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A Unified Approach for Control of Redundant Mechanical Systems Under Equality and Inequality Constraints

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Abstract— **The equality and inequality constraints on constraint force and/or the actuator force/torque arise in several robotic applications, for which different controllers have been specifically developed. This paper presents a unified approach to control a rather general class of robotic systems with closed loops under a set of linear equality and inequality constraints using the notion of projection operator. The controller does not require the kinematic constraints to be independent, i.e., systems with time-varying topology can be dealt with, while demanding minimum-norm actuation force or torque in the case that the system becomes redundant. The orthogonal decomposition of the generalized force yields the tangential (potent) and normal (impotent) components leads. The tangential component is obtained using projected inverse dynamics control law, while the optimal normal component is found through solving a quadratic programming problem, in which the equality and inequality constraints are derived to be equivalent to the originally specified ones. Finally, a case study is presented to demonstrate how the control technique can be applied to multi-arms manipulation of an object.**

I. Introduction

The equality and inequality constraints on constraint force and/or the actuator force/torque arise in control of several robotic applications. Examples include cooperative manipulators holding an object [1], [2], robotic grasping [3], robotic hand manipulation [4], walking machine [5], backlash avoidance in parallel manipulators [6], tendon-based parallel manipulators [7], [8], parallel manipulators with passive joints [9], and control of redundant parallel manipulators subject to limited torque or force capability of the actuators [10], [11]. These systems often have a time-varying topology, in which forming new independent constraints reduces the system's degrees of freedom that may cause the system to be redundantly actuated.

In the literature, different control techniques have been developed for mechanical systems subject to unilateral constraints based on the individual applications. A method for obtaining real-time solution to the force optimization problem of redundantly actuated parallel mechanism with inequality constraints was presented in [12]. The control of planar rolling contacts in multi-arm manipulation is investigated in [1]. In this control approach, a minimal set of control inputs is employed to control the trajectory of the robotic system while the surplus inputs control the contact condition. Control of unilateral systems is addressed in this work, but the input force norm is not minimized. This technique was further developed for controlling of whole arm grasp [3]. Zefran *et al.* presented how the problem of setpoint control of robots that involves unilateral constraints can be optimally solved [2]. In this technique, firstly the constrained optimal control problem is formulated as an unconstrained problem of calculus of variation and then it is solved using an integral formulation. However, the control technique is not applicable to reference trajectory tracking. Backlash free control of parallel manipulators is another area involving unilateral constraints. It was shown by Muller that backlash in redundantly actuated parallel manipulators can be avoided if the magnitude of each particular control force remains above a certain level and that its sign does not change [6]. Control of tendon-based parallel manipulators also involve unilateral constraints since tendons can only transmit limited and tractive forces [8]. Motion control of a tendon-based parallel manipulator in which the tension constraints and limiting torque constraints of actuators are taken into account is presented in [7]. Experimental results demonstrated that the proposed control laws reduce the energy consumption of the actuators. Control schemes for redundant manipulators with passive joints subject to unilateral constraints arising from taking the actuators' torque limits into account were presented in [10], [11]. Other researchers have studied control of mechanical systems subject to unilateral constraints on position rather than on force [13], [14].

This work tends to unify control approaches for a rather general class of robotic systems subject to equality and inequality constraints using the notion of projection operator. The advantages of the projection-based controller is twofold: (i) it does not require the kinematic constraints be independent, hence a fixed controller can handle systems with time-varying topology which may change the degrees-of-freedom (ii) the controller demands minimum-norm actuation force or torque in the case that the system becomes redundant. The generalized force is decomposed into the tangential (potent) and normal (impotent) components, which are mutually orthogonal. First, the tangential component is obtained using projected inverse dynamics control law. Due to the orthogonal decomposition, minimizing the Euclidean norm of the normal component is tantamount to that of the generalized force. Therefore, the problem of finding the optimal normal component is formulated as a quadratic programming problem in which the equality and inequality constraints are derived so as to be equivalent to the originally specified ones. This paper is organized as follows: Section II reviews modeling of constrained mechanical systems using the projection operator matrix. Development of optimal tracking controller satisfying the equality and inequality

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constraints is presented in Section III. Finally, Section IV illustrates how the control technique can be applied to multi-arms manipulation of an object.

II. Modeling of Mechanical Systems Under Equality and Inequality Constraints

We consider a constrained mechanical system described by generalized coordinates $\boldsymbol{q} \in \mathbb{R}^n$ and a set of m constraint equations

$$\boldsymbol{A}\dot{\boldsymbol{q}} = \boldsymbol{0} \in \mathbb{R}^m. \tag{1}$$

Here, matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ may or may not be rank-deficient, i.e., if $r = \text{rank}(\boldsymbol{A})$ then $r \leq m$. The differential equations describing the motion of the system under unilateral and equality constraints on its forces is then written as

$$\begin{aligned} \boldsymbol{M}\ddot{\boldsymbol{q}}+\boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \boldsymbol{\tau} - \boldsymbol{A}^T\boldsymbol{\lambda} && \text{(2a)}\\ \text{subject to:}\quad \boldsymbol{B}_1\boldsymbol{\lambda} - \boldsymbol{b}_1 &\leq \boldsymbol{0} && \text{(2b)}\\ \boldsymbol{B}_2\boldsymbol{\tau} - \boldsymbol{b}_2 &\leq \boldsymbol{0} && \text{(2c)}\\ \boldsymbol{C}\boldsymbol{\tau} &= \boldsymbol{0}, && \text{(2d)} \end{aligned}$$

where $\boldsymbol{\tau} \in \mathbb{R}^n$ is the vector of generalized input force, $\boldsymbol{M} \in \mathbb{R}^{n\times n}$ is the positive-definite inertia matrix, vector $\boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}) \in \mathbb{R}^n$ contains the Coriolis, centrifugal, and gravitational terms, and $\boldsymbol{\lambda} \in \mathbb{R}^m$ is the vector of generalized lagrangian multipliers. Note, that (2a) is the dynamics system subject to c_{in_1} inequality constraints (2b), c_{in_2} inequality constraints (2c), and c_{eq} equality constraints (2d). Thus $\boldsymbol{B}_1 \in \mathbb{R}^{c_{in_1}\times m}$, $\boldsymbol{B}_2 \in \mathbb{R}^{c_{in_2}\times n}$, and $\boldsymbol{C} \in \mathbb{R}^{c_{eq}\times n}$.

Dynamics of a rather general class of constrained robotic systems can be modeled as (2). For example, as will be shown later in Section IV, the inequality constraints due to frictional contact in multi-arm manipulation of an object can be written in form of (2b). Parallel manipulators with limited force/torque capability of the actuators can be also modeled as (2) if $\boldsymbol{B}_2$ and $\boldsymbol{b}_2$ are selected as

$$\boldsymbol{B}_2 = \begin{bmatrix} \boldsymbol{1}_n \\ -\boldsymbol{1}_n \end{bmatrix}, \qquad \boldsymbol{b}_2 = \begin{bmatrix} \boldsymbol{\tau}_{\max} \\ \boldsymbol{\tau}_{\min} \end{bmatrix}$$

where $\boldsymbol{\tau}_{\min}$ and $\boldsymbol{\tau}_{\max}$ are the given values corresponds to the lower-bound and upper-bound limits of the actuation force or torque. Some parallel manipulators have unactuated joints, which are known as passive joints. The vector of generalized forces of such manipulators takes the form

$$\boldsymbol{\tau} = \begin{bmatrix} \boldsymbol{\tau}_{\text{act}} \\ \boldsymbol{0}_{c_{eq}\times 1} \end{bmatrix} \tag{3}$$

where $\boldsymbol{\tau}_{\text{act}}$ and the zero vector in (3) correspond to the active joints and passive joints, respectively. Therefore, any admissible vector of generalized force must satisfy equality condition (2d), in which

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{1}_{c_{eq}} \end{bmatrix}. \tag{4}$$

Now, given the constraint matrix $\boldsymbol{A}$, we can uniquely define *symmetric* matrix $\boldsymbol{P} \in \mathbb{R}^{n\times n}$, the *null-space orthogonal projector* of $\boldsymbol{A}$, as

$$\boldsymbol{P} \triangleq \boldsymbol{1}_n - \boldsymbol{A}^+\boldsymbol{A} \tag{5}$$

where $\boldsymbol{A}^+$ is the pseudo-inverse of $\boldsymbol{A}$ and $\boldsymbol{1}_n$ is the $n\times n$ identity matrix [15]. Because $\boldsymbol{P}$ is an orthogonal projection onto the null-space of the constraint matrix—also known as the *tangent space* of the constraint manifold—any vector in $\mathcal{N}(\boldsymbol{A})$ is projected onto itself, whereas any vector perpendicular to the tangent space lies in the null-space of $\boldsymbol{P}$. The vector $\dot{\boldsymbol{q}}$ of generalized velocities belongs to the former group as $\boldsymbol{P}\dot{\boldsymbol{q}} = \boldsymbol{P}^T\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}$.

In the light of the notion of projection operator, the vector of generalized forces $\boldsymbol{\tau}$ can be decomposed into two components denoted by subscripts $_\parallel$ and $_\perp$, lying in the orthogonal subspaces, namely, the tangent space $\mathcal{N}(\boldsymbol{A})$ and the null-space of $\boldsymbol{P}$, respectively:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_\parallel \oplus \boldsymbol{\tau}_\perp. \tag{6}$$

Because $\boldsymbol{\tau}_\perp \in \mathcal{N}^\perp(\boldsymbol{A})$ and the constrained motion occurs in $\mathcal{N}(\boldsymbol{A})$, by definition, this component of the actuation generalized forces does not contribute to the motion of the system [16]. Since $\mathcal{R}(\boldsymbol{A}^T) \in \mathcal{N}(\boldsymbol{A})^\perp$, the lagrangian multiplier term in (2a) can be eliminated if both sides of the latter equation is pre-multiplied by $\boldsymbol{P}$, that is

$$\boldsymbol{P}\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{P}\boldsymbol{h} = \boldsymbol{\tau}_\parallel. \tag{7}$$

It can be shown that the lagrangian multipliers are related to the decomposed components of the generalized force [17], $\boldsymbol{\tau}_\parallel$ and $\boldsymbol{\tau}_\perp$, by

$$\boldsymbol{\lambda} = \boldsymbol{A}^{+T}\big(\boldsymbol{\tau}_\perp - \boldsymbol{\psi}(\boldsymbol{\tau}_\parallel, \boldsymbol{q}, \dot{\boldsymbol{q}})\big) \tag{8}$$

where

$$\boldsymbol{\psi} = \boldsymbol{h} + \boldsymbol{\Delta}^{-1}(\boldsymbol{\tau}_\parallel - \boldsymbol{P}\boldsymbol{h} - \boldsymbol{M}\boldsymbol{A}^+\dot{\boldsymbol{A}}\dot{\boldsymbol{q}}), \tag{9}$$

and matrix $\boldsymbol{\Delta} = \boldsymbol{1}_n + \boldsymbol{P} - \boldsymbol{M}\boldsymbol{P}\boldsymbol{M}^{-1}$ is always invertible; see the Appendix for derivation.

III. Control

The number of independent generalized coordinates of the system is the DOF of the system $d = n - r$; recall that $r \leq m$ is the number of independent constraints. This means that one can control the constrained mechanical system by only controlling an independent set $\boldsymbol{x}(\boldsymbol{q}) \in \mathbb{R}^k$ of the generalized coordinates, where

$$k \leq d.$$

Now, differentiation of the given function $\boldsymbol{x}(\boldsymbol{q})$ with respect to time yields

$$\dot{\boldsymbol{x}} = \boldsymbol{\Lambda}\dot{\boldsymbol{q}}, \quad \text{and} \quad \ddot{\boldsymbol{x}} = \dot{\boldsymbol{\Lambda}}\dot{\boldsymbol{q}} + \boldsymbol{\Lambda}\ddot{\boldsymbol{q}} \tag{10}$$

where $\boldsymbol{\Lambda} = \partial\boldsymbol{x}(\boldsymbol{q})/\partial\boldsymbol{q} \in \mathbb{R}^{n\times k}$. Since vector $\boldsymbol{x}(\boldsymbol{q})$ constitutes a set of independent functions, then rank$(\boldsymbol{\Lambda}) = k$. Now, the control objective is to find input force, $\boldsymbol{\tau}$, with minimum Euclidean norm, i.e.,

$$\|\boldsymbol{\tau}\| \mapsto \min, \tag{11}$$

such that $\boldsymbol{x}$ tracks the desired trajectory $\boldsymbol{x}^*$ while satisfying constraints (2b), (2c) and (2d). Due to the orthogonal decomposition of the generalized force, we have

$$\|\boldsymbol{\tau}\| = \|\boldsymbol{\tau}_\parallel\| + \|\boldsymbol{\tau}_\perp\|. \tag{12}$$

Since the *impotent* component of the generalized force, $\boldsymbol{\tau}_\perp$, does not contribute to the motion of the system, we firstly find the potent component of generalized force, $\boldsymbol{\tau}_\parallel$, and then find the optimal impotent component, $\boldsymbol{\tau}_\parallel$.

Consider the following control law:

$$\boldsymbol{\tau}_\parallel = \boldsymbol{P}\boldsymbol{h} + \boldsymbol{P}\boldsymbol{M}\boldsymbol{\Lambda}^+\big(-\dot{\boldsymbol{\Lambda}}\dot{\boldsymbol{q}} + \ddot{\boldsymbol{x}}^* + \boldsymbol{K}_D\dot{\boldsymbol{e}} + \boldsymbol{K}_P\boldsymbol{e}\big), \quad (13)$$

where $\boldsymbol{e} = \boldsymbol{x}^* - \boldsymbol{x}$ is the position tracking error, and $\boldsymbol{K}_P > 0$ and $\boldsymbol{K}_D > 0$ are $k \times k$ gain matrices.

Theorem 1: Tracking error of system (2a) under the control law (13) exponentially converges to zero.

PROOF Substituting (13) into (7) yields dynamics of the closed loop system as

$$\boldsymbol{P}\boldsymbol{M}\boldsymbol{\Lambda}^+\big(\ddot{\boldsymbol{e}} + \boldsymbol{K}_D\dot{\boldsymbol{e}} + \boldsymbol{K}_P\boldsymbol{e}\big) = \boldsymbol{0} \quad (14)$$

The stability proof rests on showing that matrix $\boldsymbol{P}\boldsymbol{M}\boldsymbol{\Lambda}^+$ is full rank. In a proof by contradiction, we will show that the latter matrix is indeed full rank. If the matrix is rank deficient, then there must exist a non-zero vector $\boldsymbol{\zeta}$ such that

$$\boldsymbol{P}\boldsymbol{M}\boldsymbol{\Lambda}^+\boldsymbol{\zeta} = \boldsymbol{0} \qquad \text{where} \quad \boldsymbol{\zeta} \neq \boldsymbol{0} \quad (15)$$

Denote $\boldsymbol{\xi} \triangleq \boldsymbol{\Lambda}^+\boldsymbol{\zeta}$. Then, since $\mathcal{R}(\boldsymbol{\Lambda}^+) \subseteq \mathcal{R}(\boldsymbol{P})$, we can say $\boldsymbol{\xi} \in \mathcal{R}(\boldsymbol{P})$ meaning that $\boldsymbol{P}\boldsymbol{\xi} = \boldsymbol{\xi}$. Moreover, $\boldsymbol{\xi} \neq \boldsymbol{0}$ because matrix $\boldsymbol{\Lambda}^+$ is full rank and $\boldsymbol{\zeta} \neq \boldsymbol{0}$. Therefore, pre-multiplying both sides of (15) by $\boldsymbol{\xi}^T$ yields

$$\boldsymbol{\xi}^T\boldsymbol{M}\boldsymbol{\xi} = \boldsymbol{0} \qquad \text{where} \quad \boldsymbol{\xi} \neq \boldsymbol{0}, \quad (16)$$

which is a contradiction because $\boldsymbol{M}$ is a positive-definite matrix. Consequently, matrix $\boldsymbol{P}\boldsymbol{M}\boldsymbol{\Lambda}^+$ can not be rank deficient and the only possibility for (14) to happen is that the expression inside the parenthesis of (14) is identically zero. This completes the proof by noting that the gains are positive definite. Thus $\boldsymbol{x} \to \boldsymbol{x}^*$, $\dot{\boldsymbol{x}} \to \dot{\boldsymbol{x}}^*$ and $\ddot{\boldsymbol{x}} \to \ddot{\boldsymbol{x}}^*$ as $t \to \infty$. □

Now, we need to modify the motion control law (13) so as to fulfill the equality and inequality constraints (2b)-(2d). To achieve this goal we add an $\mathcal{N}^\perp$ component, say $\boldsymbol{\tau}_\perp$, to $\boldsymbol{\tau}_\parallel$, i.e., $\boldsymbol{\tau} = \boldsymbol{\tau}_\parallel + \boldsymbol{\tau}_\perp$, so that the equality and inequality conditions on the constraint forces are satisfied. Note that, since $\boldsymbol{\tau}_\perp$ does not affect the system motion at all, the motion tracking performance of the controller is preserved. Since $\boldsymbol{\tau}_\perp \in \mathcal{N}^\perp$, we have $\boldsymbol{P}\boldsymbol{\tau}_\perp = \boldsymbol{0}$. Therefore, the problem of finding optimal normal torque can be formulated by the following *quadratic programming*

$$\min \|\boldsymbol{\tau}_\perp\| \quad (17a)$$

$$\text{subject to:} \qquad \boldsymbol{W}\boldsymbol{\tau}_\perp - \boldsymbol{w} = \boldsymbol{0} \quad (17b)$$

$$\boldsymbol{U}\boldsymbol{\tau}_\perp - \boldsymbol{u} \leq \boldsymbol{0} \quad (17c)$$

where

$$\boldsymbol{W} = \begin{bmatrix}\boldsymbol{C}\\ \boldsymbol{P}\end{bmatrix} \qquad \boldsymbol{w} = \begin{bmatrix}-\boldsymbol{C}\boldsymbol{\tau}_\parallel\\ \boldsymbol{0}\end{bmatrix}$$
$$\boldsymbol{U} = \begin{bmatrix}\boldsymbol{B}_1\boldsymbol{A}^{+T}\\ \boldsymbol{B}_2\end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix}\boldsymbol{b}_1 + \boldsymbol{B}_1\boldsymbol{A}^{+T}\boldsymbol{\psi}(\boldsymbol{\tau}_\parallel)\\ \boldsymbol{b}_2 - \boldsymbol{B}_2\boldsymbol{\tau}_\parallel\end{bmatrix} \quad (18)$$

Note that inclusion of identity $\boldsymbol{P}\boldsymbol{\tau}_\perp = \boldsymbol{0}$ in the set of equality constraints (17b) ensures that the solution lies in subspace $\mathcal{N}^\perp$. The above *quadratic optimization program* with inequality and equality constraints has global minimizer. It can be solved by minimizing the quadratic function over a polyhedron [18].

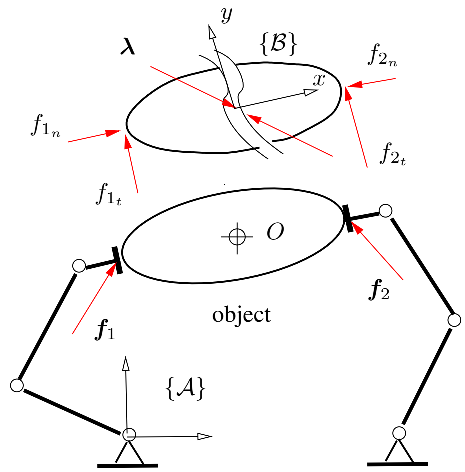

Fig. 1. Two manipulators holding an object.

In summary, computation of the control input force may proceed as the following steps:

i) Find the tangential (potent) component, $\boldsymbol{\tau}_\parallel$, from the projected inverse dynamics control law (13)
ii) With $\boldsymbol{\tau}_\parallel$ in hand, calculate the parameters of the quadratic programming (17) from the original constraint parameters specified in (2), and then solve the quadratic programming for optimal $\boldsymbol{\tau}_\perp$.

IV. An Illustrative Example

This section presents how the control technique described in Section III can be applied for the classical problem of cooperative control of two manipulators grasping an object. Fig. 1 illustrates two planar manipulators transporting an object. The goal is to determine the control joint torques possessing minimum possible norm such that the object follows a desired trajectory while maintaining the frictional contacts between the end-effectors and the object.

Let us suppose that $\boldsymbol{q}_i \in \mathbb{R}^3$ is the joint vector of the ith manipulators and $\boldsymbol{A}_i$ denotes the Jacobian matrix relating the velocity of the center of mass of the object to the joint velocities $\dot{\boldsymbol{q}}_i$. Then, the kinematic constraints can be expressed by

$$\boldsymbol{A}_1\dot{\boldsymbol{q}}_1 - \boldsymbol{A}_2\dot{\boldsymbol{q}}_2 = \boldsymbol{A}\dot{\boldsymbol{q}} = 0, \quad \text{where} \quad \boldsymbol{A} = \begin{bmatrix}\boldsymbol{A}_1 & -\boldsymbol{A}_2\end{bmatrix},$$

and $\dot{\boldsymbol{q}} = [\dot{\boldsymbol{q}}_1^T \;\; \dot{\boldsymbol{q}}_2^T]^T \in \mathbb{R}^6$. Now assume that the object body is cut right at the location of its center of mass and then the divided bodies are attached to the last links of the two manipulators. Then, the dynamics of the closed-chain system can be written as

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{h} = \boldsymbol{A}^T\boldsymbol{\lambda} + \boldsymbol{\tau} \quad (19)$$

where $\boldsymbol{M} = \text{diag}\{\boldsymbol{M}_1, \boldsymbol{M}_2\}$, $\boldsymbol{h} = [\boldsymbol{h}_1^T \quad \boldsymbol{h}_2^T]$, $\boldsymbol{\tau} = [\boldsymbol{\tau}_1^T \quad \boldsymbol{\tau}_2^T]^T$, and $\boldsymbol{\lambda}$ represents the internal force interaction

TABLE I

MANIPULATOR'S LINK PARAMETERS.

Properties	Link 1	Link 2	Link 3
Link mass (kg)	10	10	3.0
Link length (m)	0.3	0.3	0.1
Link inertia (kgm^2)	0.075	0.075	0.0025

between the object half bodies, while $\boldsymbol{M}_i$ and $\boldsymbol{h}_i$ are the inertia matrix and the nonlinear vector associated with the ith manipulator. Writing the balance of forces on the two divided bodies of the object, we get

$$\boldsymbol{f}_1 + \boldsymbol{\lambda} = \frac{1}{2}\boldsymbol{M}_o\ddot{\boldsymbol{x}}_o \tag{20a}$$

$$\boldsymbol{f}_2 - \boldsymbol{\lambda} = \frac{1}{2}\boldsymbol{M}_o\ddot{\boldsymbol{x}}_o \tag{20b}$$

where $\boldsymbol{x}_o = [x_o\ y_o\ \phi]^T$ and $\boldsymbol{M}_o \in \mathbb{R}^{3\times3}$ represent the pose and the inertia matrix of the object, respectively. Note that here $\boldsymbol{\lambda}$ is the internal force, which does not affect the motion of object, i.e., $\boldsymbol{f}_1+\boldsymbol{f}_2 = \boldsymbol{M}_o\ddot{\boldsymbol{x}}_o$. In order to model the rolling contact constraints, the normal and tangential components of the contact forces have to be calculated [19]. Let us suppose that unit vectors $\boldsymbol{n}$ and $\boldsymbol{t}$ are the inward pointing normal and the tangential for the object at the contact point of the first manipulator. Then, the normal and tangential components of the contact forces are:

$$f_{1_n} = \boldsymbol{n}^T\boldsymbol{f}_1, \qquad f_{1_n} = \boldsymbol{t}^T\boldsymbol{f}_1 \tag{21a}$$

$$f_{2_n} = -\boldsymbol{n}^T\boldsymbol{f}_2, \qquad f_{2_n} = \boldsymbol{t}^T\boldsymbol{f}_2 \tag{21b}$$

Now, according to the Coulomb's friction law, the contact is maintained and slipping will not occur if the following unilateral constraints are satisfied:

$$f_{i_n} \geq 0 \tag{22a}$$

$$\mu f_{i_n} - |f_{i_t}| \geq 0 \qquad i = 1, 2. \tag{22b}$$

where μ is coefficient of friction. Using identities (21) in inequalities (22), one can show that the inequality constrains can be written in the standard linear inequality form (2b) if the parameters are selected as

$$\boldsymbol{B}_1 = \begin{bmatrix} \boldsymbol{n}^T \\ \boldsymbol{n}^T \\ \mu\boldsymbol{n}^T - \boldsymbol{t}^T \\ \mu\boldsymbol{n}^T - \boldsymbol{t}^T \\ \mu\boldsymbol{n}^T + \boldsymbol{t}^T \\ \mu\boldsymbol{n}^T + \boldsymbol{t}^T \end{bmatrix}, \qquad \boldsymbol{b}_1 = \frac{1}{2}\begin{bmatrix} \boldsymbol{n}^T \\ -\boldsymbol{n}^T \\ \mu\boldsymbol{n}^T - \boldsymbol{t}^T \\ -\mu\boldsymbol{n}^T + \boldsymbol{t}^T \\ -\mu\boldsymbol{n}^T - \boldsymbol{t}^T \\ \mu\boldsymbol{n}^T + \boldsymbol{t}^T \end{bmatrix}\boldsymbol{M}_o\ddot{\boldsymbol{x}}_o.$$

Denoting $\boldsymbol{\Lambda} = \frac{1}{2}[\boldsymbol{A}_1\ \ \boldsymbol{A}_2]$, the acceleration term, $\ddot{\boldsymbol{x}}_o$, in the above equation can be calculated from

$$\begin{aligned}\ddot{\boldsymbol{x}}_o &= \dot{\boldsymbol{\Lambda}}\dot{\boldsymbol{q}} + \boldsymbol{\Lambda}\ddot{\boldsymbol{q}} \\ &= \dot{\boldsymbol{\Lambda}}\dot{\boldsymbol{q}} + \boldsymbol{\Lambda}\boldsymbol{M}_c^{-1}(\boldsymbol{\tau}_{\|} - \boldsymbol{P}\boldsymbol{h} + \boldsymbol{M}\boldsymbol{A}^{+}\dot{\boldsymbol{A}}\dot{\boldsymbol{q}})\end{aligned} \tag{23}$$

in which $\ddot{\boldsymbol{q}}$ is obtained from (25).

For the simulation, we assume that the two manipulators are identical and that they have uniform links with properties described in Table I. The object is with of length 0.2m, mass 7.0 kg and inertia 0.1 kgm^2. The coefficient of friction between the box and end-effectors is $\mu = 0.2$. The desired trajectories are specified as:

$$\boldsymbol{x}_o^*(t) = \begin{bmatrix} 0.4 + 0.1\sin 3t \\ 0.5 + 0.1\sin 6t \\ \pi/2 \end{bmatrix}$$

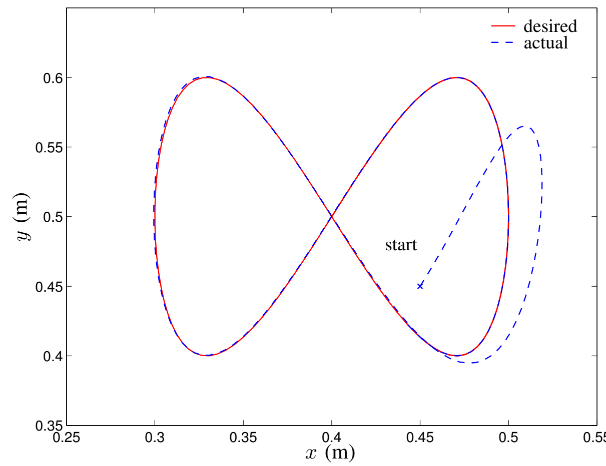

Fig. 2. Trajectories of the object's center of mass.

The technique described in Section III is applied for cooperative control of the manipulators, while the quadratic programming (17) is solved using function `quadprog` in Matlab [20]. Fig. 2 illustrates that the actual trajectories of the object converge to the the desired ones. The normal and tangential components of the contact forces are plotted in Figs. 3 and 4, respectively. It is apparent from the sign of the normal forces shown that manipulator always exerts pressing force to the object.

To ensure that the magnitude of the tangential force does not exceed the product of the normal force and the coefficient of static friction, the difference between the two quantities are calculated and the results are plotted in Fig. 5. Since the difference is always positive, the inequality constraints (22b) are satisfied. Time histories of the potent and impotent components of the first manipulator joint torques are shown in Fig. 6. Recall that $\boldsymbol{\tau}_{\|}$ is responsible to generate the desired motion trajectories, whereas $\boldsymbol{\tau}_{\|}$ is responsible to satisfy the unilateral constraints required for holding the object. For a comparison, a standard hybrid force/motion control scheme for tracking the motion reference and regulating the contact force to a constant value $\boldsymbol{\lambda}^*$ has been also applied; here $\boldsymbol{\lambda}^* = [30\ 30\ 0]^T$ is selected so that the axial contact forces be always positive. The Euclidean norms of the joint torque of the standard hybrid controller versus the proposed controller are depicted in Fig. 7. Clearly, the proposed controller always demands minimum-norm joint torques.

V. CONCLUSION

A control approach for a rather general class of robotic systems with closed loops under a set of linear equality and inequality constraints that minimizes the actuation

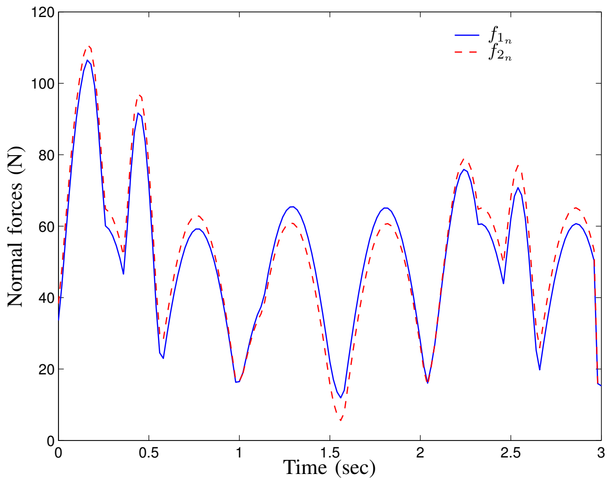

Fig. 3. The normal forces.

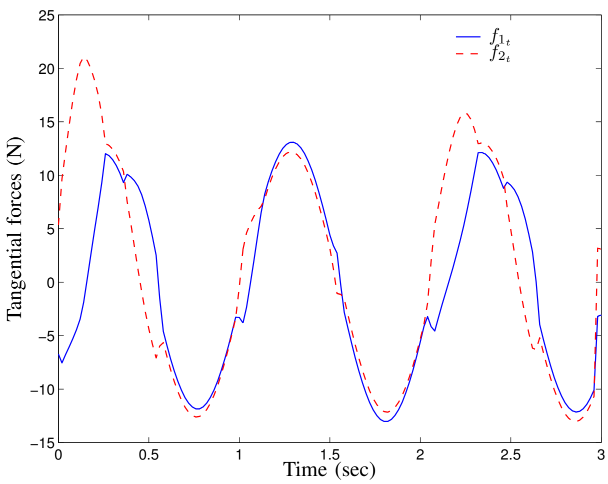

Fig. 4. The tangential forces.

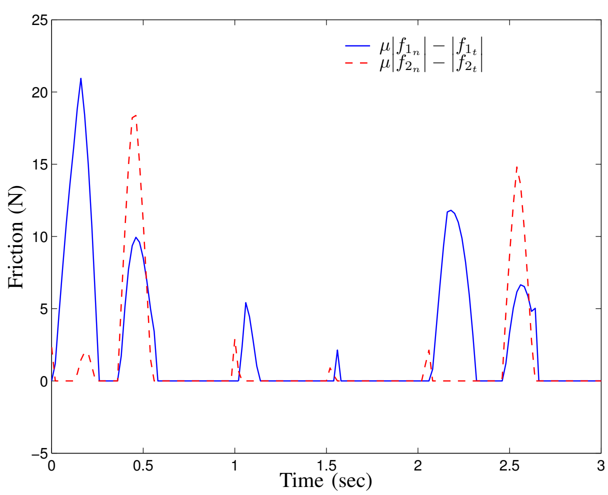

Fig. 5. The difference between critical friction and tangential force

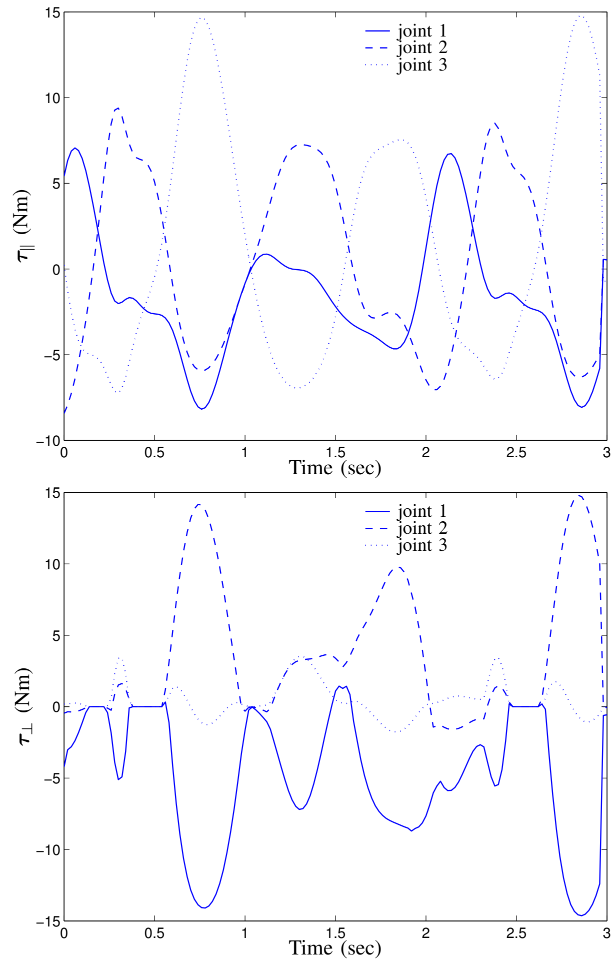

Fig. 6. Time history of the potent and impotent components of the joint torques.

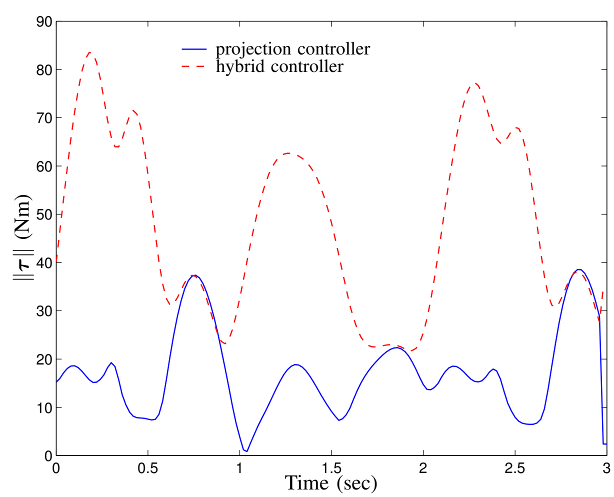

Fig. 7. Euclidean norm of input force.

force/torque was presented. The projection matrix corresponding to the null-space of constraint jacobian was used for orthogonal decomposition of the generalized force. The tangential component (potent) of generalized force is obtained using projected inverse dynamics. Since minimization of the Euclidean norm of generalized force is tantamount to that of its normal (impotent) component, the problem of finding the optimal normal component of generalized force was formulated as a quadratic programming problem after the original equality and inequality constraints are equivalently expressed linearly in terms of the the normal component. The control technique can be applied to a number of applications involving combination of unilateral and bilateral constrains such as cooperative manipulators holding an object, backlash avoidance in parallel manipulators, tendon-based parallel manipulators, parallel manipulators with passive joints, and control of redundant parallel manipulators subject to limited torque or force capability of the actuators. As an illustrative example, how the control technique can be applied for cooperative control of multi-arms grasping an object was presented.

Appendix

Time derivative of the constraint equation gives

$$\boldsymbol{A}\ddot{\boldsymbol{q}} = -\dot{\boldsymbol{A}}\dot{\boldsymbol{q}},$$

which leads to

$$(\mathbf{1}_n - \boldsymbol{P})\ddot{\boldsymbol{q}} = -\boldsymbol{A}^+\dot{\boldsymbol{A}}\dot{\boldsymbol{q}}. \tag{24}$$

Pre-multiplying both sides of equation (24) by $\boldsymbol{M}$ and then add both sides of the equation to those of (7), we arrive at

$$\boldsymbol{M}_c\ddot{\boldsymbol{q}} = \boldsymbol{\tau}_{\parallel} - \boldsymbol{P}\boldsymbol{h} - \boldsymbol{M}\boldsymbol{A}^+\dot{\boldsymbol{A}}\dot{\boldsymbol{q}} \tag{25}$$

where

$$\boldsymbol{M}_c \triangleq \boldsymbol{M} + \boldsymbol{P}\boldsymbol{M} - \boldsymbol{M}\boldsymbol{P} \tag{26}$$

is called *constraint inertia matrix*. Apparently, the term $\tilde{\boldsymbol{M}} = \boldsymbol{P}\boldsymbol{M} - \boldsymbol{M}\boldsymbol{P}$ in (26) is a skew-symmetric matrix because $\tilde{\boldsymbol{M}}^T = -\tilde{\boldsymbol{M}}$. Consequently, adding $\tilde{\boldsymbol{M}}$ to the inertia matrix in equation (26) preserves the positive definiteness property of the inertia matrix. This is because, for any vector $\boldsymbol{z} \in \mathbb{R}^n$ we can say $\boldsymbol{z}^T\boldsymbol{M}_c\boldsymbol{z} = \boldsymbol{z}^T\boldsymbol{M}\boldsymbol{z} > 0$. In the following we will retrieve the constraint force by projecting equation (2a) onto $\mathbf{1}_n - \boldsymbol{P}$, that gives

$$\boldsymbol{A}^T\boldsymbol{\lambda} = \boldsymbol{\tau}_{\perp} - (\mathbf{1}_n - \boldsymbol{P})(\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{h}) \tag{27}$$

Finally, substituting the acceleration from (25) into (27) yields (8).

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