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Explicit Dynamics Formulation of Stewart–Gough Platform: A Newton–Euler Approach

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Abstract— **Dynamic analysis of parallel manipulators plays a vital role in the design and control of such manipulators. Closed–chain kinematic structure affects the dynamics formulations by several constraints. Therefore, especially for higher degrees of freedom manipulators, manipulation of implicit and bulky dynamics formulation looses the tractability of the analysis. In this paper, a methodology and some simplification tools are introduced to achieve explicit dynamics formulation for parallel manipulators. This methodology is applied for the dynamics analysis of the most celebrated parallel manipulator, namely Stewart-Gough platform. By avoiding any recursive or component–wise derivations, the resulting dynamics formulation provides more insight for designers, and can be much easier used in any model–based control of such manipulators. In order to verify the resulting dynamics equations, Lagrange method is used to derive and compare the manipulator mass matrix. This methodology can be further used to formulate the explicit dynamics of other parallel manipulators.**

I. INTRODUCTION

In recent decades, numerous research results has been reported on the kinematics of parallel manipulators, and relatively fewer results on the dynamics of parallel manipulators. Several approaches have been proposed for the dynamic analysis of parallel manipulators. The traditional Newton–Euler formulation is used for the dynamic analysis of general parallel manipulators [1] and also for the *Stewart-Gough platform* (SGP), which is the most celebrated parallel manipulator [2]. In this formulation the equations of motion for each limb and the moving platform must be derived, which inevitably leads to a large number of equations and less computational efficiency. On the other hand, all the reaction forces can be computed, which is very useful in the design of a parallel manipulator. The Lagrangian formulation eliminates all the unwanted reaction forces at the outset, and it is usually more efficient [3]. However, because of the constraints imposed by the closed loop kinematic chains of a parallel manipulator, deriving explicit equations of motion in terms of a set of independent generalized coordinates is a prohibitive task [4]. Other approaches have also been suggested in the literature to tackle the dynamics formulation of parallel manipulators, representatives of such research can be given as [5], [6]. In most of the reported researches, partly straightforward routines were proposed in order to derive the dynamic equation using NE formulation [7], Lagrange [4], or Kane methods [6]. However, the final dynamics formulations are usually very bulky and untractable especially for a six degrees of freedom parallel manipulator like SGP. This is because of several parameter substitutions in NE and several differential component–wise operations in Lagrange and Kane methods. Therefore, dynamic analysis of such equations are usually untractable and can only be visualized by numerical methods [3] or by giving a generic physical verification for the system.

Although, writing the formulation in the following explicit form

$$\boldsymbol{M}(\boldsymbol{\mathcal{X}})\ddot{\boldsymbol{\mathcal{X}}} + \boldsymbol{C}(\boldsymbol{\mathcal{X}}, \dot{\boldsymbol{\mathcal{X}}})\dot{\boldsymbol{\mathcal{X}}} + \boldsymbol{G}(\boldsymbol{\mathcal{X}}) = \boldsymbol{F} \tag{1}$$

is reported in many papers [8], the elements of $\boldsymbol{M}, \boldsymbol{C}$, and $\boldsymbol{G}$ are rarely given clearly in these formulations. It is important to have a complete and clear representation of these elements, in order to use model–based control topologies on the manipulators [9]. In this paper the full details of such explicit formulation is given in details. Two main categories can be distinguished in the literature on the analysis of explicit of dynamic equations. First set of researches, in a generic view, proves that it is possible to derive an explicit representation of dynamic equations [10]. [11] proves that all natural mechanisms can be written in explicit form but closed–chain constraints will change them into implicit form. Representative of the second category can be cited as [3] and [4] using Lagrange formulation or [8] using NE formulation, propose useful methods to extract explicit form of equations from implicit formulations by means of term by term differentiation algorithms in the Lagrange formulation. Thanks to the advances in mathematical solvers, component–wise equations are derived and the total dynamic equations of the SGP are given. In [4] it is claimed that

> *"Although it is not difficult to obtain an explicit expression for the moving platform's $\boldsymbol{M}, \boldsymbol{V}$, and $\boldsymbol{G}$, it is impossible to obtain explicitly $\boldsymbol{M}, \boldsymbol{V}$, and $\boldsymbol{G}$ for legs. Even with a symbolic package like Mathematica, the expressions are too complicated to obtain! In this case, a step-by-step formulation must be used."*

In [12] it is reported that the NE method can result into the same equations as the Lagrange formulation for a general parallel manipulator, and as a case study it is applied on a four-bar linkage. The resulting dynamics equation is however, very bulky although the manipulator degrees of freedom is much less than that of the SGP.

In this paper, dynamics formulation of a SGP with spherical joints is studied in detail. A vector based Newton-Euler notation is proposed, which preserves the inherent kinematic structure components of the manipulator in the final resulting equations. Therefore, extraction of properties through the dynamic equations becomes more tractable and insightful. Furthermore, an intermediate variable from joint space is introduced and some matrix algebraic simplifications tools are given in order to significantly simplify the formulations. It is noticeable that using the proposed method the dynamic relation of the limbs and that of the end-effector are kept separated, and therefore, can be analyzed more tractable. The resulting dynamics equations for the whole manipulator, including the limbs and the end-effector dynamics are significantly simplified into a set of equations, which to the best knowledge of the authors, are not given elsewhere in the literature in this compact form. The

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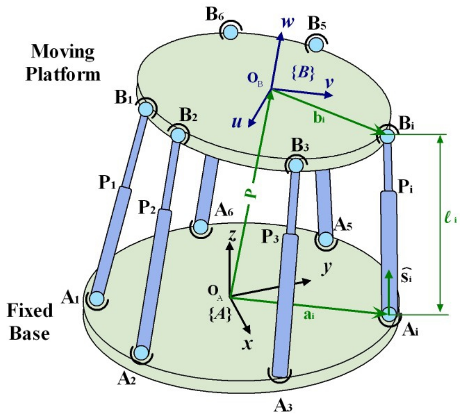

Fig. 1. A kinematic structure of SGP

proposed methodology, and the simplification tools can be used to formulate other manipulator dynamics in the same manner.

II. Kinematics

A. Mechanism Description

Figure 1 shows a schematics of a Stewart-Gough Platform (SGP) under study. In this manipulator the spatial motion of the moving platform is generated by six piston–cylinder actuators. While these limbs are generally considered identical, in this analysis they can differ in both geometry and mass properties. Each piston–cylinder actuator consists of two part connected with a prismatic joint. The actuators connected the fixed base to the moving platform by spherical joints at points A_i and B_i, $i = 1, 2, \ldots, 6$ respectively. Fixed attachment points A_i's are connected to the base, while moving attachment points B_i's are connected to the moving platform. Note that in a geometrically general Stewart Gough platform which is considered here, the attachment points are not necessarily lie in the same plane. Further more the following assumptions are considered for the manipulator. The cylinder and piston centers of masses lie on the limb's axis ($\overrightarrow{A_iB_i}$); Each limb is symmetric with respect to its axis, and therefore, its inertia matrix is diagonal with respect to the moving frame attached to the limb with $I_{xx} = I_{yy}$.

B. Position Analysis

A fixed base frame A and a moving frame B connected to the base and the moving platform respectively are considered here. The rotation matrix ${}^A\boldsymbol{R}_B$ relates these two frames by means of the three Euler angels α, β, γ that forms the Euler angle vector $\boldsymbol{\phi} = [\alpha\ \beta\ \gamma]^T$. Apart from these frames, another set of moving frames A_i's are attached to each limbs and the rotation matrixes ${}^A\boldsymbol{R}_{A_i}$ relate these frames to the base frame A. Also $\boldsymbol{x}$ is the position vector of the origin of the frame B with respect to frame A. The vector loop closure from A to B for each limb can be written as:

$$\boldsymbol{a}_i + l_i\hat{\boldsymbol{s}}_i = \boldsymbol{x} + {}^AR_B{}^B\boldsymbol{b}_i \tag{2}$$

in which, $\boldsymbol{a}_i$ is the position vector from the origin of the frame A to the fixed attachment points A_i, l_i is a scalar representing the length of each limb and $\hat{\boldsymbol{s}}_i$ is a unit vector along $\overrightarrow{A_iB_i}$. The vector Loop closure 2 can be used to derive l_i and $\hat{\boldsymbol{s}}_i$:

$$\begin{aligned} l_i &= \|\boldsymbol{x} + \boldsymbol{b}_i - \boldsymbol{a}_i\|_2 && (3)\\ \hat{s}_i &= \frac{1}{l_i}(\boldsymbol{x} + \boldsymbol{b}_i - \boldsymbol{a}_i) && (4)\end{aligned}$$

in which,

$$\boldsymbol{b}_i = {}^AR_B{}^B\boldsymbol{b}_i \tag{5}$$

Define an intermediate variable $\boldsymbol{x}_i$ as the position vector of the moving attachment points B_i. Note that this intermediate variable has a significant role in the simplification of the dynamic equation. Thus,

$$\begin{aligned} \boldsymbol{x}_i &= \boldsymbol{x} + \boldsymbol{b}_i && (6)\\ \boldsymbol{x}_i &= \boldsymbol{a}_i + l_i\hat{\boldsymbol{s}}_i && (7)\end{aligned}$$

Furthermore, the position vector of the center of mass (COM) of the cylinder and the piston of each limb denoted by $\boldsymbol{c}_{i_1}$ and $\boldsymbol{c}_{i_2}$, respectively, can be written as:

$$\begin{aligned} \boldsymbol{c}_{i_1} &= \boldsymbol{a}_i + c_{i_1}\hat{\boldsymbol{s}}_i && (8)\\ \boldsymbol{c}_{i_2} &= \boldsymbol{a}_i + (l_i - c_{i_2})\hat{\boldsymbol{s}}_i && (9)\end{aligned}$$

in which, $\boldsymbol{c}_{i_1}$ and$\boldsymbol{c}_{i_2}$ are considered at the half length of the cylinder and the piston, respectively.

C. Velocity and Acceleration Analysis

Velocity and acceleration analysis of the limbs and the moving platform is performed in this section. For the moving platform,$\dot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{x}}$ are the linear velocity and acceleration of the moving frame B, respectively. While the angular velocity $\boldsymbol{\omega}$ and acceleration $\dot{\boldsymbol{\omega}}$ of the moving platform can be defined as:

$$\begin{aligned} \boldsymbol{\omega} &= \boldsymbol{E}\dot{\boldsymbol{\phi}} && (10)\\ \dot{\boldsymbol{\omega}} &= \dot{\boldsymbol{E}}\dot{\boldsymbol{\phi}} + \boldsymbol{E}\ddot{\boldsymbol{\phi}} \end{aligned}$$

in which, $\boldsymbol{E}$ is the transformation matrix that relates the rate of the Euler angle vector $\boldsymbol{\phi}$ to the angular velocity $\boldsymbol{\omega}$. For each limb, differentiate equation 2 with respect to time and use intermediate vector $\boldsymbol{x}_i$ defined in equations 6 and 7:

$$\dot{\boldsymbol{x}}_i = \dot{\boldsymbol{x}} + \boldsymbol{\omega} \times \boldsymbol{b}_i \tag{11}$$

$$\dot{\boldsymbol{x}}_i = \dot{l}_i\hat{\boldsymbol{s}}_i + l_i\boldsymbol{\omega}_i \times \hat{\boldsymbol{s}}_i \tag{12}$$

in which, $\dot{l}_i$ is the limb length rate and $\boldsymbol{\omega}_i$ is the angular velocity of each limb. Dot multiply the above equations to $\hat{\boldsymbol{s}}_i$:

$$\dot{l}_i = \dot{\boldsymbol{x}}_i.\hat{\boldsymbol{s}}_i = [\dot{\boldsymbol{x}} + \boldsymbol{\omega} \times \boldsymbol{b}_i]^T\hat{\boldsymbol{s}}_i \tag{13}$$

Since there is no actuation torque about $\hat{\boldsymbol{s}}_i$, the limb angular velocity and acceleration vectors are normal to $\hat{\boldsymbol{s}}_i$ provided that the following conditions holds.

- Both ending joints of the limb are spherical (none of them should be universal joint otherwise angular velocity and acceleration vectors have a component along $\hat{\boldsymbol{s}}_i$).
- The limbs are symmetric with respect to their axis.
- The effects of friction in spherical joints can be neglected.

Considering these assumptions, we can conclude:

$$\boldsymbol{\omega}_i \cdot \hat{\boldsymbol{s}}_i = 0, \text{ and } \dot{\boldsymbol{\omega}}_i \cdot \hat{\boldsymbol{s}}_i = 0 \tag{14}$$

in which, $\dot{\boldsymbol{\omega}}_i$ is the angular acceleration of each limb. Next, cross multiply 11 and 12 to $\hat{\boldsymbol{s}}_i$, and use 14 by means of vector triple product (VTP) expansion rule. Hence, $\boldsymbol{\omega}_i$ can be derived as:

$$\boldsymbol{\omega}_i = \frac{1}{l_i}\hat{\boldsymbol{s}}_i \times \dot{\boldsymbol{x}}_i = \frac{1}{l_i}\hat{\boldsymbol{s}}_i \times (\dot{\boldsymbol{x}} + \boldsymbol{\omega} \times \boldsymbol{b}_i) \tag{15}$$

The velocity of the piston and cylinder center of mass can be written as:

$$\boldsymbol{v}_{c_{i_1}} = c_{i_1}(\boldsymbol{\omega}_i \times \hat{\boldsymbol{s}}_i) \tag{16}$$

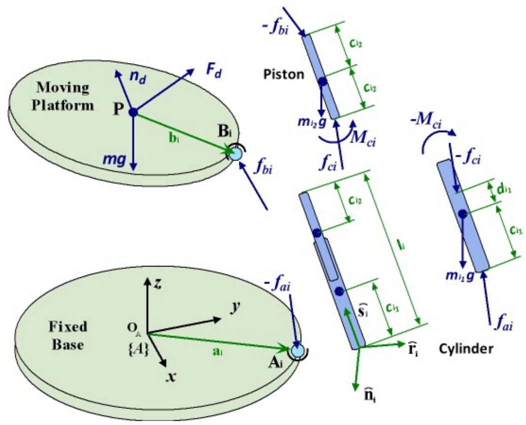

Fig. 2. Free Body diagram of different parts of SGP

$$\boldsymbol{v}_{c_{i_2}} = (l_i - c_{i2})\,(\boldsymbol{\omega}_i \times \hat{\boldsymbol{s}}_i) + \dot{l}_i\hat{\boldsymbol{s}}_i \tag{17}$$

Time derivative of 11 and 12 yields to the acceleration loop closure for each limb.

$$\ddot{\boldsymbol{x}}_i = \ddot{\boldsymbol{x}} + \dot{\boldsymbol{\omega}} \times b_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{b}_i) \tag{18}$$

$$\ddot{\boldsymbol{x}}_i = \ddot{l}_i\hat{\boldsymbol{s}}_i + \dot{l}_i\boldsymbol{\omega}_i\times\hat{\boldsymbol{s}}_i + \dot{l}_i\boldsymbol{\omega}_i\times\hat{s}_i + l_i\dot{\boldsymbol{\omega}}_i\times\hat{s}_i + l_i\boldsymbol{\omega}_i\times(\boldsymbol{\omega}_i\times\hat{s}_i) \tag{19}$$

Similarly, dot and cross multiply 18 and 19 by $\hat{\boldsymbol{s}}_i$, respectively. By some simplification based on VTP and equations 14, $\ddot{l}_i$ and $\dot{\boldsymbol{\omega}}_i$, can be derived as:

$$\begin{aligned} \ddot{l}_i &= \ddot{\boldsymbol{x}}_i \cdot \hat{\boldsymbol{s}}_i + l_i\,|\boldsymbol{\omega}_i|_2^2 \\ &= [\ddot{\boldsymbol{x}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{b}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{b}_i)]^T\hat{\boldsymbol{s}}_i + l_i\,|\boldsymbol{\omega}_i|_2^2 \end{aligned} \tag{20}$$

$$\begin{aligned} \dot{\boldsymbol{\omega}}_i &= \frac{1}{l_i}\left(\hat{\boldsymbol{s}}_i \times \ddot{\boldsymbol{x}}_i - 2\dot{l}_i\boldsymbol{\omega}_i\right) \\ &= \frac{1}{l_i}\left(\hat{s}_i\times[\ddot{\boldsymbol{x}} + \dot{\boldsymbol{\omega}}\times b_i + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{b}_i)] - 2\dot{l}_i\boldsymbol{\omega}_i\right) \end{aligned} \tag{21}$$

Also by differentiation of 16 and 17 the acceleration vectors of the piston and the cylinder center of mass can be derived as:

$$\boldsymbol{a}_{c_{i_1}} = c_{i_1}\left(\dot{\boldsymbol{\omega}}_i \times \hat{\boldsymbol{s}}_i - |\boldsymbol{\omega}_i|_2^2\,\hat{\boldsymbol{s}}_i\right) \tag{22}$$

$$\boldsymbol{a}_{c_{i_2}} = (l_i - c_{i2})\left(\dot{\boldsymbol{\omega}}_i \times \hat{\boldsymbol{s}}_i - |\boldsymbol{\omega}_i|_2^2\,\hat{\boldsymbol{s}}_i\right) + 2\dot{l}_i\,(\boldsymbol{\omega}_i \times \hat{\boldsymbol{s}}_i) + \ddot{l}_i\hat{\boldsymbol{s}}_i \tag{23}$$

III. Implicit Dynamics Equations

In this section the equations of motion for the limbs and the moving platform is derived based on Newton–Euler formulation. This yields to a set of implicit dynamic equations for SGP as:

$$\boldsymbol{f}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}, \boldsymbol{\phi}, \dot{\boldsymbol{\phi}}, \ddot{\boldsymbol{\phi}}, \boldsymbol{\tau}) = 0 \tag{24}$$

where, $\boldsymbol{\tau}$ is an array of six actuator forces.

A. Implicit Dynamics of the Moving Platform

Figure 2 shows the free body diagram of the moving platform. Let m be the mass and ${}^B\boldsymbol{I}_P$ be the Inertia tensor of the moving platform with respect to moving frame B. Furthermore, let $\boldsymbol{f}_{b_i}$ denotes the force acting from ith limb at the B_i point on the moving platform. Newton-Euler equations for the dynamics of the moving platform can be written as:

$$\sum \boldsymbol{f}_{ext} = \sum_{i=1}^{6} \boldsymbol{f}_{\boldsymbol{B}_i} + m\boldsymbol{g} + \boldsymbol{F}_d = m\ddot{\boldsymbol{x}} \tag{25}$$

$$\sum {}^P\boldsymbol{n}_{ext} = \boldsymbol{n}_d + \sum_{i=1}^{6} \boldsymbol{b}_i \times \boldsymbol{f}_{\boldsymbol{B}_i} = {}^A\boldsymbol{I}_P\,\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times {}^A\boldsymbol{I}_P\,\boldsymbol{\omega} \tag{26}$$

in which, ${}^A\boldsymbol{I}_P$ must be considered in the fixed frame $\{\mathcal{A}\}$ by:

$${}^A\boldsymbol{I}_P = {}^AR_B\,{}^B\boldsymbol{I}_P\,{}^AR_B^T \tag{27}$$

B. Implicit Dynamics of the Limbs

Figure 2 shows also the free body diagram for the cylinder and the piston. Let m_{i_1} and m_{i_2} be the mass and ${}^{A_i}\boldsymbol{I}_{c_{i_1}}$ and ${}^{A_i}\boldsymbol{I}_{c_{i_2}}$ be the inertia tensor of the cylinder and the piston, respectively. Newton-Euler equations for the dynamics of the cylinder can be written as:

$$\sum \boldsymbol{f}_{ext} = \boldsymbol{f}_{\boldsymbol{A}_i} - \boldsymbol{f}_{c_i} + m_{i_1}\boldsymbol{g} = m_{i_1}\boldsymbol{a}_{c_{i_1}} \tag{28}$$

$$\begin{aligned} \sum {}^{c_{i_1}}\boldsymbol{n}_{ext} &= c_{i_1}\left(-\hat{\boldsymbol{s}}_i \times \boldsymbol{f}_{\boldsymbol{A}_i}\right) + d_{i_1}\left(\hat{\boldsymbol{s}}_i \times -\boldsymbol{f}_{c_i}\right) - \boldsymbol{M}_{c_i} \\ &= {}^A\boldsymbol{I}_{c_{i_1}}\dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times {}^A\boldsymbol{I}_{c_{i_1}}\boldsymbol{\omega}_i \end{aligned} \tag{29}$$

in which,

$${}^{A_i}\boldsymbol{I}_{c_{i_1}} = {}^A\boldsymbol{R}_{A_i}\,{}^{A_i}\boldsymbol{I}_{c_{i_1}}\,{}^A\boldsymbol{R}_{A_i}^T \tag{30}$$

and,

$$d_{i_1} = l_i - c_{i_1} - 2c_{i_2} \tag{31}$$

Note that $\boldsymbol{f}_{A_i}$, is the internal force acting on the cylinder at point A_i, $\boldsymbol{f}_{c_i}$ are the internal forces and $\boldsymbol{M}_{c_i}$ are the constraint momentum between piston and the cylinder at point c_i, respectively. Similar to the dynamics of the cylinder, equations of motion for the piston is derived as:

$$\sum \boldsymbol{f}_{ext} = \boldsymbol{f}_{c_i} - \boldsymbol{f}_{\boldsymbol{B}_i} + m_{i_2}\boldsymbol{g} = m_{i_2}\boldsymbol{a}_{c_{i2}} \tag{32}$$

$$\begin{aligned} \sum {}^{c_{i2}}\boldsymbol{n}_{ext} &= c_{i_2}\left(-\hat{\boldsymbol{s}}_i \times \boldsymbol{f}_{c_i}\right) + c_{i_2}\left(\hat{\boldsymbol{s}}_i \times -\boldsymbol{f}_{\boldsymbol{B}_i}\right) + \boldsymbol{M}_{c_i} \\ &= {}^A\boldsymbol{I}_{c_{i_2}}\dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times {}^A\boldsymbol{I}_{c_{i_2}}\boldsymbol{\omega}_i \end{aligned} \tag{33}$$

in which,

$${}^A\boldsymbol{I}_{c_{i_2}} = {}^A\boldsymbol{R}_{\mathbf{A_i}}\,{}^{A_i}\boldsymbol{I}_{c_{i_2}}\,{}^A\boldsymbol{R}_{\mathbf{A_i}}^T \tag{34}$$

Note that the moving frame A_i is attached to the limb i. In order to simplify the formulation of ${}^A\boldsymbol{R}_{A_i}$, it is better to use $\hat{\boldsymbol{s}}_i$ and two other unit vectors which have been defined as follows to find the principal axes of frame A:

$$\hat{q}_i = \frac{\hat{\boldsymbol{s}}_i \times \boldsymbol{a}_i}{|\hat{\boldsymbol{s}}_i \times \boldsymbol{a}_i|_2} \tag{35}$$

$$\hat{\boldsymbol{r}}_i = \hat{\boldsymbol{s}}_i \times \hat{\boldsymbol{q}}_i \tag{36}$$

Therefore, the rotation matrix ${}^A\boldsymbol{R}_{A_i}$ is represented by:

$${}^A\boldsymbol{R}_{\mathbf{A_i}} = \begin{bmatrix} \hat{\boldsymbol{s}}_i \cdot \hat{\boldsymbol{x}} & \hat{\boldsymbol{q}}_i \cdot \hat{\boldsymbol{x}} & \hat{\boldsymbol{r}}_i \cdot \hat{\boldsymbol{x}} \\ \hat{\boldsymbol{s}}_i \cdot \hat{\boldsymbol{y}} & \hat{\boldsymbol{q}}_i \cdot \hat{\boldsymbol{y}} & \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{s}}_i \cdot \hat{\boldsymbol{z}} & \hat{\boldsymbol{q}}_i \cdot \hat{\boldsymbol{z}} & \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{z}} \end{bmatrix} \tag{37}$$

In which, $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ are unit vectors along principle axes of the fixed frame A. Moreover, as described before the limbs are considered to be symmetric. Hence, ${}^{A_i}\boldsymbol{I}_{c_{i_1}}$ and ${}^{A_i}\boldsymbol{I}_{c_{i_2}}$ (and generally ${}^{A_i}\boldsymbol{I}_{c_i}$) is in the form of,

$${}^{A_i}\boldsymbol{I}_{c_i} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{xx} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \tag{38}$$

In order to simplify and eliminate internal forces in equations 28–34, we may start using 28 to derive $\boldsymbol{f}_{A_i}$ as a function of $\boldsymbol{f}_{c_i}$:

$$\boldsymbol{f}_{\boldsymbol{A}_i} = m_{i_1}\boldsymbol{a}_{c_{i_1}} + \boldsymbol{f}_{c_i} - m_{i_1}\boldsymbol{g} \tag{39}$$

Substitute $\boldsymbol{f}_{A_i}$ from 39 in 29. Through some manipulation $\boldsymbol{M}_{c_i}$ results in:

$$\begin{aligned}\boldsymbol{M}_{c_i} =& - c_{i_1}\hat{\boldsymbol{s}}_i \times \left(m_{i_1}\boldsymbol{a}_{c_{i_1}} + \boldsymbol{f}_{c_i} - m_{i_1}\boldsymbol{g}\right) \\ &+ d_{i_1}\left(\hat{\boldsymbol{s}}_i \times -\boldsymbol{f}_{c_i}\right) - {}^A\boldsymbol{I}_{c_{i_1}}\dot{\boldsymbol{\omega}}_i - \boldsymbol{\omega}_i \times {}^A\boldsymbol{I}_{c_{i_1}}\boldsymbol{\omega}_i\end{aligned} \tag{40}$$

Moreover, in order to separate actuator force $\tau_i\hat{\boldsymbol{s}}_i$, from the internal force between the cylinder and the piston, $\boldsymbol{f}_{c_i}$ can be rewritten as:

$$\boldsymbol{f}_{c_i} = \boldsymbol{f}^n_{c_i} + \boldsymbol{f}^s_{c_i} = \boldsymbol{f}^n_{c_i} + \tau_i\hat{\boldsymbol{s}}_i \tag{41}$$

in which, $\boldsymbol{f}^n_{c_i}$ denotes the component of the force in a direction normal to $\hat{\boldsymbol{s}}_i$. Substitute $\boldsymbol{f}_{c_i}$ from 41, $\boldsymbol{a}_{c_{i_1}}$ from 22, and d_{i_1} from 31, into equation 40. This results in $\boldsymbol{M}_{c_i}$ as a function of $\boldsymbol{f}^n_{c_i}$ and the actuator forces τ_i. Using VTP and considering the fact that $\hat{\boldsymbol{s}}_i \times \boldsymbol{f}_{c_i} = \boldsymbol{f}^n_{c_i}$, simplifies $\boldsymbol{M}_{c_i}$ to:

$$\begin{aligned}\boldsymbol{M}_{c_i} =& - m_{i_1}c^2_{i_1}\dot{\boldsymbol{\omega}}_i - {}^A\boldsymbol{I}_{c_{i_1}}\dot{\boldsymbol{\omega}}_i - \boldsymbol{\omega}_i \times {}^A\boldsymbol{I}_{c_{i_1}}\boldsymbol{\omega}_i \\ &- (l_i - 2c_{i_2})\left(\hat{\boldsymbol{s}}_i \times \boldsymbol{f}^n_{c_i}\right) + m_{i_1}c_{i_1}\hat{\boldsymbol{s}}_i \times \boldsymbol{g}\end{aligned} \tag{42}$$

The simplified form of $\hat{\boldsymbol{s}}_i \times \boldsymbol{M}_{c_i}$ is also derived here for later use.

$$\begin{aligned}\hat{\boldsymbol{s}}_i \times \boldsymbol{M}_{c_i} =& - m_{i_1}c^2_{i_1}\left(\hat{\boldsymbol{s}}_i \times \dot{\boldsymbol{\omega}}_i\right) - \hat{\boldsymbol{s}} \times \left({}^A\boldsymbol{I}_{c_{i_1}}\dot{\boldsymbol{\omega}}_i\right) \\ &- \left(\hat{\boldsymbol{s}}_i.\left({}^A\boldsymbol{I}_{c_{i_1}}\boldsymbol{\omega}_i\right)\right)\boldsymbol{\omega}_i + (l_i - 2c_{i_2})\,\boldsymbol{f}^n_{c_i} \\ &+ m_{i_1}c_{i_1}\hat{\boldsymbol{s}}_i \times (\hat{\boldsymbol{s}}_i \times \boldsymbol{g})\end{aligned} \tag{43}$$

Because of the nature of the prismatic joints of the limbs, and in order to derive $\boldsymbol{f}_{b_i}$ in a vector form and not componentwise, we have also separated $\boldsymbol{f}_{b_i}$ into two parts:

$$\boldsymbol{f}_{b_i} = \boldsymbol{f}^n_{b_i} + \boldsymbol{f}^s_{b_i} \tag{44}$$

in which, $\boldsymbol{f}^s_{b_i}$ and $\boldsymbol{f}^n_{b_i}$ are the contributions of $\boldsymbol{f}_{b_i}$ in the direction of $\hat{\boldsymbol{s}}_i$ and a direction normal to $\hat{\boldsymbol{s}}_i$, respectively. This separation results into a better elimination of internal forces and thus simplification of dynamic equations for each limb. Note that while $\boldsymbol{f}^n_{b_i}$ and $\boldsymbol{f}^n_{c_i}$ are normal to $\hat{\boldsymbol{s}}_i$, *they are not generally parallel to each other*. By substitution of $\boldsymbol{f}_{c_i}$, $\boldsymbol{f}_{b_i}$, and $\boldsymbol{a}_{c_{i_2}}$ from 41, 44 and 23 into 32, and through some manipulation the NE equation for the piston results in:

$$\begin{aligned}\boldsymbol{f}^n_{b_i} - \boldsymbol{f}^n_{c_i} + \boldsymbol{f}^s_{b_i} =& \tau_i\hat{\boldsymbol{s}}_i - m_{i_2}\left((l_i - c_{i_2})\left(\dot{\boldsymbol{\omega}}_i \times \hat{\boldsymbol{s}}_i - |\boldsymbol{\omega}_i|^2_2\,\hat{\boldsymbol{s}}_i\right)\right. \\ &\left.+2\dot{l}_i\left(\boldsymbol{\omega}_i \times \hat{\boldsymbol{s}}_i\right) + \ddot{l}_i\hat{\boldsymbol{s}}_i\right) + m_{i_2}\boldsymbol{g}\end{aligned} \tag{45}$$

Equation 45 shows that separating $\boldsymbol{f}_{b_i}$ as described before is very useful since by dot multiplying 45 by $\hat{\boldsymbol{s}}_i$, $\boldsymbol{f}^s_{b_i}$ can be derived as a function of the actuator force and the kinematic parameters of the limb:

$$\boldsymbol{f}^s_{b_i} = \tau_i\hat{\boldsymbol{s}}_i + m_{i_2}\left(l_i - c_{i_2}\right)|\boldsymbol{\omega}_i|^2_2\,\hat{\boldsymbol{s}}_i - m_{i_2}\ddot{l}_i\hat{\boldsymbol{s}}_i + m_{i_2}\left(\hat{s}_i.\boldsymbol{g}\right)\hat{\boldsymbol{s}}_i \tag{46}$$

Finding $\boldsymbol{f}^n_{b_i}$ is a little more involved. Cross multiply 45 by $\hat{\boldsymbol{s}}_i$ *twice* results into:

$$\begin{aligned}\hat{\boldsymbol{s}}_i \times \left(\hat{\boldsymbol{s}}_i \times \boldsymbol{f}^n_{b_i}\right) - \hat{\boldsymbol{s}}_i \times \left(\hat{\boldsymbol{s}}_i \times \boldsymbol{f}^n_{c_i}\right) =& -m_{i_2}\left(l_i - c_{i_2}\right)\left(\hat{\boldsymbol{s}}_i \times \dot{\boldsymbol{\omega}}_i\right) \\ &- 2m_{i_2}\dot{l}_i\left(\hat{\boldsymbol{s}}_i \times \boldsymbol{\omega}_i\right) + m_{i_2}\hat{\boldsymbol{s}}_i \times \left(\hat{\boldsymbol{s}}_i \times \boldsymbol{g}\right)\end{aligned} \tag{47}$$

Now using VTP rule, $\boldsymbol{f}^n_{c_i}$ can be found as a function of $\boldsymbol{f}^n_{b_i}$:

$$\begin{aligned}\boldsymbol{f}^n_{c_i} =& \boldsymbol{f}^n_{b_i} - m_{i_2}\left(l_i - c_{i_2}\right)\left(\hat{\boldsymbol{s}}_i \times \dot{\boldsymbol{\omega}}_i\right) \\ &- 2m_{i_2}\dot{l}_i\left(\hat{\boldsymbol{s}}_i \times \boldsymbol{\omega}_i\right) + m_{i_2}\hat{\boldsymbol{s}}_i \times \left(\hat{\boldsymbol{s}}_i \times \boldsymbol{g}\right)\end{aligned} \tag{48}$$

Similarly, substitute $\boldsymbol{f}_{c_i}$ and $\boldsymbol{f}_{b_i}$ from equations 41 and 44, into equation 33, rearrange the equation and then cross multiply by $\hat{\boldsymbol{s}}_i$. This yields to:

$$\begin{aligned}-c_{i_2}\boldsymbol{f}^n_{b_i} =& - \hat{\boldsymbol{s}}_i \times \left({}^A\boldsymbol{I}_{c_{i_2}}\dot{\boldsymbol{\omega}}_i\right) - \left(\hat{\boldsymbol{s}}_i.\left({}^A\boldsymbol{I}_{c_{i_2}}\boldsymbol{\omega}_i\right)\right)\boldsymbol{\omega}_i \\ &+ c_{i_2}\boldsymbol{f}^n_{c_i} + \hat{\boldsymbol{s}}_i \times \boldsymbol{M}_{c_i}\end{aligned} \tag{49}$$

Now in order to derive $\boldsymbol{f}^n_{b_i}$ as a function of the kinematic parameters, substitute $\hat{\boldsymbol{s}}_i \times \boldsymbol{M}_{c_i}$ from equation 43 and $\boldsymbol{f}^n_{c_i}$ from 48 into 49. By some manipulations the resulting $\boldsymbol{f}^n_{b_i}$ can be written as:

$$\begin{aligned}\boldsymbol{f}^n_{b_i} =& \frac{1}{l_i}\hat{\boldsymbol{s}}_i \times \left(\left({}^A\boldsymbol{I}_{c_{i_1}} + {}^A\boldsymbol{I}_{c_{i_2}}\right)\dot{\boldsymbol{\omega}}_i\right) \\ &+ \frac{1}{l_i}\left(m_{i_1}c^2_{i_1} + m_{i_2}(l_i - c_{i_2})^2\right)\left(\hat{\boldsymbol{s}}_i \times \dot{\boldsymbol{\omega}}_i\right) \\ &+ \frac{1}{l_i}\left(\hat{\boldsymbol{s}}_i.\left(\left({}^A\boldsymbol{I}_{c_{i_1}} + {}^A\boldsymbol{I}_{c_{i_2}}\right)\boldsymbol{\omega}_i\right)\right)\boldsymbol{\omega}_i \\ &+ \frac{2}{l_i}m_{i_2}\left(l_i - c_{i_2}\right)\dot{l}_i\left(\hat{\boldsymbol{s}}_i \times \boldsymbol{\omega}_i\right) \\ &- \frac{1}{l_i}\left(m_{i_1}c_{i_1} + m_{i_2}\left(l_i - c_{i_2}\right)\right)\hat{\boldsymbol{s}}_i \times \left(\hat{\boldsymbol{s}}_i \times \boldsymbol{g}\right)\end{aligned} \tag{50}$$

Equations 46 and 50 together, describes $\boldsymbol{f}_{b_i}$ as a function of kinematic parameters of each limbs. These equations together with 25 and 26 completely define the equations of motion of the Stewart-Gough platform in an implicit form given by equation 24.

IV. Explicit Dynamics

While dynamics formulation in the form of equation 24 can be used to simulate inverse dynamics of the SGP, its implicit nature makes it unpleasant for dynamics analysis and control. Hence in this section we introduce a method to reformulate the dynamics equation into explicit form, comparable to that usually obtained from Lagrange formulation.

A. Explicit Dynamics of Limbs

In order to derive explicit equations for the dynamics of Stewart-Gough platform as indicated in equation 1, first we consider intermediate vector $\boldsymbol{x}_i$, which is introduced in 6 and 7 as an intermediate generalized coordinate, and manipulate the dynamic equations for each limb to convert them to the following form.

$$\boldsymbol{M}_i\ddot{\boldsymbol{x}}_i + \boldsymbol{C}_i\dot{\boldsymbol{x}}_i + \boldsymbol{G}_i = \boldsymbol{F}_i \tag{51}$$

We first introduce some relations to substitute kinematic parameters like $\dot{\boldsymbol{\omega}}_i, \ddot{\boldsymbol{\omega}}_i, \ddot{l}_i$, and $\ldots$ with intermediate vector $\boldsymbol{x}_i$ and its time derivatives. In order to do that, vector multiplications such as dot and cross products should be transformed into their corresponding matrix multiplication. For any three arbitrary vectors $\boldsymbol{a}, \boldsymbol{b}$ and$\boldsymbol{c}$ we have:

$$(\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c} = \left(\boldsymbol{a}^T\boldsymbol{b}\right)\boldsymbol{c} = \boldsymbol{c}\left(\boldsymbol{a}^T\boldsymbol{b}\right) = \left(\boldsymbol{c}\boldsymbol{a}^T\right)b = \boldsymbol{c}\boldsymbol{a}^T\boldsymbol{b} \tag{52}$$

And for cross multiplication $\boldsymbol{a} \times \boldsymbol{b}$ we have:

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{a}_\times\boldsymbol{b} = -\boldsymbol{b}_\times\boldsymbol{a} \tag{53}$$

in which, $\boldsymbol{a}_\times$ and $\boldsymbol{b}_\times$ denotes skew-symmetric matrices derived from elements of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. In order to relate $\boldsymbol{x}_i$ and its derivatives to joint variables, vector multiplications in equations 11, 12, 18 and 19 is transformed to their corresponding matrix multiplications using equations 52 and 53:

$$\begin{aligned}\dot{l}_i = \hat{\boldsymbol{s}}_i^T\dot{\boldsymbol{x}}_i \;\; ; \;\; \dot{l}_i\hat{\boldsymbol{s}}_i = \hat{\boldsymbol{s}}_i\hat{\boldsymbol{s}}_i^T\dot{\boldsymbol{x}}_i \\ \boldsymbol{\omega}_i = \frac{1}{l_i}\hat{\boldsymbol{s}}_{i\times}\dot{\boldsymbol{x}}_i \;\; ; \;\; |\boldsymbol{\omega}_i|^2_2\,\hat{\boldsymbol{s}}_i = \frac{-1}{{l_i}^2}\hat{\boldsymbol{s}}_i\dot{\boldsymbol{x}}_i^T\hat{\boldsymbol{s}}^2_{i\times}\dot{\boldsymbol{x}}_i\end{aligned} \tag{54}$$

$$
\begin{aligned}
&\ddot{l}_i - l_i \boldsymbol{\omega}_i.\boldsymbol{\omega}_i = \hat{\boldsymbol{s}}_i^T \ddot{\boldsymbol{x}}_i \;\; ; \;\; \ddot{l}_i \hat{\boldsymbol{s}}_i - l_i \left(\boldsymbol{\omega}_i \cdot \boldsymbol{\omega}_i\right) \hat{\boldsymbol{s}}_i = \hat{\boldsymbol{s}}_i \hat{\boldsymbol{s}}_i^T \ddot{\boldsymbol{x}}_i \\
&l_i \dot{\boldsymbol{\omega}}_i + 2\dot{l}_i \boldsymbol{\omega}_i = \hat{\boldsymbol{s}}_{i\times} \ddot{\boldsymbol{x}}_i \;\; ; \;\; \dot{\boldsymbol{\omega}}_i = \frac{1}{l_i}\left(\hat{\boldsymbol{s}}_{i\times}\ddot{\boldsymbol{x}}_i - 2\dot{l}_i \boldsymbol{\omega}_i\right)
\end{aligned} \tag{55}
$$

Relations 54 and 55 are the main framework in order to derive explicit dynamic equations from implicit formulations. By substitution of the kinematic parameters into dynamic equations 46 and 50 and using equations 54 and 55, and performing some simple mathematical simplifications, $\boldsymbol{f}_{b_i}^s$ and $\boldsymbol{f}_{b_i}^n$ can be converted to the following form:

$$
\boldsymbol{f}_{b_i}^s - \tau_i \hat{\boldsymbol{s}}_i = -m_{i_2} \hat{\boldsymbol{s}}_i \hat{\boldsymbol{s}}_i^T \ddot{\boldsymbol{x}}_i + \frac{m_{i_2} c_{i_2}}{l_i^2} \hat{\boldsymbol{s}}_i \dot{\boldsymbol{x}}_i^T \hat{\boldsymbol{s}}_{i\times}^2 \dot{\boldsymbol{x}}_i + m_{i_2} \hat{\boldsymbol{s}}_i \hat{\boldsymbol{s}}_i^T \boldsymbol{g} \tag{56}
$$

$$
\begin{aligned}
\boldsymbol{f}_{b_i}^n = &\frac{1}{l_i^2}\Big[\ \hat{\boldsymbol{s}}_{i\times}\left({}^A\boldsymbol{I}_{c_{i_1}} + {}^A\boldsymbol{I}_{c_{i_2}}\right)\hat{\boldsymbol{s}}_{i\times} \\
&+ \left(m_{i_1} c_{i_1}^2 + m_{i_2}(l_i - c_{i_2})^2\right) \hat{\boldsymbol{s}}_{i\times}^2 \Big] \ddot{\boldsymbol{x}}_i \\
&+ \Big[\ \frac{1}{l_i^2} \boldsymbol{\omega}_i \hat{\boldsymbol{s}}_i^T \left({}^A\boldsymbol{I}_{c_{i_1}} + {}^A\boldsymbol{I}_{c_{i_2}}\right)\hat{\boldsymbol{s}}_{i\times} \\
&- \frac{2\dot{l}_i}{l_i^3}\left(\left(m_{i_1} c_{i_1}^2 + m_{i_2}(l_i - c_{i_2})^2 - m_{i_2} l_i \left(l_i - c_{i_2}\right)\right) \hat{\boldsymbol{s}}_{i\times}^2 \right.\\
&\left. + \hat{\boldsymbol{s}}_{i\times}\left({}^A\boldsymbol{I}_{c_{i_1}} + {}^A\boldsymbol{I}_{c_{i_2}}\right)\hat{\boldsymbol{s}}_{i\times}\right)\Big] \dot{\boldsymbol{x}}_i \\
&- \frac{1}{l_i}\left[m_{i_1} c_{i_1} + m_{i_2}\left(l_i - c_{i_2}\right)\right] \hat{\boldsymbol{s}}_{i\times}^2 g
\end{aligned} \tag{57}
$$

Note that in deriving dynamic equations for the piston it is assumed that $-\boldsymbol{f}_{b_i}$ is acting upon the piston, and therefore, first negate 56 and 57 and then add them together. Then factor $\ddot{\boldsymbol{x}}_i$ and $\dot{\boldsymbol{x}}_i$ to yield to a single equation for the dynamics of the limb i, based on intermediate generalized coordinate $\boldsymbol{x}_i$. Factor this equation according to 51 leads to:

$$
\begin{aligned}
\boldsymbol{M}_i = &\frac{-1}{l_i^2}\left(\hat{\boldsymbol{s}}_{i\times}\left({}^A\boldsymbol{I}_{c_{i_1}} + {}^A\boldsymbol{I}_{c_{i_2}}\right)\hat{\boldsymbol{s}}_{i\times}\right. \\
&\left. + \left(m_{i_1} c_{i_1}^2 + m_{i_2}(l_i - c_{i_2})^2\right)\hat{\boldsymbol{s}}_{i\times}^2\right) + m_{i_2} \hat{\boldsymbol{s}}_i \hat{\boldsymbol{s}}_i^T
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
\boldsymbol{C}_i = &\frac{-m_{i_2} c_{i_2}}{l_i^2} \hat{\boldsymbol{s}}_i \dot{\boldsymbol{x}}_i^T \hat{\boldsymbol{s}}_{i\times}^2 - \frac{1}{l_i^2} \boldsymbol{\omega}_i \hat{\boldsymbol{s}}_i^T \left({}^A\boldsymbol{I}_{c_{i_1}} + {}^A\boldsymbol{I}_{c_{i_2}}\right)\hat{\boldsymbol{s}}_{i\times} \\
&+ \frac{2\dot{l}_i}{l_i^3}\left(\left(m_{i_1} c_{i_1}^2 + m_{i_2}(l_i - c_{i_2})^2 - m_{i_2} l_i \left(l_i - c_{i_2}\right)\right) \hat{s}_{i\times}^2 \right. \\
&\left. + \hat{\boldsymbol{s}}_{i\times}\left({}^A\boldsymbol{I}_{c_{i_1}} + {}^A\boldsymbol{I}_{c_{i_2}}\right)\hat{s}_{i\times}\right)
\end{aligned} \tag{59}
$$

$$
\boldsymbol{G}_i = \left(-m_{i_2} \hat{\boldsymbol{s}}_i \hat{\boldsymbol{s}}_i^T + \frac{1}{l_i}\left(m_{i_1} c_{i_1} + m_{i_2}\left(l_i - c_{i_2}\right)\right) \hat{\boldsymbol{s}}_{i\times}^2\right) \boldsymbol{g} \tag{60}
$$

$$
\boldsymbol{F}_i = -\boldsymbol{f}_{b_i} + \tau_i \hat{\boldsymbol{s}}_i \tag{61}
$$

Equation 51 with its terms defined in 58–61 gives the explicit equation for each limb based on kinematic parameters. In this equation the generalized coordinates is the position of the moving attachment point B_i.

B. Explicit Dynamics of the Moving Platform

In this section, the moving platform equations of motion given in 25 and 26, are transformed into an explicit form as:

$$
\boldsymbol{M}_{mp}\ddot{\boldsymbol{\mathcal{X}}} + \boldsymbol{C}_{mp}\dot{\boldsymbol{\mathcal{X}}} + \boldsymbol{G}_{mp} = \boldsymbol{F}_{mp} \tag{62}
$$

in which, $\boldsymbol{\mathcal{X}}$ denotes a set of generalized coordinates for the position and orientation of the moving platform:

$$
\boldsymbol{\mathcal{X}} = \left[\begin{array}{cc} \boldsymbol{x} & \boldsymbol{\phi} \end{array}\right]^T = \left[\begin{array}{cccccc} x & y & z & \alpha & \beta & \gamma \end{array}\right]^T \tag{63}
$$

and $\ddot{\boldsymbol{\mathcal{X}}}$ and $\dot{\boldsymbol{\mathcal{X}}}$ are its time derivatives. Substituting $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\omega}}$ from 10 into equation 26 and transform cross products into matrix products will yield to:

$$
\begin{aligned}
\boldsymbol{E}^T \boldsymbol{I}_{mp} \boldsymbol{E} \ddot{\boldsymbol{\phi}} + \boldsymbol{E}^T\Big(\boldsymbol{I}_{mp}\dot{\boldsymbol{\phi}} + \left(\boldsymbol{E}\dot{\phi}\right)_\times \boldsymbol{I}_{mp}\boldsymbol{E}\Big)\dot{\boldsymbol{\phi}} = \\
\boldsymbol{E}^T \sum \boldsymbol{b}_{i\times} \boldsymbol{f}_{B_i}
\end{aligned} \tag{64}
$$

Note that $\boldsymbol{E}^T$ has been multiplied to both side of equation 64 to harmonize these equations to that derived from Lagrange approach. Equation 64 together with 25 can be rewritten in a complete explicit form given in 62, whose terms are given as follows,

$$
\boldsymbol{M}_{mp} = \left[\begin{array}{cc} m_{mp}\boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{E}^T\boldsymbol{I}_{mp}\boldsymbol{E} \end{array}\right]_{6\times6} \tag{65}
$$

$$
\boldsymbol{C}_{mp} = \left[\begin{array}{cc} \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{E}^T\boldsymbol{I}_{mp}\dot{\boldsymbol{E}} + \boldsymbol{E}^T\left(\boldsymbol{E}\dot{\boldsymbol{\phi}}\right)_\times \boldsymbol{I}_{mp}\boldsymbol{E} \end{array}\right]_{6\times6} \tag{66}
$$

$$
\boldsymbol{G}_{mp} = \left[\begin{array}{c} -m_{mp}\boldsymbol{g} \\ \boldsymbol{0}_{3\times1} \end{array}\right]_{6\times1} \tag{67}
$$

$$
\boldsymbol{F}_{mp} = \left[\begin{array}{c} \sum \boldsymbol{f}_{b_i} \\ \boldsymbol{E}^T \sum b_{i\times}\boldsymbol{f}_{b_i} \end{array}\right]_{6\times1} \tag{68}
$$

in which, $\boldsymbol{I}_{3\times3}$ is a 3×3 Identity matrix.

C. Explicit Dynamics for the Manipulator

In order to derive the explicit equations of motion for the whole, first a transformation similar to what has been discussed in [13] is used. This map will transform the intermediate generalized coordinates $\boldsymbol{x}_i$, used for equations of motion of each limb, into principal generalized coordinates $\boldsymbol{\mathcal{X}}$. Then it is shown that by adding the resulting equations together with equation (64), one can eliminates the remaining internal forces $\boldsymbol{f}_{b_i}$. Hence, the equations of motion of the whole manipulator can be easily derived in an explicit form. Substitute $\boldsymbol{\omega}$ from 10 into 11 and once more use 53. This results in:

$$
\dot{\boldsymbol{x}}_i = \boldsymbol{J}_i \dot{\boldsymbol{\mathcal{X}}} \tag{69}
$$

in which,

$$
\boldsymbol{J}_i = \left[\begin{array}{cc} \boldsymbol{I}_{3\times3} & -\boldsymbol{b}_{i\times}\boldsymbol{E} \end{array}\right]. \tag{70}
$$

Furthermore, time derivative of $\boldsymbol{J}_i$ can be derived as:

$$
\begin{aligned}
\dot{\boldsymbol{J}}_i &= \left[\begin{array}{cc} \boldsymbol{0}_{3\times3} & -\left(\left(\boldsymbol{\omega}_\times \boldsymbol{b}_i\right)_\times \boldsymbol{E} + b_{i\times}\dot{\boldsymbol{E}}\right) \end{array}\right] \\
&= \left[\begin{array}{cc} \boldsymbol{0}_{3\times3} & -\boldsymbol{\omega}_\times \boldsymbol{b}_{i\times}\boldsymbol{E} + \boldsymbol{b}_{i\times}\boldsymbol{\omega}_\times \boldsymbol{E} \;\; + \boldsymbol{b}_{i\times}\dot{\boldsymbol{E}} \end{array}\right]
\end{aligned} \tag{71}
$$

Now substitute $\dot{\boldsymbol{x}}_i$ from 69 into 51, then:

$$
\boldsymbol{M}_{li}\ddot{\boldsymbol{x}}_i + \boldsymbol{C}_{li}\dot{\boldsymbol{x}}_i + \boldsymbol{G}_{li} = \boldsymbol{F}_{li} \tag{72}
$$

in which,

$$
\boldsymbol{M}_{li} = \boldsymbol{J}_i^T \boldsymbol{M}_i \boldsymbol{J}_i \tag{73}
$$

$$
\boldsymbol{C}_{li} = \boldsymbol{J}_i^T \boldsymbol{M}_i \dot{\boldsymbol{J}}_i + \boldsymbol{J}_i^T \boldsymbol{C}_i \boldsymbol{J}_i \tag{74}
$$

$$
\boldsymbol{G}_{li} = \boldsymbol{J}_i^T G_i \tag{75}
$$

$$
\boldsymbol{F}_{li} = \boldsymbol{J}_i^T \boldsymbol{F}_i \tag{76}
$$

Substitution of $\boldsymbol{F}_i$ from 61 into 76, results in:

$$
\begin{aligned}
\boldsymbol{F}_{li} &= \left[\begin{array}{c} \boldsymbol{I}_3 \\ \boldsymbol{E}^T\boldsymbol{b}_{i\times} \end{array}\right]\left[-\boldsymbol{f}_{b_i} + \tau_i \hat{\boldsymbol{s}}_i\right] \\
&= \left[\begin{array}{c} -\boldsymbol{f}_{b_i} \\ -\boldsymbol{E}^T\boldsymbol{b}_{i\times}\boldsymbol{f}_{b_i} \end{array}\right] + \boldsymbol{f}_{\tau_i}
\end{aligned} \tag{77}
$$

in which,

$$\boldsymbol{f}_{\tau_i} = \left[\begin{array}{c} \tau_i \hat{\boldsymbol{s}}_i \\ \tau_i \boldsymbol{E}^T \boldsymbol{b}_{i\times} \hat{\boldsymbol{s}}_i \end{array} \right] \tag{78}$$

Add equation 72 for all limbs together and add them to 62 to eliminate $\boldsymbol{f}_{b_i}$ from these equations. By this means the explicit dynamic formulation for the whole manipulator is derived as in equation 1:

$$\boldsymbol{M}(\boldsymbol{\mathcal{X}}) = \sum_{i=1}^{i=6} \boldsymbol{M}_{li} + \boldsymbol{M}_{mp} \tag{79}$$

$$\boldsymbol{C}\left(\boldsymbol{\mathcal{X}}, \dot{\boldsymbol{\mathcal{X}}}\right) = \sum_{i=1}^{i=6} \boldsymbol{C}_{li} + \boldsymbol{C}_{mp} \tag{80}$$

$$\boldsymbol{G}(\boldsymbol{\mathcal{X}}) = \sum_{i=1}^{i=6} \boldsymbol{G}_{li} + \boldsymbol{G}_{mp} \tag{81}$$

$$\boldsymbol{F}(\boldsymbol{\mathcal{X}}) = \sum_{i=1}^{i=6} \boldsymbol{f}_{\tau_i} \tag{82}$$

These equations completely define the detail terms of the equations of motion of SGP given in 1 in an explicit form. Moreover, these terms consists of kinematic structures of the limbs and the moving platform in a matrix form, and therefore, they are very compact and tractable. It is worth mentioning that these equations can be systematically derived without use of any symbolic manipulation software.

V. Verification

In order to verify the obtained equations of motions for SGP, $\boldsymbol{M}_i$ is derived by means of Lagrange method, and the results are compared. Let $\dot{\boldsymbol{x}}_i$ denote a generalized coordinate and $\boldsymbol{T}_i$ denote the kinetic energy of the limb, $\boldsymbol{M}_i$ can be found from,

$$T_i = \frac{1}{2} \dot{\boldsymbol{x}}_i \boldsymbol{M}_i(\boldsymbol{x}_i) \dot{\boldsymbol{x}}_i \tag{83}$$

Furthermore, the kinetic energy of the limb is:

$$\begin{aligned} T_i =& \frac{1}{2} \boldsymbol{v}_{c_{i_1}}^T m_{i_1} \boldsymbol{v}_{c_{i_1}} + \frac{1}{2} \boldsymbol{\omega}_i^T \left({}^A\boldsymbol{I}_{c_{i_1}} + {}^A\boldsymbol{I}_{c_{i_2}} \right) \boldsymbol{\omega}_i \\ &+ \frac{1}{2} \boldsymbol{v}_{c_{i_2}}^T m_{i_2} \boldsymbol{v}_{c_{i_2}} \end{aligned} \tag{84}$$

By means of relations given in 54, $\boldsymbol{\omega}$, $\boldsymbol{v}_{c_{i_1}}$ and $\boldsymbol{v}_{c_{i_2}}$ can be transformed to:

$$\begin{aligned} \boldsymbol{\omega}_i &= \frac{1}{l_i} \hat{\boldsymbol{s}}_{i\times} \dot{\boldsymbol{x}}_i \ ; \ \boldsymbol{v}_{c_{i_1}} = \frac{-c_{i_1}}{l_i} \hat{\boldsymbol{s}}_{i\times}^2 \dot{\boldsymbol{x}}_i \\ \boldsymbol{v}_{c_{i_2}} &= \left(-\frac{(l_i - c_{i_2})}{l_i} \hat{\boldsymbol{s}}_{i\times}^2 + \hat{\boldsymbol{s}}_i \hat{\boldsymbol{s}}_i^T \right) \dot{\boldsymbol{x}}_i \end{aligned} \tag{85}$$

Moreover, it can be easily shown that for any arbitrary vector $\boldsymbol{a}$,

$$\begin{aligned} \hat{\boldsymbol{s}}_{i\times}^4 \boldsymbol{a} = -\hat{\boldsymbol{s}}_{i\times}^2 \boldsymbol{a} \ &; \ \hat{\boldsymbol{s}}_i \hat{\boldsymbol{s}}_i^T \boldsymbol{a} = (\boldsymbol{I}_3 + \hat{\boldsymbol{s}}_{i\times}^2) \boldsymbol{a} \\ \hat{\boldsymbol{s}}_{i\times}^2 \hat{\boldsymbol{s}}_i \hat{\boldsymbol{s}}_i^T \boldsymbol{a} = 0 \ &; \ \hat{\boldsymbol{s}}_i \hat{\boldsymbol{s}}_i^T \hat{\boldsymbol{s}}_{i\times}^2 \boldsymbol{a} = 0 \end{aligned} \tag{86}$$

Now, substitute $\boldsymbol{\omega}$, $\boldsymbol{v}_{c_{i_1}}$ and $\boldsymbol{v}_{c_{i_2}}$ from 85 into 84 and use equations 86, to simplify:

$$\begin{aligned} T_i =& \frac{1}{2} \dot{\boldsymbol{x}}_i^T \left(\frac{-1}{l_i^2} \left(\hat{\boldsymbol{s}}_{i\times} \left({}^A\boldsymbol{I}_{c_{i_1}} + {}^A\boldsymbol{I}_{c_{i_2}} \right) \hat{\boldsymbol{s}}_{i\times} \right. \right. \\ & \left. \left. + \left(m_{i_1} c_{i_1}^2 + m_{i_2} (l_i - c_{i_2})^2 \right) \hat{\boldsymbol{s}}_{i\times}^2 \right) + m_{i_2} \hat{\boldsymbol{s}}_i \hat{\boldsymbol{s}}_i^T \right) \dot{\boldsymbol{x}}_i \end{aligned} \tag{87}$$

comparing equation 87 together with 82 to equation 58 verifies identical derivation of $\boldsymbol{M}_i$ throughout two methods. Note that the other terms in the dynamics equations can be verified in a similar manner. Although for those terms Lagrange formulation will lead to an extensive manipulation which is not given here due to the limited space.

VI. Conclusions

Closed–chain kinematic structure of parallel manipulators causes the dynamic equation of such manipulators to be very bulky and intractable. On the other hand having an explicit formulation for the dynamic equations of such manipulator is essentially needed for model–based control routines. In this paper, A vector based Newton-Euler formulation is proposed, which preserves the inherent kinematic structure components of the manipulator in the final resulting equations. This method is applied to the most celebrated parallel manipulator, namely the Stewart–Gough platform. Key elements to derive the explicit dynamic equation in a tractable form are to define an intermediate variable from joint space and some matrix algebraic manipulation tools. In the proposed method the equations are not derived componentwise, and therefore, the resulting equations are reduced into a concise vector based representations. Furthermore, $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{G}$ matrices are extracted and fully given for both limbs and the end-effector in a concise form. The proposed methodology, and the simplification rules can be used to derive other manipulator dynamics.

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