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Simple Models in Trajectory Planning of Human-Like Reaching Movements

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***Abstract*— The paper deals with modeling of human-like reaching movements. Several models are under study. First, we consider a model of reaching movement that corresponds to the minimization of control effort. It is shown that this model is represented by the well-known Beta function. The representation can be used for the construction of fractional order models and also for modeling of asymmetric velocity profiles. The natural boundary conditions, defined in this part of the paper, can also be used in modeling asymmetric velocity profiles. Finally, we consider the minimum time formulation of the optimization problem and (for the n-th order integrator) find its analytical solution in the general form.**

I. INTRODUCTION

Modeling of human-like reaching movements is a very important research problem in modern robotics and computer animation. In this paper, we address this problem using an optimization approach where the trajectory of the human arm is predicted by minimizing, over the movement time T, an integral performance index $\mathcal{J}$ subject to boundary conditions imposed on start and end points. In our analysis, we will consider a simplified, one-dimensional model of human movements where the configuration dependence of the human arm is ignored and the motion considered at the hand level. This simplified model is not an end in itself. It just helps to introduce and clarify different issues, such as the smoothness and the asymmetry of the hand velocity profiles, arising in the computational representation of reaching movements.

Several modeling issues are studied in this paper. First, in Section II we consider a generalization of the classical minimum jerk model of reaching movements [1], relate it to the lowest polynomial approach, and derive the solution in the form of the regularized incomplete Beta function. The solution was first obtained in [2]. Here, we present a different (more solid and compact) derivation and show how this representation can be useful for modeling of asymmetric velocity profiles. The fractional order models of reaching movements are also discussed and illustrated in this section.

Next, in Section III, we address the boundary conditions of the optimization task. Since the optimal solution is defined not only by the optimality criterion but also by the boundary conditions, the latter is not less important than the former. However, the role of the boundary conditions is rarely discussed in the literature on reaching movements. We analyzed it in [3]. However, the system under consideration in [3] was somewhat complex (the manipulation of multi-mass flexible objects in virtual dynamic environments). Certainly, the boundary conditions can be (and even, methodologically speaking, should be) communicated first for the free-space reaching movements. In this section we first introduce natural boundary conditions. We show then that a weighted formulation of the natural boundary conditions can be used for modeling of coordinate jumps and also for fitting asymmetric velocity profiles.

At last, in Section IV, we address the formation of reaching movements with bounded control actions and consider the minimum time formulation of the optimization problem. Here, for the model under consideration (n-th order integrator) we first establish the general analytical solution that is not available in the literature on optimal control. We show then that this solution produces smooth velocity profiles for $n > 2$. We thus argue that, contrary to popular opinion, the minimum time model can be a reasonable candidate for modeling human-like reaching movements. Finally, conclusions are summarized in Section V.

II. MINIMUM EFFORT MODEL

Consider a generalized criterion of optimality minimizing the squared n-th derivative of the hand position x over the movement time:

$$\mathcal{J} = \frac{1}{2}\int_0^T \left(\frac{d^n x}{dt^n}\right)^2 dt. \tag{1}$$

From the control theoretical point of view this is the minimum effort criterion for the controlled system $x^{(n)}(t) = u$, describing the hand movement in the free space. Here, u is the control input, and n is the order of the motor dynamics associated with the hand. This order implies $2n$ boundary conditions necessary for the correct formulation of the optimization problem. The state vector is composed of the hand position and its first $n-1$ derivatives. Assuming that the hand is at rest in the beginning and in the end of the reaching movement, one assigns

$$x(0)=0,\quad \dot{x}(0)=0,\quad \ddot{x}(0)=0,\ldots,x^{(n-1)}(0)=0, \tag{2}$$
$$x(T)=L,\quad \dot{x}(T)=0,\quad \ddot{x}(T)=0,\ldots,x^{(n-1)}(T)=0, \tag{3}$$

where L stands for the length of reaching movement.

As can be shown, the Euler-Lagrange equation for the criterion (1) is $x^{(2n)}(t) = 0$. The general solution for this equation is given by the polynomial $x(t) = \sum_{i=0}^{2n-1} c_i\, t^i$, where the coefficients c_i are established from the boundary conditions (2,3). Thus, the optimization problem under consideration is equivalent to the construction of the lowest order polynomial satisfying the boundary conditions (2,3).

To clarify the analytical structure of the solution, we recast the optimization task as the optimal control problem. Let us introduce the state vector $\boldsymbol{x} = \{x, \dot{x}, \ldots, x^{(n-1)}\}^{\mathrm{T}}$ and define the control input $u = x^{(n)}$. The state dynamics are the n-th

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order integrator $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u$, where $\boldsymbol{b} = \{0, \ldots, 0, 1\}^{\mathrm{T}}$, and the elements of the matrix $\boldsymbol{A}$ are defined as

$$[\boldsymbol{A}]_{ij} = \begin{cases} 1 & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The analytical solution for the minimum effort control problem, seeking the control minimizing $\mathcal{J} = \frac{1}{2}\int_0^T u^2\,dt$ for the dynamic system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}+\boldsymbol{b}u$ and the given boundary conditions $\boldsymbol{x}(0)$ and $\boldsymbol{x}(T)$, is well-established in the control literature [4]. It can be represented as

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\left(\left\{\boldsymbol{I} - \boldsymbol{W}(t)\boldsymbol{W}^{-1}(T)\right\}\boldsymbol{x}(0) + \boldsymbol{W}(t)\boldsymbol{W}^{-1}(T)e^{-\boldsymbol{A}T}\boldsymbol{x}(T)\right), \tag{5}$$

where

$$\boldsymbol{W}(t) = \int_0^t e^{-\boldsymbol{A}s}\boldsymbol{b}\boldsymbol{b}^{\mathrm{T}}e^{-\boldsymbol{A}^{\mathrm{T}}s}ds. \tag{6}$$

where T stands for the operation of transposition. For the n-th order integrator the matrix exponent is defined as

$$\left[e^{\boldsymbol{A}s}\right]_{ij} = \begin{cases} \dfrac{s^{j-i}}{(j-i)!} & \text{if } j \geq i, \\ 0 & \text{if } j < i, \end{cases} \tag{7}$$

and by direct calculations one finds

$$[\boldsymbol{W}(t)]_{ij} = \frac{(-1)^{i+j}\, t^{2n+1-i-j}}{(2n+1-i-j)(n-i)!(n-j)!}. \tag{8}$$

As shown in [5], the matrix $\boldsymbol{W}(t)$ can be decomposed as $\boldsymbol{W}(t) = t\boldsymbol{P}(-t)\boldsymbol{H}\boldsymbol{P}(-t)$, where

$$[\boldsymbol{P}(t)]_{ij} = \begin{cases} \dfrac{t^{n-j}}{(n-i)!} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

and $\boldsymbol{H}$ is the Hilbert matrix

$$[\boldsymbol{H}]_{ij} = \frac{1}{(2n+1-i-j)}, \tag{10}$$

the inverse of which is known to be

$$\left[\boldsymbol{H}^{-1}\right]_{ij} = \frac{(-1)^{i+j}(2n-i)!(2n-j)!}{(2n+1-i-j)\{(n-i)!(n-j)!\}^2(i-1)!(j-1)!}. \tag{11}$$

The elements of $\boldsymbol{W}^{-1}(t)$ can now be easily established:

$$\left[\boldsymbol{W}^{-1}(t)\right]_{ij} = t^{-1}\boldsymbol{P}^{-1}(-t)\boldsymbol{H}^{-1}\boldsymbol{P}^{-1}(-t) = \frac{(2n-i)!(2n-j)!}{(2n+1-i-j)(n-i)!(n-j)!(i-1)!(j-1)!t^{2n+1-i-j}}. \tag{12}$$

For the rest-to-rest movements $\boldsymbol{x}(0) = \boldsymbol{0}$, $\boldsymbol{x}(T) = \{L, 0 \ldots, 0\}^{\mathrm{T}}$, and $e^{-\boldsymbol{A}T}\boldsymbol{x}(T) = \boldsymbol{x}(T)$. The components of the vector $\boldsymbol{x}(t)$, which are the derivatives of the corresponding order of $x(t)$, are found from (5,7,8,12) by direct computation:

$$x_i(t) \triangleq x^{(i-1)}(t) = \frac{L}{T^{i-1}}\sum_{j=i}^{n}\sum_{s=1}^{n} \frac{(-1)^{j+s}(2n-1)!(2n-s-1)!\tau^{2n+1-i-s}}{(j-i)!(2n+1-s-j)\{(n-s)!\}^2(n-j)!(n-1)!(s-1)!}, \tag{13}$$

where $\tau = t/T$. To find the velocity $\dot{x}(t)$, we set $i = 2$ and, after some manipulations with the binomial expansion for $(1-\tau)^{n-1}$, obtain $\dot{x}(t) = L\dot{\bar{x}}(\tau)/T$, where the normalized velocity

$$\dot{\bar{x}}(\tau) = \frac{(2n-1)!\tau^{n-1}}{(n-1)!}\sum_{s=1}^{n}\frac{(-1)^{n+s}\tau^{n-s}}{(n-s)!(s-1)!} = \frac{\tau^{n-1}(1-\tau)^{n-1}}{B(n,n)}, \tag{14}$$

and

$$B(n,n) = \frac{(n-1)!(n-1)!}{(2n-1)!} \tag{15}$$

is the symmetric Beta function. Therefore, the normalized solution $\bar{x}(\tau) \triangleq x(t)/L$ can be formally represented as

$$\bar{x}(\tau) = \frac{\int_0^\tau p^{n-1}(1-p)^{n-1}dp}{B(n,n)} \triangleq \frac{B(\tau; n, n)}{B(n,n)} \triangleq \bar{B}(\tau; n, n). \tag{16}$$

This expression is known as the regularized incomplete Beta function [6]. The first 10 solutions for the normalized velocity $\dot{\bar{x}}(\tau)$ are plotted in Figure 1. While the solutions are continuous functions of time for any fixed n, in the limiting case of $n \to \infty$ the solution is discontinuous.

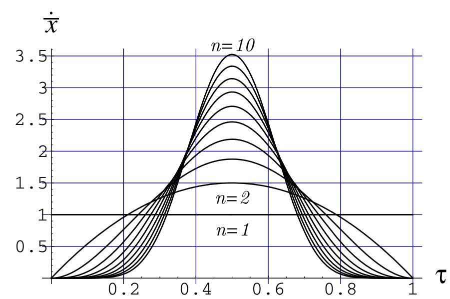

Fig. 1. Normalized velocities in the minimum effort model for $n = 1, \ldots, 10$.

Several comments are in order.

1°. It is noticed in [1] that while, in general, the minimum jerk model ($n = 3$) is a good candidate for mimicking human-like movements, the minimum snap model ($n = 4$) sometimes also provide a reasonable fit to the experimental data of unconstrained reaching movements. In this connection, it should be noted that the Beta function is defined not only for integer but also for real n. This suggests that the criterion (1) can be generalized to non-integer orders using, for example, Riemann-Liouville fractional integrals and derivatives [7]. It also suggests that in fitting experimental data of reaching movements we can use real numbers $n \in [3, 4]$ (see Figure 2). The assumption of real n sounds biologically plausible because the natural sensors and actuators in the human body are likely to deal with the fractional derivatives.

2°. It is well established that for relatively fast movements the velocity profile of the hand trajectory tends to a symmetric bell-shaped form [8], [9]. However, for relatively slow movements the velocity profiles tends to be left-skewed [10], [11]. There also exists experimental evidence that for very fast movements the velocity profiles can be even slightly skewed to the right [10], [12].

Note that the symmetry of the Beta function with respect to its last two arguments in (14,16) is explained by the symmetric placement of the boundary conditions. It can

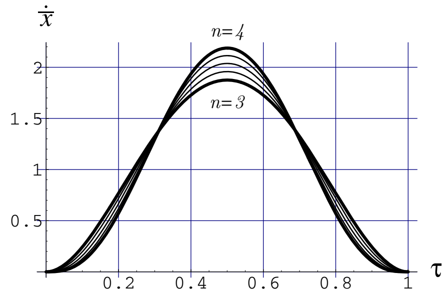

Fig. 2. Normalized velocity profiles for the integer (thick lines: $n = 3$ and $n = 4$) and fractional (thin lines: $n = 3.25$, $n = 3.5$, and $n = 3.75$) orders.

be shown that for the asymmetric placement (n_a boundary conditions at the start point and n_b at the end point, $n_a + n_b = 2n$) the solution is expressed through the asymmetric Beta function

$$\bar{x}(\tau) = \frac{\int_0^\tau p^{n_a-1}(1-p)^{n_b-1}dp}{B(n_a, n_b)} \triangleq \frac{B(\tau; n_a, n_b)}{B(n_a, n_b)}, \quad (17)$$

where

$$B(n_a, n_b) = \frac{(n_a-1)!(n_b-1)!}{(n_a+n_b-1)!}. \quad (18)$$

The asymmetric Beta functions are illustrated in Figure 3.

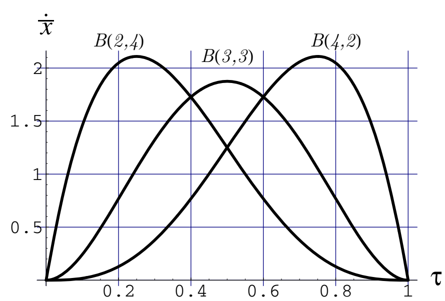

Fig. 3. Normalized velocity profiles given by the symmetric and asymmetric Beta functions.

3°. Note that while the function (17) is the lowest order polynomial satisfying the asymmetric boundary conditions, it does not have a variational meaning. Strictly speaking, it does not minimize (1) because in the classical variational formulation the boundary conditions at the end-points (those that are imposed directly, as in (2,3), and those obtained automatically from the variational formulation and sometimes called the natural boundary conditions [13]) are always placed symmetrically.

III. Natural boundary conditions

This section addresses the formation of the boundary conditions. They are an important part of the optimization problem because the optimal solution is defined not only by the form of the optimality criterion but also by the boundary conditions, and they are not less important than the criterion itself.

In the optimization problem considered in Section II we fixed first $n-1$ derivatives of the hand position, which, as far as human movements are concerned, might not be always plausible from the physiological point of view[1]. Note that in experiments human subjects are normally not literally requested to produce zero end-point accelerations; they are only instructed to move comfortably from one point to another and stop there. Therefore, it does not seem reasonable to constrain the initial and final hand accelerations to zero in the optimization models. We can reliably pose the boundary conditions only for position and velocities. The remaining boundary conditions should be established differently.

Consider again the optimality criterion given by (1) but assume now that instead of (2,3) we fix only the first $k \le n-1$ derivatives of the hand position:

$$x(0) = 0, \quad \dot{x}(0) = 0, \ldots, \quad x^{(k)}(0) = 0, \quad (19)$$

$$x(T) = L, \quad \dot{x}(T) = 0, \ldots, \quad x^{(k)}(T) = 0. \quad (20)$$

The variation of the functional (1) is defined as

$$\delta\mathcal{J} = \int_0^T x^{(2n)}(t)\,\delta x\,dt + \left[\sum_{i=0}^{n-1} x^{(2n-1-i)}(t)\,\delta x^{(i)}(t)\right]_0^T. \quad (21)$$

Taking into account that $\delta x^{(i)}(0) = \delta x^{(i)}(T) = 0$ for $i = 0, \ldots, k$, from the necessary condition of optimality, $\delta\mathcal{J} = 0$, one obtains the Euler-Lagrange equation $x^{(2n)}(t) = 0$, and establishes the following boundary conditions:

$$x^{(n)}(0) = 0, \ x^{(n+1)}(0) = 0, \ldots, \ x^{(2n-2-k)}(0) = 0, \quad (22)$$

$$x^{(n)}(T) = 0, x^{(n+1)}(T) = 0, \ldots, x^{(2n-2-k)}(T) = 0. \quad (23)$$

Note that in the calculus of variations the boundary conditions (22,23), obtained from the condition $\delta\mathcal{J} = 0$, are called natural [13]. The solution of the optimization problem is the polynomial of degree $2n-1$ whose coefficients are established from the directly imposed (19,20) and the natural (22,23) boundary conditions.

Consider, for the purpose of illustration, the case of $n = 3$ and $k = 1$. In this model we fix the boundary conditions for the position and velocities of the hand, while the hand accelerations in the beginning and the end of movement are defined automatically from the minimization of the squared hand jerk over the movement time. This differs from the conventional formulation where the boundary conditions for accelerations are set to zero. The minimum hand jerk model with the boundary conditions (19,20) and (22,23) yields

$$x = L\tau^2(2.5 - 2.5\tau^2 + \tau^3), \quad (24)$$

where $\tau = t/T$, while the conventional minimum hand jerk model gives

$$x = L\tau^3(10 - 15\tau + 6\tau^2). \quad (25)$$

Both these models produce single-peaked velocity profiles. However, the maximal velocity in the former model is $1.5625L/T$, while in the latter one it is $1.875L/T$.

It should be noted that the solution (24) features non-zero initial and final acceleration jumps because in this model $\ddot{x}(0) \neq 0$ and $\ddot{x}(T) \neq 0$. To avoid possible confusion, it should be said that no coordinate jumps are physical, of course. Strictly speaking, the exact modeling of this

[1]For example, in the minimum snap model ($n = 4$) the end-point jerk are supposed to be zero. However, it is not evident that human can do the end-point control of the 3rd derivative of the hand position.

phenomenon requires more sophisticated models considered at different time scales as the coordinate (acceleration, in our case) change is considerably high in a short instance of time. However, in simplified models, such as the one considered in our paper, this behavior can be modeled by the jumps in the boundary conditions.

The assumption of zero initial and final hand accelerations can be too restrictive when dealing with fast movements in highly dynamic reaching tasks [3]. For the free-space movements the applicability of the models (24) or (25) is likely to depend on the movement duration T. For relatively slow movements the central nervous system (CNS) has enough time to nullify the end-point accelerations and this defines the model (25). For relatively fast movements the end-point accelerations are placed at will and one gets the model (24).

To unify the minimum hand jerk model (24) with natural boundary conditions for accelerations with the conventional one (25), it is reasonable to assume that the CNS controls the boundary conditions. Intuitively, the way they are controlled is changed from task to task and, in general, the more dynamic the reaching task the looser the control of the boundary conditions.

To reflect the cost of control of the boundary conditions, one can replace the optimality criterion (1) by

$$\mathcal{J} = \frac{1}{2}\int_0^T \left(\frac{d^n x}{dt^n}\right)^2 dt + \frac{1}{2}\sum_{i=0}^{n-1} w_{i0}\left(x^{(i)}(0) - \tilde{x}_0^i\right)^2 + w_{iT}\left(x^{(i)}(T) - \tilde{x}_T^i\right)^2. \quad (26)$$

Here, $\tilde{x}_0^i$ and $\tilde{x}_T^i$ are the "desired" steady state values for the derivatives of the hand trajectory at the boundary points, and $w_{i0} \geq 0$ and $w_{iT} \geq 0$ are the corresponding weight coefficients assigned by the CNS. For simplicity, we assume that $\tilde{x}_0^i = 0$ for $i = 0, \ldots, n$, $\tilde{x}_T^i = 0$ for $i = 1, \ldots, n$, and $\tilde{x}_T^0 = L$.

Consider now the criterion of optimality given by (26). Note that now the separation of the boundary conditions into the directly imposed and the natural ones is no longer necessary, as we assume that all of them are controlled by the CNS and all of them are established from the condition $\delta\mathcal{J} = 0$. In other words, *all the boundary conditions are natural* now. It is straightforward to show that the variation of the functional (26) is defined as

$$\delta\mathcal{J} = \int_0^T x^{(2n)}(t)\,\delta x\,dt + \left[\sum_{i=0}^{n-1} x^{(2n-1-i)}(t)\,\delta x^{(i)}(t)\right]_0^T + \sum_{i=0}^{n-1} w_{i0}\left(x^{(i)}(0) - \tilde{x}_0^i\right)\delta x^i(0) + w_{iT}\left(x^{(i)}(T) - \tilde{x}_0^i\right)\delta x^i(T). \quad (27)$$

From $\delta\mathcal{J} = 0$ one obtains the Euler-Lagrange equation, $x^{(2n)}(t) = 0$, and establishes the natural boundary conditions. Since the variations $\delta x^i(t)$ are not restricted at the boundary points, one obtains:

$$-x^{(2n-1-i)}(0) + w_{i0}\left(x^{(i)}(0) - \tilde{x}_0^i\right) = 0, \quad (28)$$

$$x^{(2n-1-i)}(T) + w_{iT}\left(x^{(i)}(T) - \tilde{x}_T^i\right) = 0, \quad (29)$$

where $i = 0, \ldots, n-1$. It is worth pointing out two limiting cases. First, if $w_{i0} \to \infty$ (the CNS places a very high gain) the initial state for the i-th derivative coincides with the desired one, as the condition (28) is reduced to $x^{(i)}(0) = \tilde{x}_0^i$. If, on the other hand, the CNS cannot afford to control $x^{(i)}(0)$ then $w_{i0} \to 0$ and we get $x^{(2n-1-i)}(0) = 0$.

Let us continue the illustrative example of $n = 3$ and assign $\tilde{x}_0^i = 0$, $i = 0, 1, 2$, $\tilde{x}_T^0 = L$, and $\tilde{x}_T^i = 0$, $i = 1, 2$. Assuming the tight control for the initial and final position and velocity, we set $w_{i0} = w_{iT} = \infty$, $i = 0, 1$. By considering, for simplicity, the symmetric assignment of the accelerations weights, define $w \triangleq w_{20} = w_{2T}$. It is not difficult to show that the optimal solution, parameterized by the acceleration weight w is defined as

$$x = \frac{L\tau^2\left(30 + 10\bar{w}\tau - 15(2+\bar{w})\tau^2 + 6(2+\bar{w})\tau^3\right)}{(12+\bar{w})}, \quad (30)$$

where $\tau = t/T$, and $\bar{w} = wT$. For $w = 0$ we get the model (24) with natural boundary conditions for accelerations, and for $w \to \infty$ we get the conventional minimum hand jerk model (25). Changing the acceleration weight from zero to infinity defines a family of curves (all in the class of the minimum hand jerk models). The normalized velocity and acceleration profiles of this family are shown in Fig. 4.

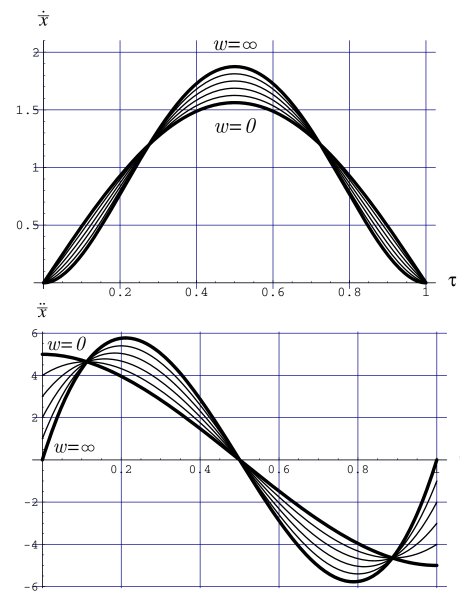

Fig. 4. Velocity and acceleration profiles for the family of curves (30).

Several comments are in order.

1°. Asymmetric velocity profiles can be obtained by asymmetric placement of the weight coefficients ($w_{i0} \neq w_{iT}$). Note also that the optimality criterion (26) can be generalized by the introduction of matrix weight coefficients.

2°. It is not clear how specifically the CNS may assign the weight coefficients. Intuitively, the human perception and control of position is better than that of velocity, that of velocity is better than that of accelerations, and so on, and it is plausible to assume that $w_{i,0} > w_{i+1,0}$ and $w_{i,T} > w_{i+1,T}$. The specific assignment of the weight coefficients must depend on how fast the reaching movement under inspection is. In this connection, it should be noted that the analysis of reaching movements is often conducted under the subtle assumption that the CNS operates without undue stress, away from the limits of neuromuscular performance [1]. But this assumption is not specific.

It is reported that for comfortable reaching movements in free space the movement time is in the range of $[0.25, 0.8]$s for the traveling distance varying from 0.2m to 0.5m [14]. Since the conventional (minimum hand jerk or and the minimum torque change) models are experimentally verified for these ranges, one can presume that in such movements the boundary conditions for the hand accelerations are zero. However, if the movement time is decreased the CNS obviously cannot afford to keep the initial hand accelerations zero. Therefore, for certain ranges of time the reaching movements will feature the acceleration jumps (soft impacts). And if the movement duration is decreased further, the movements will feature even the velocity jumps (hard impact). One can see such impulsive movements in the exercises of karate fighters.

Obviously, the movement time T cannot be reduced arbitrarily without violating the constraints on the magnitude of the control input. These constraints were ignored so far in our paper. Taking them into account would lead to different models with possibly different criteria of optimality. One of these models is the minimum time model.

IV. Minimum time model

In this section, the hand trajectory $x(t)$ is determined by minimizing the performance index

$$\mathcal{J} = \int_0^T dt = T, \tag{31}$$

where T is the movement time to be obtained. For the simplicity of formulation, we will consider the rest-to-rest time-optimal control where the boundary conditions are defined by (2,3) and the scalar control input is bounded as

$$|u(t)| \leq U. \tag{32}$$

For a linear controllable system, existence and uniqueness of the time-optimal solution are guaranteed, and the solution is "bang-bang" control with a finite number of switches [15]. It is known [16] that for the system under consideration (n-th order integrator) the optimal solution is symmetric about $T/2$, and the number of switches is $n+1$ ($t_0 = 0, t_1, t_2, \ldots, t_n = T$). However, to the best of our knowledge, the analytical form of the general (with respect to the order n) solution does not seem to be available in the literature.

Assuming $L > 0$, we set $u(t) = (-1)^s U$ for the time interval $t \in [t_s, t_{s+1}]$, $s = 0, 1, \ldots, n-1$, and, by integrating $x^{(n)} = u(t)$, define the $(n-k)$-th derivative ($k = 1, \ldots, n$) of the hand position:

$$x^{(n-k)}(t) = \frac{(t-t_s)^k}{k!}(-1)^s U + \sum_{p=0}^{k-1} x^{(n-k+p)}(t_s)\frac{(t-t_s)^p}{p!}. \tag{33}$$

By setting $t = t_{s+1}$ and considering the resulting recurrent relationship with the zero initial condition (2), one can represent $x^{(n-k)}(t_s)$ in the following non-recurrent form

$$x^{(n-k)}(t_s) = \frac{U}{k!}\sum_{i_1=1}^{s}\sum_{i_2=1}^{s}\cdots$$
$$\cdots\sum_{i_k=1}^{s}(-1)^{\min\{i_1,i_2,\ldots,i_k\}-1}\Delta t_{i_1}\Delta t_{i_2}\ldots\Delta t_{i_k}, \tag{34}$$

where $\Delta t_i = t_i - t_{i-1}$, and $s = 1, 2, \ldots, n$. Now, substitute $x^{(n-k)}(t_n)$, $k = 1, 2, \ldots, n$, into the boundary condition (3). After some algebra, one obtains the following system of n equations

$$2\left\{\sum_{i=1}^{n-1}(-1)^i t_i^k\right\} + (-1)^n t_n^k = 0, \quad k = 1, \ldots, n-1, \tag{35}$$
$$2\left\{\sum_{i=1}^{n-1}(-1)^i t_i^n\right\} + (-1)^n t_n^n = (-1)^n n! L/U, \tag{36}$$

with respect to the unknown switching times $t_1, t_2, \ldots, t_n$. The system (35) is homogenous, and with the introduction of the non-dimensional switching times $\tau_i = t_i/t_n$ it is converted to the following form

$$\sum_{i=1}^{n-1}(-1)^i \tau_i^k = (-1)^{n+1}/2, \quad k = 1, \ldots, n-1. \tag{37}$$

The solution for this system is given by

$$\tau_i = \sin^2\left(\frac{\pi i}{2n}\right), \quad i = 1, \ldots, n-1. \tag{38}$$

This can be shown by converting (38) to the double angle representation, substituting the result into (37), and using the known (see section 4.4.2 of [17]) identity

$$\sum_{k=0}^{n-1}(-1)^k \cos^m \frac{\pi k}{n} = \begin{cases} \frac{1}{2}\left\{1 - (-1)^{m+n}\right\} & \text{for } m = 0, 1, 2, \ldots, n-1 \\ \frac{n}{2^{n-1}} & \text{for } m = n \end{cases} \tag{39}$$

for $m < n$.

To define the movement duration $T \triangleq t_n$, rewrite (36) as

$$2\left\{\sum_{i=1}^{n-1}(-1)^i \tau_i^n\right\} + (-1)^n = (-1)^n n! \frac{L}{UT^n}. \tag{40}$$

By using the identity (39) for $m = n$, it can be shown that for the switching times (38) one has

$$\sum_{i=1}^{n-1}(-1)^i \tau_i^n = -\frac{(-1)^n}{2} + \frac{(-1)^n n}{2^{2n-1}}. \tag{41}$$

Therefore, the minimal movement time is defined as

$$T = \sqrt[n]{2^{2(n-1)}(n-1)! L/U}, \tag{42}$$

and the control switching times $t_0, t_1, t_2, \ldots, t_n$ are computed as

$$t_i = \sqrt[n]{\frac{2^{2(n-1)}(n-1)! L}{U}}\, \sin^2\left(\frac{\pi i}{2n}\right), \quad i = 0, \ldots, n. \tag{43}$$

To the best of our knowledge, this general solution has not been reported in the literature.

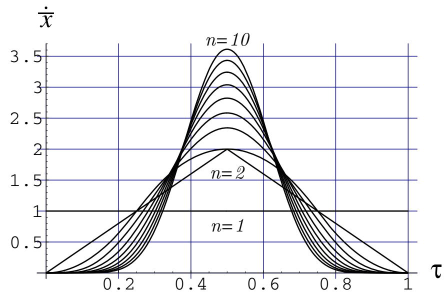

Fig. 5. Normalized velocities in the minimum time model for $n = 1, \ldots, 10$.

The first 10 solutions for the normalized velocity profiles in the minimum time model are plotted in Figure 5. Comparing these plots with those shown in Figure 1, one can say that qualitatively the minimum time model is similar to the minimum effort model.

Several comments are in order.

1°. It should be noted that in modeling of rapid reaching movements the minimum time model is often rejected (in favor of, say, the minimum hand jerk model) on the ground that it features acceleration jumps, which is not compatible with experimental data [12], [18]. The rejection of the minimum time model only for this reason is premature because it is based on the fallacious comparison of the minimum time model for $n = 2$ with the minimum effort model for $n = 3$.

2°. The minimum time model for $n = 3$ does not feature acceleration jumps (the discontinuities here are at the jerk level). The velocity profile in this model looks like the typical bell-shaped pattern observed in experiments, with the normalized maximal velocity being between those in the minimum hand jerk ($n = 3$) and snap ($n = 4$) models (see Table I). Thus, the minimum time model for $n = 3$ is a reasonable candidate for mimicking human-like reaching movements.

TABLE I

NORMALIZED VELOCITY AT THE MIDDLE POINT: $\dot{x}(1/2)$.

	minimum effort model	minimum time model
$n = 1$	1.0	1.0
$n = 2$	1.5	2.0
$n = 3$	1.875	2.0
$n = 4$	2.188	2.343
$n = 5$	2.461	2.584
$n = 6$	2.707	2.823
$n = 7$	2.933	3.039
$n = 8$	3.142	3.242

3°. An interesting approach to modeling of human-like reaching movements is proposed in [19]. The approach is based on the minimum acceleration model and can be linked to the minimization of kinetic energy. In the generalized form, for the controlled system $d^n x/dt^n = u(t)$ the trajectory is found by minimizing the criterion $\mathcal{J} = \frac{1}{2}\int_0^T \left(d^{n-1}x/dt^{n-1}\right)^2 dt$ under the constraints $|u(t)| \leq U$. This is a singular optimal control problem. We conjecture that this approach is equivalent to the minimum time model when the movement duration is specified by (42). This fact can be easily verified for $n = 2$ and $n = 3$ but remains to be proven for the general case.

V. Conclusions

Several issues related to modeling human-like reaching movements were studied in this paper. First, a generalized mathematical model, based on the minimization of square derivatives of the hand position over the movement duration, has been established in the analytical form represented by the Beta function. This representation can be used for the construction of fractional order models and also for modeling of asymmetric velocity profiles. Next, we addressed a natural (from the viewpoint of the calculus of variations) formation of the boundary conditions. From the mathematical point of view, the structure of the optimal solution is defined not only by the form of the optimality criterion but also by the boundary conditions of the optimization task. The natural boundary conditions, defined in our paper, can also be used in modeling asymmetric velocity profiles. Finally, addressing modeling reaching movements with bounded control actions, we have considered the minimum time formulation of the optimization problem and (for the n-th order integrator) established its analytical solution.

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