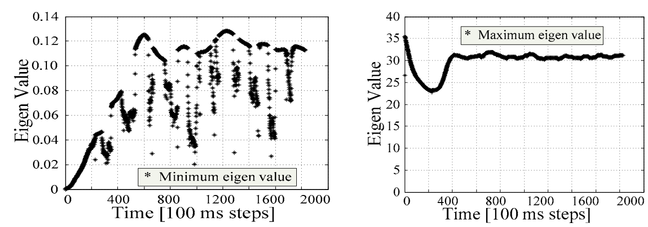
(a) (b)

Fig. 6 Eigen values of the SLAM state error covariance matrix when all the estimated landmarks and one *a priori* known landmark are observed.



(a) (b)

Fig. 7 Eigen values of the SLAM state error covariance matrix when all the estimated landmarks and two *a priori* known landmarks which are not collinear with the vehicle position or any of the estimated landmarks are observed (stating with zero initial uncertainty).



(a) (b)

Fig. 8 Eigen values of the SLAM state error covariance matrix when all the estimated landmarks and two *a priori* known landmarks which are not collinear with the vehicle position or any of the estimated landmarks are observed starting with infinitely large initial uncertainties

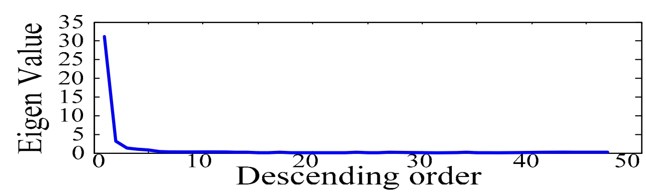


Fig. 9 Eigen value variation of the SLAM state error covariance matrix at time step 1800 when only all the estimated landmarks are observed.

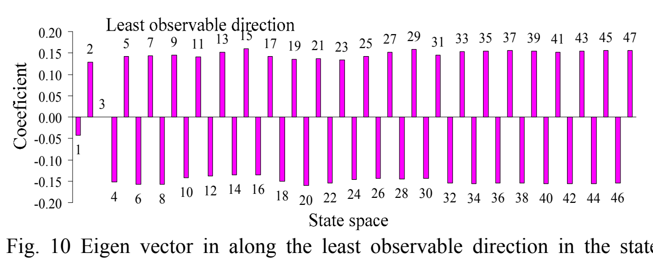


Fig. 10 Eigen vector in along the least observable direction in the state space of the SLAM problem at time step 1800 when only all the estimated landmarks are observed.

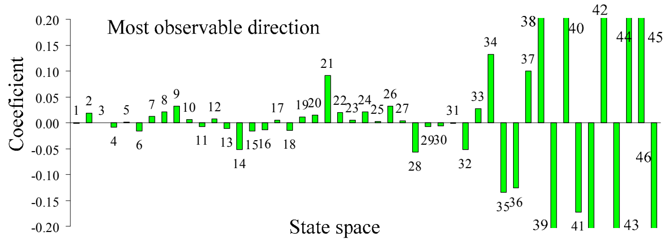


Fig. 11 Eigen vector in along the most observable direction in the state space of the SLAM problem at time step 1800 when only all the estimated landmarks are observed.

B. Experiments

This subsection describes the SLAM experiments ([10]) using the car park dataset of the University of Sydney. Car park dataset was obtained by driving a utility vehicle equipped with GPS, wheel and steering encoders and a laser range finder. We use the car park data set to check the consistency of the localization error estimates when SLAM is made locally weakly observable (by observing at least 2 known landmarks and all the estimated landmarks). We use the GPS measured landmark locations as known landmarks. Fig. 12 shows the map of estimated landmarks and the vehicle. It can be observed that the estimated vehicle path and the landmarks are consistent with the true vehicle path and the landmark locations as measured by GPS.

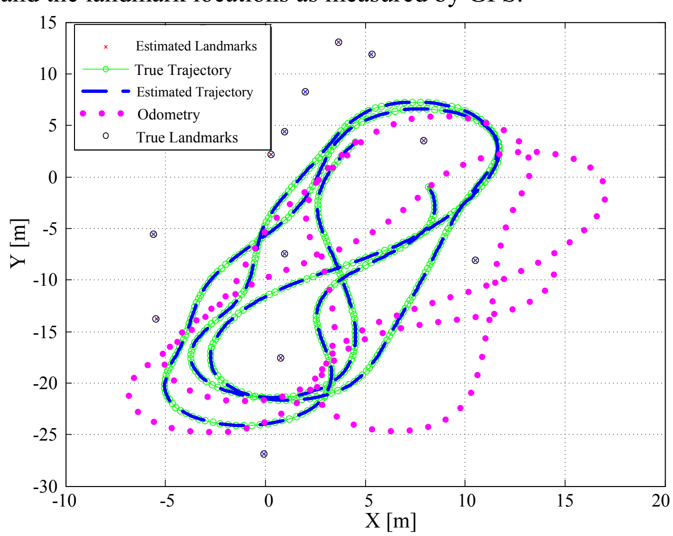


Fig. 12 SLAM results from the car park data set when the nonlinear observability rank condition is satisfied .

Fig. 13 and 15 demonstrate the variation of minimum and maximum eigen values when (1) only all the estimated landmarks are observed and initialized with zero state uncertainty, and (2) all the estimated landmarks and two *a priori* known landmarks which are not collinear with either the vehicle or any other estimated landmarks are observed and initialized with very large state uncertainty respectively for the University of Sydney car park data set. Fig 13 (b) shows an increasing maximum eigen value whereas Fig. 15 (b) shows a decreasing maximum eigen value. Hence, establishing SLAM is estimable and stochastically observable even when initialized with very large state uncertainties when all the estimated landmarks and two *a priori* known landmarks which are not collinear with either the vehicle or any other estimated landmarks are observed.

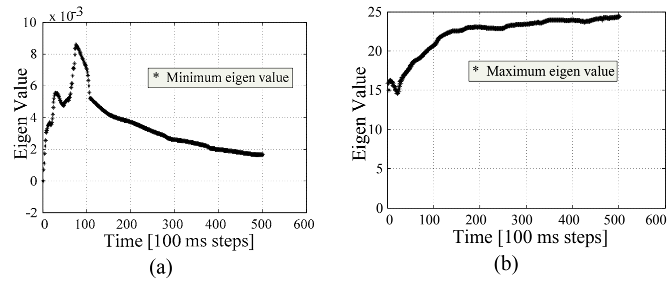


Fig. 13 Eigen values of the SLAM (starting with zero initial state uncertainty) state error covariance matrix when only all the estimated landmarks are observed in the experiment.

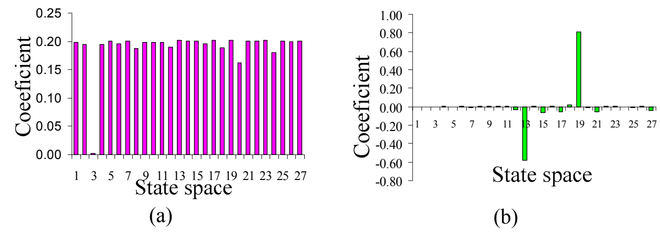


Fig. 14 Eigen vectors of the SLAM (starting with zero initial state uncertainty) state error covariance matrix when only all the estimated landmarks are observed in the experiment. (a) Least observable direction, (b)-Most observable direction

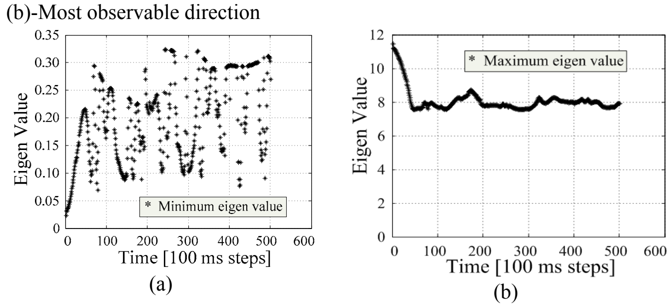


Fig. 15 Eigen values of the SLAM (starting with infinitely large initial state uncertainty) state error covariance matrix when all the estimated landmarks and two *a priori* known landmarks are observed.

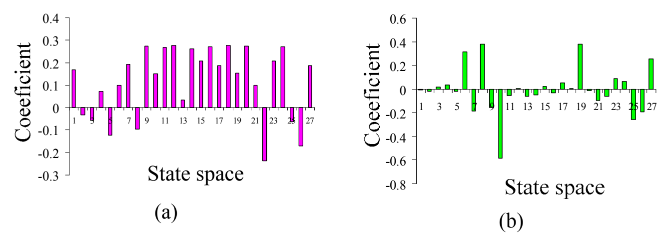


Fig. 16 Eigen vectors of the SLAM (starting with infinitely large initial state uncertainty) state error covariance matrix when only all the estimated landmarks are observed in the experiment. (a) Least observable direction, (b)-Most observable direction

Fig. 14 and 16 show the least and most observable directions of SLAM state space for SLAM initialized with zero initial uncertainty and SLAM initialized with very large uncertainty. The results verify that we can change the stochastically observable and unobservable directions in the SLAM state space by changing the vehicle path with respect to landmarks, by selecting which landmarks to observe and by modifying the observation model (as done in simulations, experiments and theoretical discussion in Section IV).

VI. Conclusion

Since, SLAM is a stochastic state estimation problem we argue that it is important to evaluate its observability in a stochastic sense. We have described in this paper an interesting and useful insight into the stochastic observability of the SLAM problem using eigen values and eigen vectors of its state error covariance matrix.

We highlighted that the Fisher Information Matrix Associated with Observations (IMAO) must be nonsingular if the *n* landmark SLAM problem initialized with completely unknown initial conditions be solvable. We have shown that the eigen space corresponding to the stochastically unobservable states of the state error covariance matrix of the SLAM problem initialized with unknown initial conditions are in the null space of the IMAO of the SLAM problem.

We show that there are three vectors (given by (20), (21) and (22)) along the stochastically unobservable state space in the *n* landmark SLAM problem initialized with unknown initial conditions if only the estimated landmarks are observed. We also show that there are two vectors (given by (20) and (21)) along the stochastically unobservable state space in the *n* landmark SLAM problem initialized with unknown initial conditions if only the estimated landmarks and the vehicle's heading are observed. We also establish that there is a vector (given by (25)) along the stochastically unobservable state space in the *n* landmark SLAM problem initialized with unknown initial conditions if only the estimated landmarks and the vehicle's longitudinal and lateral coordinates are observed.

Furthermore, we show that there is a vector (given by (29)) along the stochastically unobservable state space in the *n* landmark SLAM problem initialized with unknown initial conditions if only the estimated landmarks and *a priori* known landmark are observed. Finally we establish that, when all the estimated landmarks and two *a priori* known landmarks which are nether collinear with the vehicle nor any one of two *a priori* known landmarks are collinear with any estimated landmark and the vehicle are observed; the *n* landmark SLAM problem initialized with unknown initial conditions has no completely stochastically unobservable direction in the state space.

We have also used simulations and experiments to show that stochastically observable directions of state space and their degree of stochastic observability can be modified as required in a particular application by changing the vehicle path with respect to the landmarks in the environment, by selecting which landmarks to observe and by modifying the observation model. Therefore it is shown that depending on the application one can improve the stochastic observability of required landmarks of the map by selecting certain vehicle paths, modifying observation model and changing the sensor configuration thus improving the performance of SLAM in surveying, surveillance and mapping applications.

Appendix

Properties of positive definite matrices (from [8]).

1. Every positive definite matrix is invertible and its inverse is also positive definite.
2. Let $\mathbf{A} = ()_{n\times n}$ be positive definite. If $\mathbf{C} = ()_{n\times m}$, then $\mathbf{C}^*\mathbf{AC}$ is positive semi-definite. Furthermore,

$$rank(\mathbf{C}^*\mathbf{AC}) = rank(\mathbf{C}) \quad (40)$$

$$null(\mathbf{C}^*\mathbf{AC}) = null(\mathbf{C}) \quad (41)$$

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