The 2010 IEEE/RSJ International Conference on
Intelligent Robots and Systems
October 18-22, 2010, Taipei, Taiwan

Stability of Time-varying Control for an Underactuated Biped Robot Based on Choice of Controlled Outputs

Ting Wang and Christine Chevallereau

Abstract— **This paper studies the effect of controlled outputs selection on the walking stability for an under-actuated planar biped robot. The control is based on tracking reference motions expressed as function of time. First, the reference motions are adapted at each step in order to create an hybrid zero dynamic system. Second, the stability of the walking gait under closed-loop control is evaluated with the linearization of the restricted Poincaré map of the hybrid zero dynamics. It shows that for the same robot, and for the same reference trajectory, the stability of the walking can be modified by some pertinent choices of controlled outputs. Third, we find that, at the desired moment of impact for one step, the height of swing foot is nearly zero for all the stable walking, though the configurations of the robot are not the desired configurations. Based on this, we propose a new method to choose the controlled outputs to obtain the stable walking for robot. As a result, two stable domains for the controlled outputs selection are obtained. Furthermore, we point out that, this kind of control law, which based on reference motion as a function of time, can produce better convergent property than that based on reference motion as a function of a state of robot via pertinent choices of controlled outputs.**

I. INTRODUCTION

The primary objective of this paper is to present a feedback controller that achieve an asymptotically stable, periodic walking gait for an under-actuated planar biped robot. The biped studied consists of five links, connected to form two legs with knees and a torso. It has point feet without actuation between the feet and ground, so the ZMP heuristic is not applicable, and thus under actuation must be explicitly addressed in the feedback control design.

The control of this robot is based on tracking reference motions. There are two groups of method which depend on the differences of reference trajectory. The first one is based on reference trajectory as function of time and the other one is not. In the second method, for example, the method of virtual constraints, which has been proved very successful in designing feedback controllers for stable walking in planar bipeds [9], [5], [13], [15]. A recent paper [7] extended it to the case of spatial robots. In this method, a state quantity of the biped, which is strictly monotonic (i.e., strictly increasing or decreasing) along a typical walking gait, is used to replace time in parameterizing a periodic motion of the biped. When such a control has converged, the configuration of the planar robot at the impact is the desired one but this approach involves parameterized reference trajectories. In fact, this parametrization method is not usual in robotics. In addition, it is impossible to find the strictly monotonic state for some robots, for example, robot-semiquad [1], quadruped with a curve gait, or biped with frontal motion [11], [8].

Ting Wang and Christine Chevallereau are with the IRCCyN, CNRS, Ecole centrale de Nantes, 1 rue de la Noe, 44321 Nantes, cedex 03, France, Ting.Wang@irccyn.ec-nantes.fr Christine.Chevallereau@irccyn.ec-nantes.fr

Based on these observations, we want to propose a tool to analyze the stability of a control based on tracking reference motion as function of time and to propose solution to obtain stable walking. It has been observed that for the same robot, and for a same known cyclic motion, a control law based on a reference trajectory as function of the state of a robot produces a stable walking, while a control law based on reference motion as a function of time produces an unstable walking [4]. In fact, for the control law based on reference motion as a function of time, most periodic walking gaits are unstable when the controlled outputs are selected to be the actuated coordinates. In [7], the effect of output selection on the zero dynamics is discussed first time, but this is based on parameterized reference trajectory. Next, this approach is used successfully for the control based on time-variant reference trajectory in [14], in which a pertinent choice of outputs is proposed, leading to stabilization without the use of a supplemental event-based controller.

This paper focus on how to choose the control outputs pertinently to improve the walking stability of biped, which is extended from [14]. Poincaré method is used to analyze the stability of limit cycles for hybrid zero dynamics. The numerical simulation shows the effect of controlled outputs selection on the walking stability. A new method to choose the control outputs is proposed based on a condition of the swing foot height at the desired moment of impact. As a result, two stable domains for the controlled outputs selection are obtained. We compared the control property of this method with the method of virtual constraints [9], [5], [13], [15]. It shows the velocity converge more quickly with our method.

II. MODEL

A. Description of the robot

The biped considered walks in a vertical xz plane. It is composed of a torso and two identical legs. In the simulation, the robot Rabbit is considered [16]. Each leg is composed of two links articulated by a knee. The knees and the hips are one-degree-of-freedom rotational joints.

The gait is composed of single support phases separated by impact phases. During the single support phase, the vector $q = [q_1, \ldots, q_5]'$ (Fig. 1) describes the configuration of the biped. The knee and hip relative angles are actuated, but the ankle joint is not actuated. We define the vector of actuated variables $q_a = [q_2, \ldots, q_5]'$ and unactuated variable $q_u = q_1$.

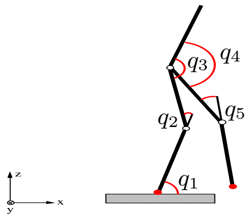

Fig. 1. The studied biped

B. Dynamic model

The dynamic models for single support and impact (i.e., double support) are derived here by assuming support on leg 1. The models for support on leg 2 can be written in a similar way. The Euler-Lagrange equations yield the dynamic model for the robot in the single support phase as

$$D(q)\ddot{q} + H(q,\dot{q}) = B\,\Gamma = \begin{bmatrix} 0_{1\times 4} \\ I_{4\times 4} \end{bmatrix} u, \tag{1}$$

where $D(q)$ is the positive-definite (5×5) mass-inertia matrix, $H(q,\dot{q})$ is the (5×1) vector of Coriolis and gravity terms, B is an (5×4) full-rank, constant matrix indicating whether a joint is actuated or not, and u is the (4×1) vector of input torques. The double support phase is assumed to be instantaneous. During the impact, the biped's configuration variables do not change, but the generalized velocities undergo a jump. Defining $-$ and $+$ denotes the moment just before and after impact respectively. As shown in [16], this jump is linear with respect to the joint velocity before the impact $\dot{q}^-$.

$$\dot{q}^+ = I(q^-)\dot{q}^-, \tag{2}$$

Considering the exchange of legs, the configuration after impact becomes :

$$q^+ = Eq^-, \tag{3}$$

where E is a (5×5) matrix which describes the transformation of two legs.

Define state variables as $x = [q, \dot{q}]'$, and let $x^+ = [q^+, \dot{q}^+]'$ and $x^- = [q^-, \dot{q}^-]'$. Then a complete walking motion of the robot can be expressed as a nonlinear system with impulse effects, and written as

$$\Sigma : \begin{cases} \dot{x} = f(x) + g(x)\,u & x^- \notin S \\ x^+ = \Delta(x^-) & x^- \in S \end{cases}, \tag{4}$$

where $S = \{(q,\dot{q}) | z_{sw}(q) = 0,\ x_{sw}(q) > 0\}$ is the switching surface, z_{sw}, x_{sw} describe coordinates of swing foot, and u denotes the input torques.

$$f(x) = \begin{bmatrix} \dot{q} \\ -D^{-1}(q)H(q,\dot{q}) \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0_{5\times 4} \\ D^{-1}(q)\,B \end{bmatrix},$$

and

$$x^+ = \Delta(x^-) = \begin{bmatrix} E \\ EI(q^-) \end{bmatrix} x^-.$$

III. CONTROL LAW

A. The output

Since the robot is equipped for m actuators, only m outputs can be controlled. In many papers, the controlled variables are simply the actuated variables. In fact, the choice of the controlled variables directly affects the behavior of the robots [7], [14]. For simplicity, we limit our analysis to the case that the controlled variables v are linear expression of the configuration variables. They are expressed as:

$$v = M\begin{bmatrix} q_u \\ q_a \end{bmatrix} = \begin{bmatrix} M_1 & M_2 \end{bmatrix}\begin{bmatrix} q_u \\ q_a \end{bmatrix}, \tag{5}$$

where M is a (4×5) constant matrix. To obtain a simple expression of q with respect to v and q_u, we impose M_2 be invertible. The m outputs that must be zeroed by the control law are :

$$y = v - v^d(t) \tag{6}$$

where $v^d(t)$ is the desired evolution of the controlled variables. The function $q^d(t)$ corresponding to a cyclic motion of the robot, which has been obtained by [6]. The cyclic motion has been defined for one cyclic step from $t = 0$ to $t = T$. In the control strategy this desired trajectory is restarted at each impact. A more precise definition of the output is :

$$\begin{cases} y = v - v^d(\tau) & x^- \notin S \\ \dot{\tau} = 1 & x^- \notin S \\ \tau^+ = 0 & x^- \in S \end{cases}. \tag{7}$$

For this cyclic motion, the reference motion v^d is defined by

$$v^d(\tau) = Mq^d(\tau) = M_1 q_u^d + M_2 q_a^d \tag{8}$$

Since the reference trajectory is expressed as the function of time, the torques depend on the state of the robot and of time τ. To be able to consider the closed loop state as an autonomous system, we will extend the state as $X = [x, \tau]'$, and the studied system can be written as:

$$\Sigma : \begin{cases} \dot{X} = f_e(X) + g_e(X)\,u(X) & x^- \notin S \\ X^+ = \Delta_e(X^-) & x^- \in S \end{cases}, \tag{9}$$

where

$$f_e(X) = \begin{bmatrix} f(x) \\ 1 \end{bmatrix}, \quad g_e(X) = \begin{bmatrix} g(x) \\ 0_{1\times 4} \end{bmatrix}, \quad \Delta_e = \begin{bmatrix} \Delta \\ 0 \end{bmatrix}.$$

Considering $q_u = q_1$ and (5), the current configuration of the robot can be expressed as:

$$q = \begin{bmatrix} 1 & 0_{1\times 4} \\ -{M_2}^{-1}M_1 & {M_2}^{-1} \end{bmatrix}\begin{bmatrix} q_u \\ v \end{bmatrix}, \tag{10}$$

In equation (1), D and H are defined as:

$$D = \begin{bmatrix} D_{11(1\times 1)} & D_{12(1\times 4)} \\ D_{21(4\times 1)} & D_{22(4\times 4)} \end{bmatrix}, \quad H = \begin{bmatrix} H_{1(1\times 1)} \\ H_{2(4\times 1)} \end{bmatrix} \tag{11}$$

Then dynamic model in single support (1) is rewritten as:

$$\begin{cases} (D_{11} - D_{12}{M_2}^{-1}M_1)\ddot{q}_u = -D_{12}{M_2}^{-1}\ddot{v} - H_1(q,\dot{q}) \\ u = (D_{21} - D_{22}{M_2}^{-1}M_1)\ddot{q}_u + D_{22}{M_2}^{-1}\ddot{v} + H_2(q,\dot{q}) \end{cases} \tag{12}$$

To zero the output in (6), the control input is defined as:

$$\ddot{v} = \ddot{v}^d(\tau) - \frac{K_d}{\varepsilon}(\dot{v} - \dot{v}^d) - \frac{K_p}{\varepsilon^2}(v - v^d), \qquad (13)$$

where $K_p > 0$, $K_d > 0$, and $\varepsilon > 0$. Using the control (13), the desired torques u in (12) can be calculated.

B. The Zero dynamics

The zero dynamics [10] is defined to describe the behavior of the system when the outputs are assumed to be zero. There are two objectives of introducing the zero dynamics. Firstly, we want to study the effect of the uncontrolled variable q_u on the property of the closed-loop system with a perfect control law. Our second objective is to analyze the stability of walking in a reduced space. If the output is zero then using (6) and the first line of (12), the zero dynamics is defined by:

$$\begin{cases} v(t) = v^d(\tau) \\ (D_{11} - D_{12}{M_2}^{-1}M_1)\ddot{q}_u = -D_{12}{M_2}^{-1}\ddot{v}^d(\tau) - H_1(q, \dot{q}) \end{cases} \qquad (14)$$

The zero dynamics can also be expressed with the variables q_u and q_a. Using the equation (8), we have:

$$M_1 q_u + M_2 q_a = v = M_1 q_u^d + M_2 q_a^d \qquad (15)$$

Since M_2 is invertible, there exists:

$$\begin{cases} q_a(\tau) = q_a^d + M_2^{-1}M_1(q_u^d - q_u) \\ (D_{11} - D_{12}{M_2}^{-1}M_1)\ddot{q}_u = -D_{12}({M_2}^{-1}M_1\ddot{q}_u^d \\ +\ddot{q}_a^d) - H_1(q, \dot{q}) \end{cases} \qquad (16)$$

This equation clearly shows that the behavior of the robot will be affected by the value of ${M_2}^{-1}M_1$. In fact, according to the definition of the controlled outputs v in (5), the introduction of M_1 permits to take into account the tracking error of the uncontrolled variable q_u. In the following we impose that $M_2 = I_{4\times4}$, which is a identity matrix. When $M_1 = [0, 0, 0, 0]'$, the controlled variables are simply the actuated variables q_a (see (5)).

We can clearly see that the dynamic properties of the swing phase zero dynamics depend on the particular choice of the reference motion $v^d(\tau)$ or $q^d(\tau)$. For the same desired periodic motion, the choice of the controlled variables directly affects the zero dynamics in (16). It will be proved with the simulation results in section V-A.

C. The Hybrid Zero dynamics

According to [16, Chap. 5], while the feedback control law (12) and (13) have created a zero dynamics of the stance phase dynamics, it has not created a *hybrid zero dynamics*, that is, the zero dynamics considering the impact model (2). If the control law could be modified so as to create a hybrid zero dynamics, then the study of the swing phase zero dynamics (14) and the impact model would be sufficient to determine the stability of the complete system of the robot, thereby leading to a reduced-dimension stability test [12].

The reference motions are modified stride to stride so that they are compatible with the initial state of the robot at the beginning of each step [7]. The new output for the feedback control design is

$$y_c = v(\tau) - v^d(\tau) - v_c(\tau, y_i, \dot{y}_i). \qquad (17)$$

This output consists of the previous output (6), and a correction term v_c that depends on (6) evaluated at the beginning of the step, specifically, $y_i = v(0) - v^d(0)$ and $\dot{y}_i = \dot{v}(0) - \dot{v}^d(0)$. The values of y_i and $\dot{y}_i$ are updated at the beginning of each step (or at impact) and held constant throughout the step. The function v_c is taken to be a three-times continuously differentiable function of τ such that[1]

$$\begin{cases} v_c(0, y_i, \dot{y}_i) & = & y_i \\ \dot{v}_c(0, y_i, \dot{y}_i) & = & \dot{y}_i \\ v_c(\tau, y_i, \dot{y}_i) & \equiv & 0, \quad \tau \geq \frac{T}{2}. \end{cases} \qquad (18)$$

With v_c designed in this way, the initial errors of the output and its derivative are smoothly joined to the original virtual constraint at the middle of the step, and v_c doesn't introduce any discontinuity on the desired trajectory. In particular, for any initial error, the initial reference motion v^d is exactly satisfied by the second part of the step : $\tau \geq \frac{T}{2}$.

Under the new control law defined by (17), the behavior of the robot is completely defined by the impact map and the swing phase zero dynamics (14), where v^d is replaced by $v^d + v_c$, since $M_2 = I_{4\times4}$, this equation becomes :

$$\begin{cases} v(\tau) = v^d(\tau) + v_c(\tau, y_i, \dot{y}_i) \\ (D_{11} - D_{12}M_1)\ddot{q}_u = -D_{12}(\ddot{v}^d + \ddot{v}_c) - H_1(q, \dot{q}) \end{cases} \qquad (19)$$

The zero dynamics manifold is defined by $Z = \{(\tau, q, \dot{q}) | y_c(q) = 0, \ \dot{y}_c(q) = 0\}$, this manifold can be parametrized by a vector of dimension 3: $(\tau, q_u, \dot{q}_u)$. When the reference trajectory is a function of the state variable [16], the zero dynamics manifold is of dimension 2, in our case, the supplementary variable τ must be considered. By introducing v_c, the resulted walking motion can remain in the manifold Z in the presence of impact phase.

IV. STABILITY ANALYSIS

The stability analysis of walking can be done with the Poincaré method. Since with the chosen control law, the state of the robot remains on the zero dynamics manifold, the stability analysis can be done in a reduced space.

Different Poincaré section can be considered. Usually, for biped, the Poincaré section is defined just before the impact. This choice implies that the perturbation of the state that are introduced for the calculation of the Jacobian of the Poincaré map is such that the two legs touch the ground. As a consequence the determination of the perturbation is not obvious. To avoid this, we will consider the Poincaré section at $\tau = 0.75T$, at this instant the swing leg tip does not touch the ground and since $\tau = 0.75T \geq \frac{T}{2}$ the value of the controlled variable are not affected by v_c.

A restricted Poincaré map is defined from $S_\tau \cap Z$ to $S_\tau \cap Z$, where $Z = \{(\tau, q, \dot{q}) | y_c(q) = 0, \ \dot{y}_c(q) = 0\}$ and

[1] In our specific application, we used a four order polynomial for $0 \leq \tau \leq \frac{T}{2}$; continuity of position, velocity and acceleration is ensured at $\tau = \frac{T}{2}$.

$S_\tau = \{(\tau, q, \dot{q}) | \tau = 0.75T\}$ is the Poincaré section. The key point is that since in Z the state of the robot can be parametrized by three independents variables, in $S_\tau \cap Z$, the state of the robot can be represented using only two independent variables, $x^z = [q_u(0.75T), \dot{q}_u(0.75T)]'$, where q_u denotes the unactuated joint.

The known cyclic motion $q^d(\tau)$ gives a fixed point $x^{z*} = ({q_u}^d(0.75T), \dot{q}_u^d(0.75T))$ for the proposed control law for any value of M_1.

The restricted Poincaré map $P^z : S_\tau \cap Z \rightarrow S_\tau \cap Z$ induces a discrete-time system $x^z_{k+1} = P^z(x^z_k)$. From [12], for ε sufficiently small in (13), *the linearization of P^z about a fixed-point determines exponential stability of the full order closed-loop robot model*. Define $\delta x^z_k = x^z_k - x^{z*}$, the Poincaré map linearized about the fixed-point x^{z*} gives rise to a linearized system,

$$\delta x^z_{k+1} = A^z \delta x^z_k, \qquad (20)$$

where the (2×2) square matrix A^z is the Jacobian of the Poincaré map [14].

A fixed-point of the restricted Poincaré map is locally exponentially stable, if, and only if, the eigenvalues of A^z have magnitude strictly less than one [16, Chap. 4].

V. EXAMPLES

Using optimization techniques developed in [6], an optimal cyclic motion has been defined for the robot Rabbit described in section II-A. The corresponding stick-diagram of the walking gait and the joint profiles of each angle have been presented in [14]. To investigate the influence of the choice of the controlled outputs (via the matrix M_1) on the stability of the control law for a particular desired cyclic motion, $v^d(\tau)$, $\dot{v}^d(\tau)$ and $\ddot{v}^d(\tau)$ have to be known. Since q^d are known, according to (8), the desired reference motion are defined by :

$$\begin{cases} v^d(\tau) = M q^d(\tau) \\ \dot{v}^d(\tau) = M \dot{q}^d(\tau) \\ \ddot{v}^d(\tau) = M \ddot{q}^d(\tau) \end{cases} \qquad (21)$$

A. Stability Analysis with Different Choices of M_1

It is shown that the hybrid zero dynamics depends on the choice of the output (19). We will explore the effect of M_1 on the stability of the control law.

To study the stability of this control law around the periodic motion, we need to compute the eigenvalues of A^z in (20), which are noted as $\lambda_{1,2}$. Generally, it is possible to use an optimization technique to find a vector M_1 such that the maximal norm eigenvalues of A^z is less than one, as:

$$max\,|\lambda_{1,2}| < 1, \qquad (22)$$

Here we use an exploration technique to illustrate the effect of the choice of the output. We fix arbitrary three components of M_1 to zero and $max\,|\lambda_{1,2}|$ are drawn as function of the fourth component of M_1. The results are shown in Fig. 2, where the red points note that $max\,|\lambda_{1,2}| < 1$. In order to find these stable points, we search more precisely of $M_1(j)$,

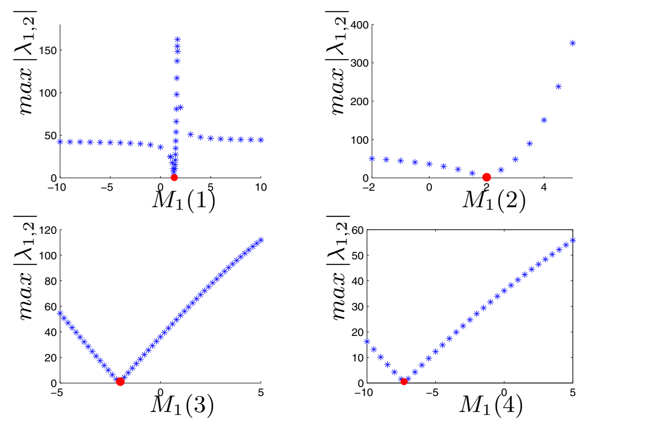

Fig. 2. $max\,|\lambda_{1,2}|$ versus $M_1(j), j = 1, 2, 3, 4$, when the other three components of M_1 are zero.

$j = 1, 2, 3, 4$ near the $max\,|\lambda_{1,2}| = 1$. We can find three main results from Fig. 2.

1) When M_1 get into some domains, the eigenvalues of A^z can change to infinity, as a result, the swing foot can't touch the ground, but when M_1 get out of it, the eigenvalues of A^z diminish instantaneously, as shown in $M_1(1)$ of Fig. 2. We observed that these domains are due to the singularity of the controller.

Proof: If we want to calculate the required torques u in (12), we must obtain $\ddot{q}_u$ using the first line of (12), which can be rewritten as:

$$\ddot{q}_u = \frac{-D_{12}\ddot{v} - H_1(q, \dot{q})}{D_{11} - D_{12}M_1}, \qquad (23)$$

If there exist values of M_1 such that:

$$D_{11} - D_{12}M_1 = 0, \qquad (24)$$

the controller is singularity.

During the swing phase, D_{11} and D_{12} are almost constant. At the desired impact moment T^d, there are $D_{11} \approx 24.2445$, $D_{12} \approx [13.4347, 3.6283, 0.5375, 0.0410]$. When $M_1 = [M_1(1), 0, 0, 0]$, (24) is satisfied for $M_1(1) = 1.8046$. As shown in $M_1(1)$ of Fig. 2, there is a jump of $max\,|\lambda_{1,2}|$ near this value. Using the same method, we can deduce that when $M_1 = [0, 6.6821, 0, 0]'$, $M_1 = [0, 0, 45.1060, 0]'$ and $M_1 = [0, 0, 0, 592.7751]'$, the controller will be singular too.

2) For some M_1, there exists a minimum $max\,|\lambda_{1,2}|$ which leads to stable walking.

3) The $max\,|\lambda_{1,2}|$ is very sensitive to some change of M_1. When M_1 is modified a little, the stable walking gait may become unstable.

Here points 2) and 3) show that it is possible but difficult to choose M_1 leading to stable walking. In order to further illustrate that, we choose different values of M_1 to test stability. Each component of M_1 is sampled between -10 and 10 with a step of 2.5 to build 9^4 vectors M_1. For each vector M_1, the effective eigenvalues of the Poincaré map, $max\,|\lambda_{1,2}|$, is calculated. Except some cases in which the swing foot can't touch the ground because of the singularity of the controller, there are only 19 stable cases and 5363 unstable cases. The vectors M_1 leading to stable walking are scattered and there is no obvious condition of stability can be determined. Is there a method to help in the choice of M_1? In the next subsection, we will try to find a way of settling this problem.

B. A Method for Pertinent Choice of M_1

For all cases corresponding to different M_1 which were presented in the previous paragraph, we choose some walking characteristics to observe whether there exists difference between the stable and unstable cases. They are:

- position of CoM (Center of Mass) at the desired impact moment T^d,
- kinetic energy just before impact and after impact (see [2] and [3]),
- errors of uncontrolled variable q_u during the swing phase and at impact moment,
- height of swing foot Z_{sw} at T^d.

We have not observed any special relations between the effective eigenvalues of the Poincaré map $max\,|\lambda_{1,2}|$ and the first three walking characteristics, but we found $Z_{sw} \approx 0$ at T^d for all stable cases, that is, $max\,|\lambda_{1,2}| < 1$, see Fig. 3.

To calculate Z_{sw} at T^d, we use the equation (16) to write:

$$v(T^d) = v^d(T^d), \tag{25}$$

and we suppose that there exists an error Δq_u on the underactuated variable of biped q_u, $\Delta q_u = q_u(T) - {q_u}^d(T)$. Considering (25), (10) and $M_2 = I_{4\times4}$, there is:

$$q(T^d) = q^d(T^d) + M_T \Delta q_u \tag{26}$$

where $M_T = [1, -M_1]'$. We can compute $Z_{sw}(T^d)$ without simulation of the biped walking. We suppose $\Delta q_u = 10^{-2}$, which is adequate to estimate the real Δq_u in the walking simulation. Then we get $Z_{sw}(T^d)$ for all the case of $M_1(j) \in [-10, 10]$, $j = 1, 2, 3, 4$, which was presented in the previous subsection. The result is shown in Fig. 3 (we only present the cases of $max\,|\lambda_{1,2}| < 50$), where the red points denote $max\,|\lambda_{1,2}| < 1$, that is, the stable cases.

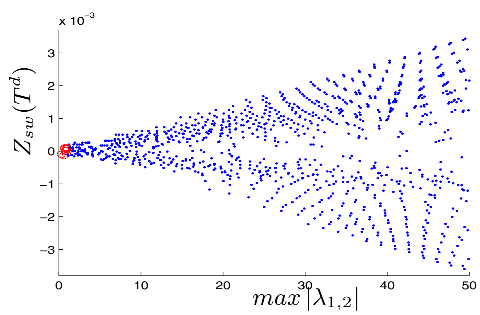

Fig. 3. The height of the swing foot at the end of the step $Z_{sw}(T^d)$ with respect to the effective eigenvalues of the Poincaré map, $max\,|\lambda_{1,2}|$.

Based on Fig. 3, we conjecture that:

A necessary condition for stable walking is $Z_{sw}(T^d) \approx 0$.

This condition implies that the swing foot still can touch ground at T^d with an error of the unactuated variable. If there is an error on the impact moment, $T \neq T^d$, there will exist not only the error of configuration and velocity of unactuated variable Δq_u, $\Delta \dot{q}_u$, but also that of controlled variables Δv, $\Delta \dot{v}$. These errors can lead to unstable walking. The necessary condition $Z_{sw}(T^d) \approx 0$ avoids the error on the impact moment, so the errors Δv and $\Delta \dot{v}$ are avoided.

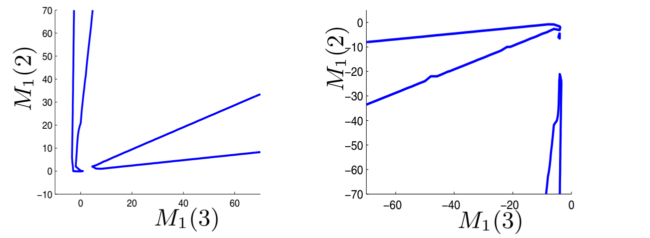

Fig. 4. The stable domain of $M_1(2)$ and $M_1(3)$ for two groups of solution, which is described with contour line $max\,|\lambda_{1,2}| = 1$

Next is how to choose M_1 with this necessary condition. According to (26), Z_{sw} can be written with Taylor series:

$$Z_{sw}(q(T^d)) \approx Z_{sw}(q^d(T^d)) + a\Delta q_u + b{\Delta q_u}^2 \tag{27}$$

where $a = \frac{\partial Z_{sw}(q^d(T^d))}{\partial q} M_T$ and $b = \frac{1}{2}{M_T}' \frac{\partial^2 Z_{sw}(q^d(T^d))}{\partial q^2} M_T$. Since the function of $Z_{sw}(q(T^d))$ is highly nonlinear, here we considered the first two terms of the Taylor series, not only the first one.

If a and b satisfy :

$$a = b = 0 \tag{28}$$

$Z_{sw}(q(T^d))$ is close to zero for any Δq_u.

Two constraint equations exist in (28) for four components of M_1, so only two components can be chosen. The condition (28) leads to two groups of solution of M_1, because the equation $b = 0$ is a second order equation as function of M_1. For each group of solution, $M_1(1)$ and $M_1(4)$ are deduced from $M_1(2)$ and $M_1(3)$. Then we search different $M_1(2)$ and $M_1(3)$ to analyze the stability of the system , we can obtain large stable domain of M_1 for each group of solution, as shown in Fig. 4, it is described with contour line $max\,|\lambda_{1,2}| = 1$. The system is always stable as long as M_1 is chosen in these domains.

C. Compare with The Method of Virtual Constraints

Here we will compare our method with the method of virtual constraints, which has been proved effectively in designing feedback controllers for stable walking in planar bipeds [9], [5], [13], [15]. It is worth mentioning that experimental tests based on the method of virtual constraints has been carried out successfully with Rabbit by the research team of J. Grizzle [16]. Here the compare of two method is based on the same model of the robot, the same desired trajectory and the same method of stability analysis. It is shown that stability of an orbit is independent of the choice of the output, as long as the constraints yield a nonsingular controller [16, Chap. 6]. As a result, the control properties can not be modified in this way. On the contrary, it can be improved by a good choice of controlled outputs with our method.

In general, the effective eigenvalues of the Poincaré map is the smaller the better for the stability, so we can choose M_1 according to the desired property. For example, we choose $M_1 = [1.9121, -4, -3, -1.4090]$ from the second solution of Fig. 4, the result of stability analysis is $max|\lambda_{1,2}| = 0.2392$. Then the planar biped's model in closed-loop is simulated with this M_1 and the method of virtual constraints. The initial errors $0.01rad$ and $0.1rad/s$ are introduced on each joint and it's velocity respectively. As shown in Fig. 5, for the method of virtual constraints, the configuration of the robot at the impact is the desired one but the velocity converge more slowly. With our method, the velocity converge to zero after walking four steps, and the control input u also follows the desired torque after these four steps. The torques of the swing leg (see Fig. 1) are shown in Fig. 6.

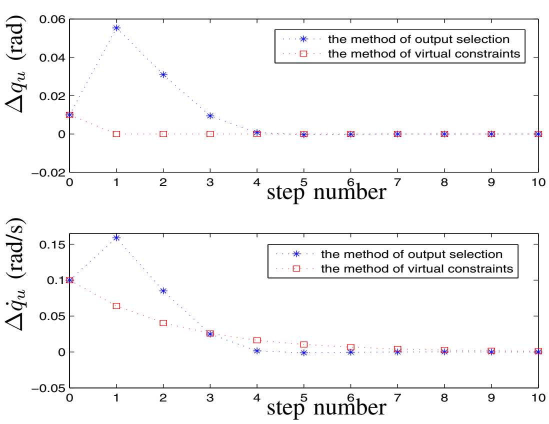

Fig. 5. The difference of the real values and desired values of q_u at the end of each step.

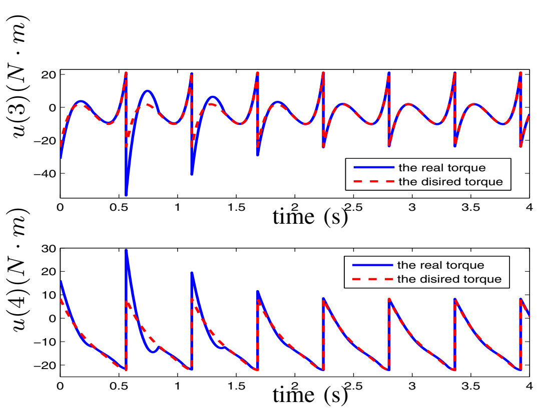

Fig. 6. The torques of the swing leg, where $u(3)$ notes the torque of torso and $u(4)$ notes the torque of knee.

As stated above, the stability of walking can be improved by pertinent choice of controlled outputs, furthermore, it is possible to produce better convergent property than the previous method for this planar bipedal model.

VI. CONCLUSIONS

In this paper, a simple planar bipedal model has been studied, with the objective of developing a time-variant feedback control law that induces asymptotically stable walking, without relying on the use of large feet. We showed that the property of zero dynamics of the walking model is affected by the choice of the controlled outputs. In addition, based on the numerical results, we conjectured a necessary condition for stable walking is that the height of the swing foot at the desired impact moment is close to zero. With this necessary condition, two large stable domains are obtained. Finally, the simulation results proved the validity and superiority of this method.

VII. ACKNOWLEDGMENT

The authors want to thank J. Grizzle and C.-L. Shih for their contribution to this study.

REFERENCES

[1] Y. Aoustin, C. Chevallereau, and A. Formal'sky, "Numerical and experimental study of a virtual quadrupedal walking robot - semiquad," *Multibody System Dynamic*, 2005.

[2] F. Asano and Z.-W. Luo, "Asymptotically stable gait generation for biped robot based on mechanical energy balance," in *Proceedings of the 2007 IEEE/RSJ International Conference on Intelligent Robots and Systems*, San Diego, USA, October 2007, pp. 3328–3333.

[3] ——, "Asymptotic stability of dynamic bipedal gait with constraint on impact posture," in *Proceedings of 2008 IEEE International Conference on Robotics and Automation*, USA, May 2008, pp. 1246–1251.

[4] C. Chevallereau, *Modern Control Theory*. Esculapio, Bologna (Italy), 1999, ch. 7,"Under actuated biped robot".

[5] C. Chevallereau, G. Abba, Y. Aoustin, F. Plestan, E. Westervelt, and et al., "Rabbit: A testbed for advanced control theory," *IEEE Control Systems*, vol. 23, no. 5, pp. 57–78, 2003.

[6] C. Chevallereau and Y. Aoustin., "Optimal reference trajectories for walking and running of a biped robot," *Robotica*, vol. 19, no. 5, pp. 557–569, 2001.

[7] C. Chevallereau, J. Grizzle, and C. Shih, "Asymptotically stable walking of a five-link underactuated 3d bipedal robot," *IEEE Trans. on Robotics*, vol. 25, no. 1, pp. 37–50, February 2009.

[8] T. Fukuda, M. Doi, Y. Hasegawa, and H. Kajima, *Fast Motions in Biomechanics and Robotics*. Springer-Verlag, 2006, chapter Multi-Locomotion Control of Biped Locomotion and Brachiation Robot.

[9] J. Grizzle, G. Abba, and F. Plestan, "Asymptotically stable walking for biped robots: Analysis via systems with impulse effects," *IEEE Transactions on Automatic Control*, vol. 46, pp. 51–64, January 2001.

[10] A. Isidori, *Nonlinear Control Systems*, 3rd ed. Berlin: Springer-Verlag, 1995.

[11] A. Kuo, "Stabilization of lateral motion in passive dynamic walking," *International Journal of Robotics Research*, vol. 18, no. 9, pp. 917–930, 1999.

[12] B. Morris and J. Grizzle, "Hybrid invariant manifolds in systems with impulse effects with application to periodic locomotion in bipedal robots," *IEEE Transactions on Automatic Control*, vol. 54, no. 8, pp. 1751–1764, 2009.

[13] F. Plestan, J. Grizzle, E. Westervelt, and G. Abba, "Stable walk-ing of a 7-dof biped robot," *IEEE Transactions on Robotics and Automation*, vol. 19, no. 4, pp. 653–668, 2003.

[14] T. Wang, C. Chevallereau, and C.-L. Shih, "Choice of output for time-variant walking control for a five-link underactuated planar biped robot," in *Proceedings of 2009 IEEE-RAS International Conference on Humanoid Robots*, Paris, France, December 2009.

[15] E. Westervelt, J. Grizzle, and D. Koditschek, "Hybrid zero dynamics of planar biped walkers," *IEEE Transactions on Automatic Control*, vol. 48, no. 1, pp. 42–56, 2003.

[16] E. Westervelt, J. Grizzle, C. Chevallereau, J. Choi, and B. Morris, *Feedback Control of Dynamic Bipedal Robot Locomotion*, ser. Control and Automation. Boca Raton: CRC Press, June 2007.