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3 Known Landmarks are Enough for Solving Planar Bearing SLAM and Fully Reconstruct Unknown Inputs

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Abstract— **In this paper we show that for an observer moving in the plane with no other information than the measurement of relative bearing to three known landmarks, it is possible to completely reconstruct its position and velocity. In particular this applies to the case where no model of the vehicle, nor odometry or acceleration measurements are available. Furthermore, in the same hypotheses, the position of any further landmark can be reconstructed from its bearing only. These results are more general than what is currently known on nonlinear observability of the SLAM problem, which relies on known observer velocities. Our results are also more general than the 2D version of known structure–from–motion observability results, which assume unknown but constant velocities. The proposed method is used to build a nonlinear observer directly applicable to a range of problems from computer vision to autonomous visual navigation.**

I. INTRODUCTION

For over twenty years, research in Simultaneous Localization and Mapping (SLAM) has been progressing. It was first introduced by R Smith and P Cheeseman in [1] and has been an active research field ever since. However, it was only recently that the observability analysis of the bearing only problem has been completely presented and discussed in literature ([2], [3], [4], and [5]). More specifically, a rigorous disturbance observability analysis, a problem also known as Unknown Input Observability (UIO) (or as Disturbance Observability (DO)), is still lacking in the current literature.

The classic observability problem, called Unknown State Observability (USO), regards the possibility of retrieving information on the state of a system given that input and output functions are completely known [6]. When applied to the SLAM problem, it identifies the conditions of problem solvability using a unified framework with the control problem [2].

Different characterizations of the observability problem represent a field of active research, e.g. bearing only observability analysis in the context of on-orbit space applications can be seen in [7]. A preliminary analysis that consider the motion of targets can be seen in [8], and the investigation of the multi–robot localization problem can be seen in [9]. Other recent studies focus on robust (or adaptive) control topics, like in [10], where the observability rank condition is applied in order to investigate on-line parameter identification problems concerning self calibration of the odometry.

Depending on the application, being able to reconstruct unknown input disturbances can be as important as the online parameter identification problem. In fact, UIO is one of the main topics in control theory and was introduced by Basile and Marro in [11] and Guidorzi and Marro in [12]. While it is usually assumed the inputs are completely known, in practice, under many situations, some of the input variables may not be completely available and, for this reason, it is appropriate to distinguish inputs between control inputs and disturbances. UIO is known to be a difficult task and is commonly associated with the problems of robust model based fault detection, a problem that was introduced in [13] and further extended to the detection of both sensor and actuator faults in [14]. Disturbance rejection can play a crucial role concerning performance and convergence of systems under feedback control, what is usually the case of autonomous vehicles.

Here we investigate the solvability of the planar bearing SLAM problem whenever input disturbances or unknown inputs are present. This article contribution is to show that if 3 landmark positions are known, not only the SLAM problem is solvable (as already discussed in literature, e.g. in [2]), but it is also possible to completely reconstruct any kind of input disturbance, even those that do not act directly on system inputs (e.g. vehicle drift). Our results are more general than the 2D version of known structure–from–motion observability results reported in [15] where inputs are assumed constant. Here, the only assumption made is that input disturbances are analytic, an assumption that is coherent with possible applications. In particular, we apply the observability rank condition ([6]) to investigate state observability and left–invertibility concurrently considering a polynomial expansion of input disturbances, and then, we apply logical induction to extend results for any analytic disturbance. Moreover, we investigate configuration singularities that may render the problem non observable.

Using an unified framework with the control problem, results presented permit the construction of nonlinear observers with direct application in many problems from computer vision to autonomous navigation, such as feature tracking, visual odometry, input reconstruction, fault tolerant visual servoing, active perception, optimal control, model independent control and others.

II. PROBLEM DEFINITION

Consider a vehicle moving on a plane where some arbitrary fixed right-handed reference frame $< W >$ with origin in ${}^{W}O$ and axes ${}^{W}X$, ${}^{W}Z$ is defined. The configuration of any generic vehicle is described by $\xi_r = [x_r, z_r, \theta_r]^T$, where $P_r = (x_r, z_r)$ is the position in $< W >$ of a reference point in the vehicle and θ_r is the robot heading with respect to the

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${}^WX$ axis (see figure 1). The vehicle moves in an unknown environment with the aim of localizing itself and the environment object features (or landmarks) represented as points of the motion plane. Following the convention presented in [2], these landmarks are distinguished between those belonging to objects with unknown position, named *targets*, and those belonging to objects whose absolute position w.r.t. $< W >$ is known, which are named *markers*. Wherever necessary, we may use the notation P_t to specify the position of a target and P_m to specify that of a marker.

A. System definition

In this paper we will consider vehicles whose dynamics are slow enough to be neglected, and kinematics can be described by

$$\dot{\xi}_r = G(\xi_r)\, u_r + B(\xi_r)\, d_r\,, \tag{1}$$

where u_r are input controls, d_r is a generalized input disturbance, $G(\xi_r)$ contains the velocity fields that describe vehicle kinematics and $B(\xi_r)$ is a known disturbance input matrix. Generic movements on the plane can be described by $u_r = [v_f\ v_h\ \omega]^T$, $d_r = [d_f\ d_h\ d_\omega]^T$ and

$$G(\xi_r) = B(\xi_r) = [g_f\ g_h\ g_\omega] = \begin{bmatrix} \cos\theta_r & -\sin\theta_r & 0 \\ \sin\theta_r & \cos\theta_r & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

While generic movements could be represented by a 3×3 identity matrix I, such choice introduces the knowledge of the world reference frame: v_f and d_f are assumed to be in the direction of axis ${}^WX$, while v_h and d_h are in direction of ${}^WZ$. This leads to inconsistencies in the observability analysis. To illustrate the problem, consider an USO analysis with one marker only and $G = I$. This problem is completely observable whereas the same analysis using G defined as (2) is not. As a matter of fact, the unobservable space in the second problem is characterised by the reference frame orientation, which is assumed to be known when G is described using the identity matrix. Therefore, in order to achieve generic results and do not introduce any knowledge apart from the position of 3 markers, all disturbance observability analysis performed here will consider generic movements as described in (2).

Specific vehicle kinematics and disturbance models can be described by choosing G and B composed of subspaces of (2) and the proper choice of u_r and d_r. For example, a boat can be modeled as a unicycle vehicle with disturbance along the nonholonomic direction to represent the undesired lateral motion defining $G(\xi_r) = [g_f\ g_\omega]$, $B(\xi_r) = [g_h]$, $u_r = [v_f,\ \omega]^T$ and $d_r = d_\omega$.

The vehicle is equipped with a sensor head such that its measurements are angles in the horizontal plane between the line joining the obstacle features (landmarks) with the head position, and the forward direction of the vehicle. The measurement process is modeled by equations of the form

$$y_i = h_i(\xi_r) = \arctan 2\left(\frac{z_r - z_i}{x_r - x_i}\right) - \theta_r + \pi\,, \quad i = 1, \ldots, q\,, \tag{3}$$

where $P_i = (x_i, z_i)$ describe the absolute position of landmark i and $q = N + M$ is the total number of landmarks (see figure 1). From these, N correspond to target observations and M correspond to measurements relating to markers. Wherever necessary, we may use the notation P_t to specify the position of a target and P_m to specify that of a marker. Note that equation (3) is not defined whenever vehicle and landmark positions coincide.

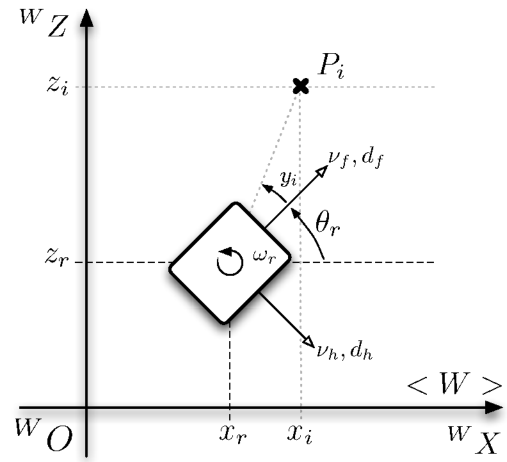

Fig. 1. Fixed frame $< W >$, vehicle state $(x_r,\ z_r,\ \theta_r)$, generalized velocities (v_f, v_h, ω), input disturbances (d_f, d_h, d_ω) and i_{th} landmark position $P_i = (x_i,\ z_i)$.

In this paper we study the solvability of the planar bearing SLAM problem whenever input disturbances (or unknown inputs) are present. In particular we will show that if 3 marker positions are known, not only the SLAM problem is solvable, but it is possible to completely reconstruct any kind of input or disturbance, even those that do not act directly on system inputs. SLAM consists in reconstructing the vehicle configuration and the state of targets $\xi_{t,i} = P_{t,i}$ concurrently. For this purpose, we introduce a system Σ, which state $\xi_\Sigma = (\xi_r, \xi_{t,1}, \xi_{t,2}, \ldots, \xi_{t,N})$ describes the SLAM problem. Moreover, as in this paper we deal with a wider problem, the one of reconstructing the disturbance d_r along with SLAM, we will consider an augmented state $\xi = (\xi_\Sigma, d_r)$ when necessary.

B. Concurrent Observability and Invertibility

Consider a generic driftless control–affine system affected by the external disturbance vector d:

$$\dot{\xi} = G(\xi)u + B(\xi)d,\ \xi \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ d \in \mathbb{R}^\eta,$$

where $B(x)$ is a known disturbance input matrix.

System left-invertibility regards the possibility of reconstructing an unknown input (or disturbance) from the knowledge of the outputs [16]. One can pose input invertibility and state observability problems concurrently considering the augmented state $\xi = (\xi_\Sigma, d_r)$. The problem is defined as follows:

Problem 1: Unknown disturbance and unknown state reconstruction: Given known output function $y(\cdot)$, determine the initial state $\bar{\xi}_0$ (or the set of initial states), and disturbance $\bar{d}(t)$ for all $t \in [0, T]$ (or the set of inputs), such that $y(t) = y(\bar{\xi}_0, \bar{d}(\cdot), t), \forall t \in [0, T]$;

Let the Lie derivative of order k of a scalar function $\lambda(x)$ along the vector-field $f(x)$ be denoted by $L_f{}^{(k)}\lambda(x)$. Considering the generic dynamics $\dot{\xi} = f(\xi, d, u)$, the observability space associated with ξ is given by

$$\Theta = \left\{ h_i, L_f h_i, L_f{}^{(2)} h_i, \ldots \right\}, i = 1 : q,$$

where q is the number of system outputs. The corresponding observability codistribution is $\mathrm{dO} = \mathrm{span}(O_\Sigma)$, where O_Σ is called observability matrix and is defined as

$$O_\Sigma = \left\{ \partial_\xi h_i, \partial_\xi L_f h_i, \partial_\xi L_f{}^{(2)} h_i, \ldots \right\}, i = 1 : q.$$

In [6] it is demonstrated that a nonlinear system is locally weakly observable if the observability rank condition $rank(O_\Sigma) = dim(\xi)$ is verified. Here, we will consider the observability definitions as presented in [6].

III. DISTURBANCE OBSERVABILITY ANALYSIS USING THE RANK CONDITION

In this section we show how the observability rank condition can be used to study state observability and left invertibility concurrently. In order to apply the rank condition UIO problems (see section II-B), assumptions regarding input disturbance dynamics must be made. Hence, here we consider input disturbances that can be expressed by an analytical function, i.e., it can be expressed by an infinitely differentiable function such that if it is equal to its Taylor expansion in some neighbourhood of every point.

Let's define the polynomial disturbance ${}^K d(t)$ as the partial Taylor expansion of $d(t)$ around d_0 as ${}^K d(t) = \sum_{i=0}^{K} d_0{}^{(i)} \frac{t^i}{i!}$ and $d^{(K+1)} = 0$. The augmented system ${}^K\Sigma$ that is composed of original state ξ_Σ and polynomial disturbance ${}^K d(t)$ is described by the augmented state

$${}^K\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \vdots \\ \xi_{K+2} \end{bmatrix} = \begin{bmatrix} \xi_\Sigma \\ d \\ d^{(1)} \\ \vdots \\ d^{(K)} \end{bmatrix}, {}^K\xi \in \mathbb{R}^{n+(K+1)\eta},$$

with corresponding system dynamics

$${}^K\dot{\xi} = \begin{bmatrix} \dot{\xi}_\Sigma \\ \xi_3 \\ \xi_4 \\ \vdots \\ 0 \end{bmatrix}.$$

The observability analysis of ${}^K\xi$ can be accomplished by applying the rank condition to its corresponding observability matrix ${}^K O_\Sigma$. However, the following question arises: if ξ_Σ and d observable under the assumption of a constant $d^{(K)}$, does it implies that ξ_Σ and d are observable for any analytic $d(t)$?

To investigate the extension of an observability analysis carried out for ${}^K\Sigma$, it is possible to apply the mathematical induction method starting from the basis case ${}^0\xi$ and studying the inductive step from ${}^k\xi$ to ${}^{k+1}\xi$. The procedure can be summarized as follows:

- ${}^0\xi$ is observable: investigate the observability problem for ${}^0\xi$;
- ${}^k\xi$observable $\rightarrow$ ${}^{k+1}\xi$observable: investigate if ${}^k\xi$ being observable implies in the observability of ${}^{k+1}\xi$.

If the above conditions are demonstrated, one may conclude that $\xi = (\xi_\Sigma, d(t))$ is observable for any analytic $d(t)$. It is important to note that the logical induction presents sufficient conditions for system observability, but not necessary ones.

IV. DISTURBANCE OBSERVABILITY OF PLANAR BEARING ONLY PROBLEMS

In this section we apply the procedure proposed in section III to the disturbance observability analysis of bearing only localization and mapping problems.

In order to simplify the following demonstrations we define the subspace $O_{\lambda,x} = \left\{ \partial_x L_f{}^{(\lambda)} h_i \right\}, i = 1 : q$, the displacements $\Delta \mathrm{x}_i = x_r - x_i$ and $\Delta \mathrm{z}_i = z_r - z_i$, and the cartesian distance $\rho_i = \sqrt{(\Delta \mathrm{x}_i)^2 + (\Delta \mathrm{z}_i)^2}$.

A. Observability analysis

Considering a vehicle for which inputs are completely unknown, inputs are treated as a disturbance itself. We can represent any possible disturbance by considering the following dynamics:

$$\dot{\xi}_r = f_r(\xi_r, d_r) = \left[g_f \; g_h \; g_\omega \right] d_r, \tag{4}$$

that corresponds to a disturbance input matrix $B(\xi_r)$ chosen to describe omnidirectional kinematics and a disturbance vector $d_r = \left[d_f, d_h, d_\omega \right]^T$ that comprehends all the possible planar generalized velocities.

The augmented system composed of vehicle state and input disturbances is

$$\xi = \left[\xi_r{}^T, d_r^T \right]^T. \tag{5}$$

and system output is described by measurements of 3 markers (see equation 3) as

$$y = \left[y_1 \; y_2 \; y_3 \right]^T. \tag{6}$$

Proposition 1: Consider the problem described by equations (4), (5) and (6). Apart from singularities, both vehicle state and analytic input disturbances are locally weakly observable if 3 landmark positions are known.

Proof: The problem is first studied assuming constant inputs, and then, results are extended assuming analytic inputs. The constant input augmented state that comprehends both vehicle configuration and unknown inputs is defined as ${}^0\xi = \left(\xi_r, d_f, d_h, d_\omega \right)$ with corresponding system dynamics

$$\dot{\xi} = f(\xi_r, d_r) = \left[f_r{}^T \; 0 \; 0 \; 0 \right]^T.$$

If 3 landmark bearings are being measured (q=3), the following subspace ${}^0O_\Sigma$ is sufficient for studying system observability:

$${}^0O_\Sigma = \begin{bmatrix} O_{0,\xi_r} & 0 \\ * & O_{1,d_r} \end{bmatrix}, \tag{7}$$

where O_{0,ξ_r} is composed of the following derivatives:

$$\partial_{\xi_r} h_i = \left[\; -\Delta \mathrm{z}_i / \rho_i{}^2 \quad \Delta \mathrm{x}_i / \rho_i{}^2 \quad -1 \; \right],$$

and O_{1,d_r} is composed of the following derivatives:

$$\partial_{d_r} L_f{}^{(1)} h_i = \left[\begin{array}{ccc} \frac{\Delta x_i \sin(\theta_r) - \Delta z_i \cos(\theta_r)}{\rho_i{}^2} & \frac{\Delta x_i \cos(\theta_r) + \Delta z_i \sin(\theta_r)}{\rho_i{}^2} & -1 \end{array} \right].$$

Apart from configuration singularities, $rank(O_{0,\xi_r}) = 3$ and $rank(O_{1,\xi_r}) = 3$, matrix rank of ${}^0O_\Sigma$ is 6 and the system is locally weakly observable.

Now, we investigate if this result can be extended to any analytic d_f, d_h and d_ω. As ${}^0\xi$ is observable, we must verify if ${}^k\xi$observable $\rightarrow$ ${}^{k+1}\xi$observable. Therefore, let's analyze what happens with ${}^kO_\Sigma$ when we apply the inductive step from ${}^k\xi$ to ${}^{k+1}\xi$:

$${}^kO_\Sigma = \left[\begin{array}{cc} * & 0 \\ * & O_{1,d_r} \end{array} \right] \rightarrow {}^{k+1}O_\Sigma = \left[\begin{array}{ccc} * & 0 & 0 \\ * & O_{1,d_r} & 0 \\ * & * & O_{1,d_r} \end{array} \right].$$

Notice that $rank({}^{k+1}O_\Sigma) = rank({}^kO_\Sigma) + rank(O_{1,d_r})$. If ${}^kO$ is observable, then:

$$rank\left({}^{k+1}O_\Sigma\right) = dim\left({}^k\xi\right) + 3 = dim\left({}^{k+1}\xi\right),$$

and we can conclude that ${}^k\xi$ observable implies in ${}^{k+1}\xi$ being also observable. Hence, for analytic unknown inputs v_f, v_h and ω, both vehicle state and velocities are locally weakly observable if 3 landmark positions are known. ∎

B. Observability singularities

Proposition 1 is valid apart from vehicle and disturbance configurations that render matrix ${}^kO_\Sigma$ singular. First we will analyse the singurities of ${}^0O_\Sigma$ only, then we will investigate the configurations that render the complete O_Σ singular for any analytic disturbance.

Given the block triangular form of ${}^0O_\Sigma$, its singularity analysis can be decoupled into the investigation of submatrices O_{0,ξ_r} and $O_{1,u}$. The set of vehicle configurations that render these 3×3 square matrices singular are coincident and describes a circumference (denoted by symbol $\mathscr{C}$) that passes through all three landmark positions. Indeed, from a geometrical point of view, for any point $P_r \in \mathscr{C}$, angles between couple of chords sharing endpoint P_r are constant.

While this set of configurations present a problem for observers which construction is based in the minimal sufficient subspace ${}^0O_\Sigma$ only, it does not render the complete observability matrix singular. Indeed, it can be verified that the problem becomes not observable (singular) for the set of vehicle and input disturbance configurations (ξ_r and d_r) for which vehicle trajectory is described by $\mathscr{C}$. More precisely, the problem is singular when $P_r \in \mathscr{C}$, instantaneous motion is tangent to $\mathscr{C}$ and the following relation is true:

$$\frac{\left\| d_f + d_h \right\|}{\omega} = \frac{\left\| d_f{}^{(1)} + d_h{}^{(1)} \right\|}{\omega^{(1)}} = \ldots = r \qquad (8)$$

where r is the radius of $\mathscr{C}$.

Figure 2 illustrates system configurations that render the problem non observable. In this figure we consider ${}^0O_\Sigma$ till $\partial_x L_f{}^{(2)} h_i$ and d_r respects condition (8). $\mathscr{C}$ is represented by the thick like and thin lines describe some solutions of $M_i = 0$, where M_i denote the i_{th} 3×3 minor of ${}^0O_\Sigma$. It can be seen that solutions intersect at two points $Z(\theta_r)$ and $Z'(\theta_r) \in \mathscr{C}$, that are symmetric w.r.t. the center of $\mathscr{C}$. $Z(\theta_r)$ and $Z'(\theta_r)$ are the points of $\mathscr{C}$ for which instantaneous motion direction (that is a function of d_f, d_h and θ_r) is tangent to the circle. Arrows indicate the instantaneous motion direction.

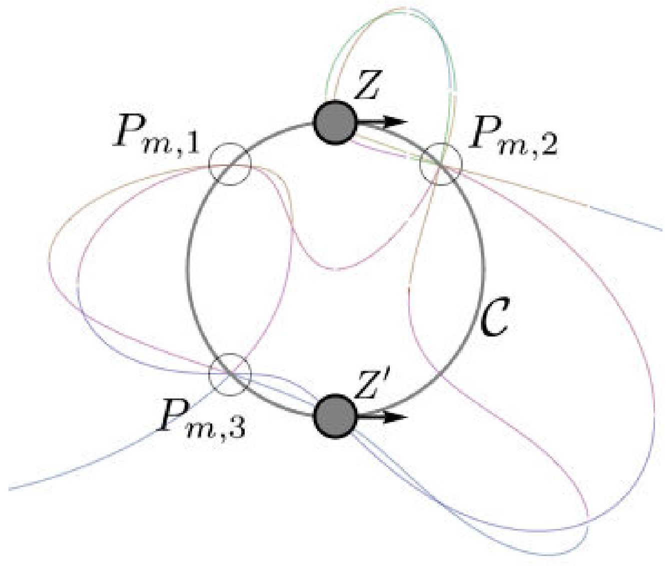

Fig. 2. Observability singularities

Note that when vehicle and landmark position are coincident, as the output representation is undefined, the observability problem is also undefined.

C. Extension of results

1) Any type of vehicle with partially known inputs: Proposition 1 can be extended for problems of partially known inputs on the state form

$$\dot{\xi}_r = f_r(\xi_r, d_r) + G(\xi_r) u_r, \qquad (9)$$

where vehicle kinematics is described by $G(\xi_r)$ and known inputs by u_r. As a matter of fact, given a generic observability matrix O_Σ^* that describes the original problem (equation (4)), the correspondent O_Σ that describes the partially known input problem (equation (9)) is composed of O_Σ^* and new lines described by covectors $L_{g_*}^{(k)} h_i$, $g_* \in G$. Hence, $rank(O_\Sigma) \geq rank(O_\Sigma^*)$ and we can conclude that if O_Σ^* is full rank, then O_Σ is also full rank and hence observable, independently of vehicle kinematics G and known inputs u_r considered.

2) Mapping targets: Proposition 1 can be extended for problems involving any number of targets. Problems that are observable when 3 marker positions are known are also observable even when targets (unknown landmarks) are being mapped. The extension of results can be done in a similar manner to the observability analysis proposed in [2]. Given a generic observability matrix O_Σ^* that describes a problem with M markers and no targets, the correspondent O_Σ that consider the same problem with M markers and N targets can be written as

$$O_\Sigma = \begin{bmatrix} O_{0,\xi_r} & O_{0,\xi_t} & O_{0,d_r} \\ \vdots & \vdots & \vdots \end{bmatrix},$$

where any O_{i,ξ_t} has the following form:

$$O_{i,\xi_t} = \left[\begin{array}{ccc} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ - & - & - \\ O_{i,\xi_{t,1}} & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & O_{i,\xi_{t,N}} \end{array} \right] \begin{array}{l} \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} M \text{ markers} \\ \\ \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} N \text{ targets} \end{array},$$

where O_{i,ξ_r} and O_{i,d_r} are similar to O^*_{i,ξ_r} and O^*_{i,d_r}, with the only difference that there are new lines corresponding to the targets. Given that

$$O_{0,\xi_{th}} = \begin{bmatrix} \frac{-\Delta z_h}{\rho_h{}^2} & \frac{\Delta x_h}{\rho_h{}^2} \end{bmatrix}$$

has rank 2 (apart from when output is undefined, i.e., $P_r \to P_{t,h}$), we can conclude that $rank(O_\Sigma) \geq rank(O^*_\Sigma) + 2q_t$. Hence, if O^*_Σ is full rank (observable), then O_Σ is also full rank, and, observable. This result does not depend on the number of targets considered.

V. SIMULATION RESULTS

Here we first describe the nonlinear observer used during simulations and then provide results that validate the disturbance observability analysis for the different cases presented.

A. Observer Design

In this section we discuss the construction of the nonlinear observer (or filter) used during simulations to estimate the augmented state ξ. The problem is introduced and briefly discussed, for further demonstrations and a recent review of nonlinear observers, please refer to [17].

Considering the estimated state $\bar{\xi}$, the observer is described by the auxiliary system

$$\begin{cases} \dot{\bar{\xi}} = f(\bar{\xi}) + \sum_i g_i(\bar{\xi}) u_i + v \\ \bar{y} = h(\bar{\xi}). \end{cases}$$

We will also use the notation $\tilde{*}$ to represent the symbolic operation $* - \bar{*}$, e.g. $\tilde{g}(.) = g(\xi) - g(\bar{\xi})$. Consider the observer error $e = \xi - \bar{\xi}$. Error dynamics are given by

$$\dot{e} = \tilde{f}(.) + \sum_i \tilde{g}(.) u_i - v,$$

where v is the observer compensator term. Let's choose v as

$$v = -KJ(\bar{\xi})^+ (Y - H(\bar{\xi})),$$

where,

$$Y = \begin{bmatrix} y \\ \dot{y} \\ \vdots \end{bmatrix}, H(\bar{\xi}) = \begin{bmatrix} h(\bar{\xi}) \\ \dot{h}(\bar{\xi}) \\ \vdots \end{bmatrix},$$

$J(\bar{\xi})^+$ is the pseudoinverse of $J(\xi) = \partial_\xi H(\xi)$ evaluated at $\bar{\xi}$, and K is a positive definite constant gain matrix. Vectors Y and H are built using the minimum number of derivatives required for the system to be observable when singularities are not considered. If the system is locally weakly observable and we consider the observer as a local problem, it is possible to rewrite v as

$$v = K(e + \varepsilon(\xi, \bar{\xi})),$$

where $\varepsilon(\xi, \bar{\xi})$ (or simply ε) is the error between the pseudoinverse approximation of e and the real error e.

Choosing as Lyapunov candidate $V = \frac{1}{2} e^T e$, we have

$$\dot{V} = e^T \left(\tilde{f}(.) + \sum_i \tilde{g}(.) u_i - K(e + \varepsilon) \right) \qquad (10)$$

If $\| \left(1 - \frac{\varepsilon}{e}\right) \|$, $\tilde{f}(.)$ and $\tilde{g}(.)$ are bounded, K can be chosen based on (10) in order to guarantee stability. One may note that if at any given moment, the term $\| \left(1 - \frac{\varepsilon}{e}\right) \|$ extrapolates the worst case admissible, convergence would not be guaranteed anymore. However, the choice of an worst ε, that is related to a maximum $\|e(0)\|$ admissible, guarantees convergence for $\bar{\xi}(0)$ in a local neighbourhood of the real state $\xi(0)$.

Remark 1: In presence of unknown input disturbances, we would expect nonlinear observers that do not take in consideration the disturbance to converge to wrong solutions, or not to converge at all. While this is the case for constant gain observers, this may not be true if one uses adaptive gain techniques (such as the well known Kalman gain, for example). It can be seen that for $K \to \infty$ such observer degenerates into a filtered approximate inverse solution of $y \to \xi$ without considering system dynamics. Consequently, the use of adaptive gain techniques would mask the disturbance effect in the observer estimated state evolution. The use of adaptive gains may guarantee convergence in cases where constant gains wouldn't, even if at the expense of renouncing the knowledge of system dynamics. Hence, while nothing excludes the possibility of using adaptive gain techniques during practical applications, here we chose to use constant gains in order make the differences between the different problems presented evident from the results of the presented simulations.

B. Simulations

During simulations an unicycle-like vehicle performs an arbitrary trajectory and receives external disturbance characterized by a constant input acting along its nonholonomic direction. In order to localize the robot while reconstructing input disturbances we present a comparison of different cases that illustrate different observability problems that were presented or discussed here:

- **Case 1 - USO using unicycle kinematics**: in this case results regard the observer reconstruction of vehicle state ξ_r considering unicycle-like vehicle kinematics. System input u is completely known and disturbance reconstruction is not an output of the observer. Results can be seen in figures 3-b, 4-b, 5-b and 6-b;
- **Case 2 - UIO considering unicycle kinematics for disturbance**: in this case results regard the observer reconstruction of vehicle state and vehicle input, i.e.

$\xi = \{\xi_r, d_r\}$. System input is completely unknown and is considered a disturbance itself. However, here we consider that $B(\xi_r)$ describe the kinematics of an unicycle–like vehicle. Results can be seen in figures 3-c, 4-c, 5-c and 6-c;
- **Case 3 - UIO with partially known inputs (using unicycle kinematics) and omnidirectional kinematics for disturbance**: in this case results regard the observer reconstruction of state and disturbance with partially known inputs. Here consider the generic disturbance input matrix $B(\xi) = I$ and $d \in \mathbb{R}^3$, permitting us to reconstruct disturbances in all directions of the input space. Partially known input $u_r = (v_f, \omega)^T$ and velocity vectors in G corresponding to the unicycle-like vehicle kinematics. Results can be seen in figures 3-d, 4-d, 5-d and 6-d;

For completeness, we also report the use of the triangulation method for direct computation of the state–output inverse $y \rightarrow \xi_r$ using 3 measurements. In this case only vehicle localization is performed as direct disturbance reconstruction is not a direct output of the method. An approximate reconstruction of inputs can be roughly obtained considering the derivative of the vehicle configuration.

All observers are realized as described in section V-A and simulations are performed using the following system parameters:

- Observers initial estimated state is $\bar{\xi}_r(0) = (0\,\text{m}, 0\,\text{m}, 0\,\text{rad})^T$.
- During simulations we consider a triangular K. To present a fair comparison between problems, all observers use the same gains: $K_1 = K_2 = 1.5$ and $K_i = 0.5, \forall i \neq 1, 2$.
- We consider a choice of $\left(1 + \frac{\varepsilon}{e}\right) \simeq 1$ and $\max(\|d\|) = 1$.
- Real vehicle trajectory is described by inputs $(v, \omega)^T$=$(0.1\,\text{m/s}, -0.1\,\text{rad/s})^T$ and input disturbance $B(x)d = (0\,\text{m/s}, -0.1\,\text{m/s}, 0\,\text{rad/s})^T$.
- Output $y_i, \forall i$ is affected by a measurement noise that is described by a random Gaussian variable with standard deviation $\sigma = 0.005\,\text{rad}$.
- No pre-filtering is used.

In these simulations the number of landmarks used is $q = 3$. For the sake of simplicity, the number of targets is $N = 0$. Results are presented as follows:

- figure 3 shows time history of real and estimated vehicle positions P_r; Real trajectory is represented with a red line and estimated trajectory with a blue line.
- figure 4 shows time history of estimation errors for variables x_r (red), z_r (green) and θ_r (blue).
- figure 5 shows time history of velocity reconstruction estimation errors $\dot{\tilde{x}}_r$ (red), $\dot{\tilde{z}}_r$ (green) and $\tilde{\omega}$. Given that velocity reconstruction is not a direct output of the triangulation method, figure 5-a shows the derivative of vehicle estimated state with the method. Similarly, for case 1, figure 5-b represent the constant error between known velocities and real ones.
- figure 6 shows time history of bearing tracking errors, i.e., the difference between estimated measurement and real ones. Figure 6-a shows the time history of measurement noise.

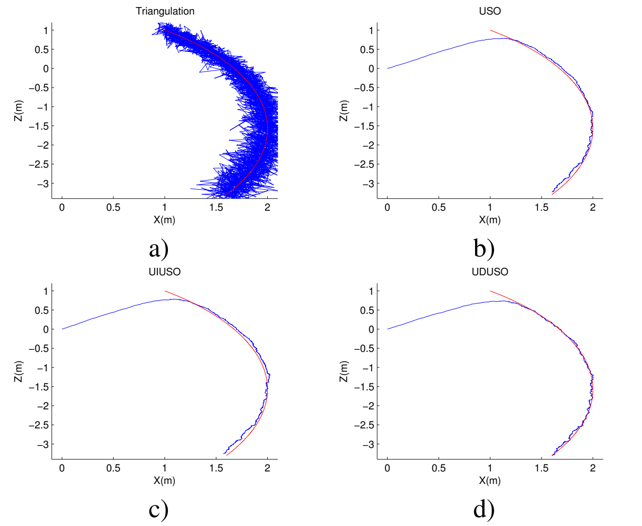

Fig. 3. Estimated trajectories: (a) triangulation; (b) case 1; (c) case 2; and (d) case 3;

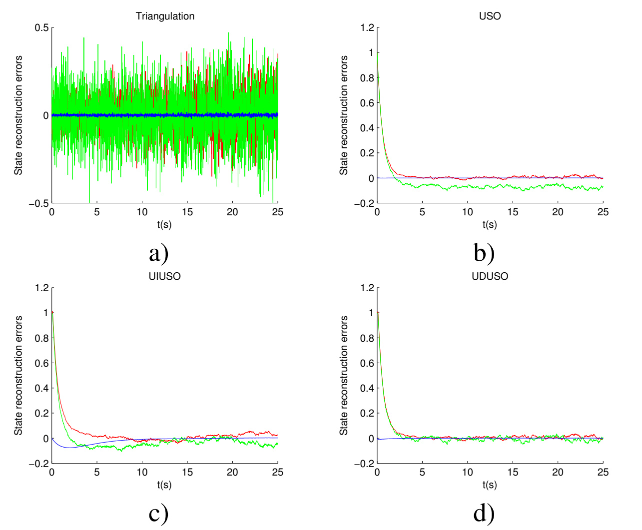

Fig. 4. Estimation errors: (a) triangulation; (b) case 1; (c) case 2; and (d) case 3;

Please note that different scales are used in each figure for each case studied. E.g. in figure 5, the order of errors for the triangulation case is 10^2 while for cases 1,2 and 3 it is 10^{-1}.

It can be observed that triangulation results are the ones most affected by measurement noise. This is a consequence from the fact that no filter is applied to measurements or results. The use of an observer constitute a filter itself, what explains the results obtained in the cases where an observer is applied.

In case 2, vehicle drift is not considered as a disturbance and, as expected, errors do not converge to zero. However, it performs much better than the USO observer (case 1), that presents the worst convergence case. As expected, case 3 presents the best results, and is the only observer for which errors converge to zero.

Presented simulations illustrate some advantages of using Disturbance Observers (DO): DO can compensate model-

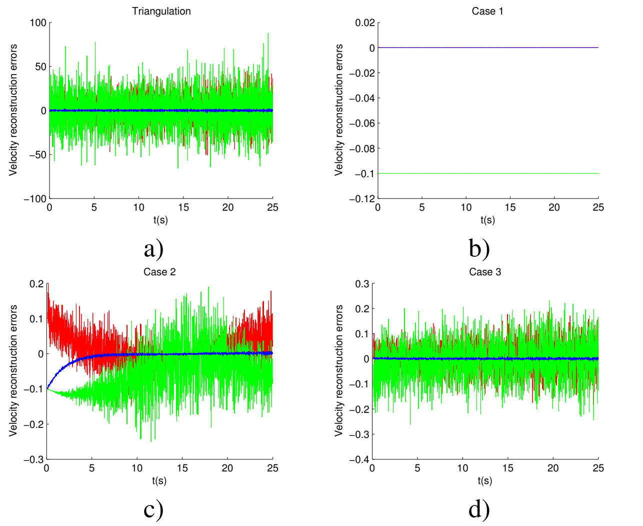

Fig. 5. Velocity reconstruction errors: (a) triangulation; (b) case 1; (c) case 2; and (d) case 3;

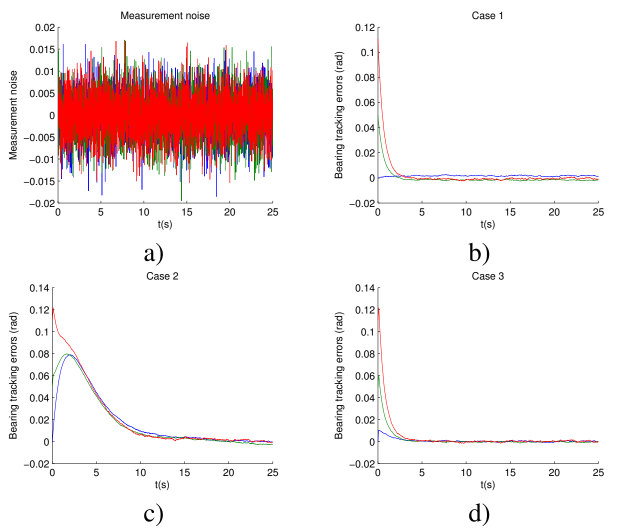

Fig. 6. Bearing tracking errors: (a) measurement noise; (b) case 1; (c) case 2; and (d) case 3;

ing uncertainties, as a matter of fact, observers can be constructed without considering any specific model at all; DO can be used to robustly track measurements (feature tracking) without requiring any previous knowledge of inputs or the system; disturbance reconstruction can guarantee convergence of constant gain observers; DO makes robust controllers with disturbance rejection a possibility; finally, disturbance observers can be applied even in cases where no input knowledge is available, e.g. visual odometry. However, disturbance observers may be used with caution, e.g. during transitory behavior the reconstruction of disturbance introduces oscillations in estimated state and equivalent observers may take longer to converge.

VI. CONCLUSIONS AND FUTURE WORKS

Here we present a disturbance observability analysis of the bearing only SLAM problem for the case of 3 known landmarks (markers) and demonstrate the conditions and singularities regarding the problem observability. Moreover, we discuss the extension of results to the partially known input and unknown landmarks (targets) cases . Finally, we validate theoretical results under simulation using constant gain non-linear observers.

Regarding future works, authors are currently investigating the application of optimal control as a potential solution to observability problems regarding the presented singularity cases. The optimization of observability indexes is a field of active research in active perception and vision. Furthermore, authors are studying the use of disturbance rejection and point-to-point stability of vehicles using unknown input observers concurrently with feedback control.

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