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Underactuated Control for Nonholonomic Mobile Robots by Using Double Integrator Model and Invariant Manifold Theory

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Abstract— **In a stabilizing control for nonholonomic mobile robots with two independent driving wheels, a nonholonomic double integrator in the kinematic model is first considered as a controlled object model. Then, a quasi-continuous exponential stabilizing control method is proposed as one of underactuated control methods by using invariant manifold theory. Next, to extend the velocity input control in a kinematic level to the torque input control in a dynamical level, an extended nonholonomic double integrator consisting of the kinematic and dynamical models is treated as a controlled object model. A quasi-continuous exponential stabilizing controller is further derived for such an extended model by using the same way as used in the kinematic level control. The effectiveness of the present method is proved with some demonstrative simulations.**

I. INTRODUCTION

It is known that, for systems described by symmetric affine systems that are based on a kinematic model in nonholonomic systems, underactuated control is well achieved by using a discontinuous model [1] or applying a switching control (e.g., a logical switching method [2], a two-stage switching method such as sliding model control [3], [4]. Among them, a switching control and a quasi-continuous exponential stabilizing control based on invariant manifolds [5] are proposed for a power form system with two-inputs and three-states or with two-inputs and n-states [6], in which such control methods are considered as general forms for sliding mode control that is known as a conventional switching control for nonlinear systems. Similar considerations are applied for a kinematic model with four-inputs and six-states in a chained form, and for a dynamical model with two-inputs and five-states in a kind of second-order chained forms [7]. Furthermore, a discontinuous time-invariant feedback control is adopted for a chained form [8] and a recursive algorithm is given for an n-th order power form system [9],[10].

In this paper, for a double integrator model that is known as an alternative canonical model in nonholonomic systems [11], a quasi-continuous exponential stabilizing control method is considered by following the invariant manifold method as mentioned above. In particular, from the viewpoints of nonholonomic systems, stabilizing controllers are newly derived for a case of a kinematic model with two-inputs and three-states, and similarly for a case of a dynamical model with two-inputs and five states that is viewed as an extended double integrator model. The effectiveness of them is demonstrated thorough simulations for a mobile robot with two independent driving wheels, in which for the kinematic case three states are controlled by using two inputs, whereas for the dynamical case five states are controlled by using only two inputs. Thus, an alternative underactuated control approach is given for stabilizing such a nonholonomic mobile robots.

It should be noted that there exists a similar study for an affine system with a drift term in Khennouf & Canudas de Wit [7], but their canonical system is not an extended double integrator system. The present quasi-continuous exponential stabilizing control law fundamentally adopts a kind of two-stage control methods, in which the first step control is to realize an attractive control to an invariant manifold, the second step control is to stabilize the invariant manifold and finally each control law is directly superimposed, or is superimposed after modifying each control law slightly to construct a continuous stabilizing control law.

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II. INVARIANT MANIFOLD FOR NONHOLONOMIC DOUBLE INTEGRATOR SYSTEM

Let the controlled object be described by the following nonholonomic double integrator system:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1 \end{aligned} \tag{1}$$

and consider a stabilizing control problem such that $\boldsymbol{x}(t) = [x_1\ x_2\ x_3]^T$ becomes zero as $t \to \infty$. Here, all the states are assumed to be measurable.

To derive an invariant manifold, the feedback law given by

$$\begin{aligned} u_1 &= -kx_1 \\ u_2 &= -kx_2, \quad k > 0 \end{aligned} \tag{2}$$

is assumed to be applied to Eq. (1).

Then, solving the time response of the closed-loop system gives

$$\begin{aligned} x_1(t) &= x_1(0)e^{-kt} \\ x_2(t) &= x_2(0)e^{-kt} \end{aligned} \tag{3}$$

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Therefore, since $x_3(t)$ is given by

$$\begin{aligned}\dot{x}_3(t) &= x_1(0)e^{-kt}u_2 - x_2(0)e^{-kt}u_1 \\ &= -kx_1(0)x_2(0)e^{-2kt} + kx_1(0)x_2(0)e^{-2kt} \\ &\equiv 0 \end{aligned} \quad (4)$$

it follows that

$$x_3(t) = x_3(0) \quad (5)$$

From the constant term of this $x_3(t)$, it is possible to select

$$S(\boldsymbol{x}) = x_3(t) \quad (6)$$

as one candidate of an invariant manifold.

Under the above conditions, let the feedback law given by

$$\begin{aligned} u_1 &= -kx_1 \\ u_2 &= -kx_2, \quad k > 0 \end{aligned} \quad (7)$$

be applied to Eq. (1). Differentiating $S(\boldsymbol{x})$ with respect to time yields

$$\begin{aligned}\dot{S}(\boldsymbol{x}) &= \dot{x}_3(t) \\ &= x_1u_2 - x_2u_1 \\ &= -kx_1x_2 + kx_1x_2 \equiv 0 \end{aligned} \quad (8)$$

and under the feedback law given by Eq. (7), it holds that

$$S(\boldsymbol{x}) = \text{Const.} \quad (9)$$

so it is seen that $S(\boldsymbol{x})$ is reduced to one invariant manifold.

From these, since if $S(\boldsymbol{x}) = 0$ can be assured at time $t = T$, then it can hold that $S(\boldsymbol{x}) = 0$ for $t \geq T$, applying Eq. (7) to Eq. (1) gives

$$\dot{x}_1(t) = -kx_1, \quad \dot{x}_2(t) = -kx_2$$

so that $x_1(t)$ and $x_2(t)$ are asymptotically stable, i.e., $x_1 \to 0$ and $x_2 \to 0$ as $t \to \infty$. Of course, it is seen from Eq. (6) that $x_3 \to 0$, because it has been already assured that $S(\boldsymbol{x}) = 0$.

III. Attractive Control to Invariant Manifold

An attractive controller is described here to the manifold derived in the previous section.

For the case of $x_1^2(0)+x_2^2(0) \neq 0$, i.e., when $x_1(0) \neq 0$ or $x_2(0) \neq 0$, to obtain the first-step control law for realizing $S(\boldsymbol{x}) = 0$, selecting the Lyapunov function of $S(\boldsymbol{x})$ as

$$V(\boldsymbol{x}) = \frac{1}{2}S^2(\boldsymbol{x}) \quad (10)$$

and setting the control input as

$$\begin{aligned} u_1 &= fS(\boldsymbol{x})x_2(t) \\ u_2 &= -fS(\boldsymbol{x})x_1(t), \quad f > 0 \end{aligned} \quad (11)$$

it follows that

$$\begin{aligned}\dot{V}(\boldsymbol{x}) = S\dot{S} &= S[x_1u_2 - x_2u_1] \\ &= -S[x_1^2fS + x_2^2fS] \\ &= -fS^2W(\boldsymbol{x}) \leq 0 \end{aligned} \quad (12)$$

where

$$W(\boldsymbol{x}) \stackrel{\triangle}{=} x_1^2(t) + x_2^2(t) \quad (13)$$

Note here that, under the control law of Eq. (11), this $W(\boldsymbol{x})$ becomes

$$\begin{aligned}\dot{W} &= 2(x_1\dot{x}_1 + x_2\dot{x}_2) \\ &= 2(x_1u_1 + x_2u_2) \equiv 0 \end{aligned} \quad (14)$$

namely, it is seen that $W(\boldsymbol{x}(t)) = W(\boldsymbol{x}(0))$.

Therefore, Eq. (12) becomes negative definite, so that $S(\boldsymbol{x}) \to 0$ as $t \to \infty$, as long as $W(\boldsymbol{x}(0)) \neq 0$.

This controller is implementable even if $x_1(0) \equiv 0$ and $x_2(0) \neq 0$. Of course, it is not implementable for the case of $x_1(0) = 0$ and $x_2(0) = 0$, in which the controller makes the system unstable.

IV. Quasi-continuous Exponential Stabilizing Control

To construct a quasi-continuous exponential stabilizing controller, the attractive controller given in section III is superimposed on the feedback controller given in section II with slight modifications.

Now, let the control inputs be set as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = f\frac{S}{W}\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} - k\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (15)$$

Then, it follows that

$$\begin{aligned}\dot{W} &= 2(x_1\dot{x}_1 + x_2\dot{x}_2) \\ &= 2(x_1u_1 + x_2u_2) \\ &= 2[x_1(f\frac{S}{W}x_2 - kx_1) + x_2(-f\frac{S}{W}x_1 - kx_2)] \\ &= -2k(x_1^2 + x_2^2) \equiv -2kW \end{aligned} \quad (16)$$

and

$$\begin{aligned}\dot{S} = \dot{x}_3 &= x_1u_2 - x_2u_1 \\ &= x_1(-f\frac{S}{W}x_1 - kx_2) - x_2(f\frac{S}{W}x_2 - kx_1) \\ &= -f\frac{S}{W}(x_1^2 + x_2^2) \\ &= -fS \end{aligned} \quad (17)$$

so that their time-responses become

$$\begin{aligned} W(t) &= W(\boldsymbol{x}(0))e^{-2kt} \\ S(t) &= S(\boldsymbol{x}(0))e^{-ft} \end{aligned} \quad (18)$$

From these, $W(t) \to 0, S(t) \to 0$ as $t \to \infty$, and therefore $x_1(t), x_2(t)$ and $x_3(t)$ asymptotically converge to the origin.

Note here that Eq. (15) may diverge, because $W(\boldsymbol{x}(t)) \to 0$ if $x_1 \to 0, x_2 \to 0$. However, noting that

$$\frac{S(\boldsymbol{x})}{W(\boldsymbol{x})} = \frac{S(\boldsymbol{x}(0))e^{-ft}}{W(\boldsymbol{x}(0))e^{-2kt}} \quad (19)$$

the numerator is shown to converge faster than the denominator if $f > 2k$, and therefore the whole will converge to zero if $f > 2k$ and $S(\boldsymbol{x}(0))/W(\boldsymbol{x}(0))$ is finite so that the input will asymptotically converge to zero.

Note also that the system is not stable with the controller deigned in Eq. (15) for the case that $x_1(0) = 0$, $x_2(0) = 0$, and $x_3(0) \neq 0$.

V. Invariant Manifold for Extended Nonholonomic Double Integrator System

Let the controlled object be represented by the following extended nonholonomic double integrator system:

$$\begin{aligned} \dot{x}_1 &= y_1 \\ \dot{x}_2 &= y_2 \\ \dot{x}_3 &= x_1 y_2 - x_2 y_1 \\ \dot{y}_1 &= u_1 \\ \dot{y}_2 &= u_2 \end{aligned} \tag{20}$$

and consider a stabilizing problem such that $\boldsymbol{x}(t) = [x_1\ x_2\ x_3\ y_1\ y_2]^T$ is settled to zero as $t \to \infty$. Here, all the states are assumed to be measurable.

To derive an invariant manifold for this system, assume that the following state feedback law

$$\begin{aligned} u_1 &= -2ky_1 - k^2 x_1 \\ u_2 &= -2ky_2 - k^2 x_2 \end{aligned} \tag{21}$$

is applied to Eq. (20).

Now, defining the state vector of the linear partial system in (20) as

$$\boldsymbol{x}_{sub}(t) \overset{\triangle}{=} [x_1\ x_2\ y_1\ y_2]^T \tag{22}$$

its closed-loop linear partial system becomes

$$\begin{aligned} \dot{\boldsymbol{x}}_{sub}(t) &= A\boldsymbol{x}_{sub}(t), \\ A &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k^2 & 0 & -2k & 0 \\ 0 & -k^2 & 0 & -2k \end{bmatrix} \end{aligned} \tag{23}$$

so that its time response is written by

$$\boldsymbol{x}_{sub}(t) = e^{At}\boldsymbol{x}_{sub}(0) \tag{24}$$

where

$$e^{At} = \begin{bmatrix} e^{-kt} & 0 & te^{-kt} & 0 \\ 0 & e^{-kt} + kte^{-kt} & 0 & te^{-kt} \\ -k^2 te^{-kt} & 0 & e^{-kt} - kte^{-kt} & 0 \\ 0 & -k^2 te^{-kt} & 0 & e^{-kt} - kte^{-kt} \end{bmatrix} \tag{25}$$

Therefore, the closed-loop linear partial system is reduced to

$$\begin{aligned} x_1(t) &= x_1(0)[e^{-kt} + kte^{-kt}] + y_1(0)te^{-kt} \\ x_2(t) &= x_2(0)[e^{-kt} + kte^{-kt}] + y_2(0)te^{-kt} \\ y_1(t) &= x_1(0)[-k^2 te^{-kt}] + y_1(0)[e^{-kt} - kte^{-kt}] \\ y_2(t) &= x_2(0)[-k^2 te^{-kt}] + y_2(0)[e^{-kt} - kte^{-kt}] \end{aligned} \tag{26}$$

Furthermore, it is easy to derive that

$$\begin{aligned} x_3(t) = x_3(0) &- \frac{x_1(0)y_2(0)}{2k}[e^{-2kt} - 1] \\ &+ \frac{y_1(0)x_2(0)}{2k}[e^{-2kt} - 1] \end{aligned} \tag{27}$$

From this constant term, it is found that

$$S(\boldsymbol{x}) = x_3(t) + \frac{1}{2k}x_1(t)y_2(t) - \frac{1}{2k}x_2(t)y_1(t) \tag{28}$$

can be derived as one candidate for the invariant manifold.

Actually, applyig the following feedback law

$$\begin{aligned} u_1 &= -k^2 x_1 - 2ky_1 \\ u_2 &= -k^2 x_2 - 2ky_2, \quad k > 0 \end{aligned} \tag{29}$$

to the original system (20) and then taking its time derivative to examine the behavior of $S(\boldsymbol{x})$, it follows that

$$\begin{aligned} \dot{S}(\boldsymbol{x}) &= \dot{x}_3(t) + \frac{1}{2k}[\dot{x}_1(t)y_2(t) + x_1(t\dot{y}_2(t)] \\ &\quad - \frac{1}{2k}[\dot{x}_2(t)y_1(t) + x_2(t)\dot{y}_1(t)] \\ &= x_1y_2 - x_2y_1 + \frac{1}{2k}[-k^2x_1x_2 - 2kx_1y_2 \\ &\quad + k^2x_1x_2 + 2kx_2y_1] \equiv 0 \end{aligned} \tag{30}$$

and moreover it holds under the above feedback control that

$$S(\boldsymbol{x}) = \text{Const.} \tag{31}$$

subsequently, $S(\boldsymbol{x})$ becomes one invariant manifold.

Thus, for a switching method in two-step control, if it holds that $S(\boldsymbol{x}) = 0$ at any time $t = T$, then it keeps that $S(\boldsymbol{x}) = 0$ for $t \geq T$.

On the other hand, when the above feedback law is adopted at $t \geq T$, it is easily found that $x_1(t), x_2(t), y_1(t)$ and $y_2(t)$ are all asymptotically stable, i.e., $x_1 \to 0, x_2 \to 0, y_1 \to 0, y_2 \to 0$ as $t \to \infty$. Then, it can be also seen that $x_3 \to 0$ because $S(\boldsymbol{x}) = 0$ has been already satisfied.

VI. Attractive Control to Invariant Manifold $S(\boldsymbol{x})$ in Extended System

One type of attractive controllers is described here to the manifold derived in section V, depending on the initial states $x_1(0)$ and $y_1(0)$.

An attractive control to $S(\boldsymbol{x}) = 0$ is performed in the first-step for $t < T$, and at the stage of $|S(\boldsymbol{x})| \leq \epsilon$ in practice the following second-step control

$$\begin{aligned} u_1 &= -k^2 x_1 - 2ky_1 \\ u_2 &= -k^2 x_2 - 2ky_2 \end{aligned} \tag{32}$$

is applied for $T \leq t$.

In order to obtain the first step control law, letting the Lyapunov function be selected as

$$V(\boldsymbol{x}) = \frac{1}{2}S^2(\boldsymbol{x}) \tag{33}$$

and setting the control input as

$$\begin{aligned} u_1 &= -2ky_1 \\ u_2 &= -fS(\boldsymbol{x})/x_1(t) - 2ky_2, \quad f > 0 \end{aligned} \tag{34}$$

it follows that

$$\begin{aligned} \dot{V}(\boldsymbol{x}) = S\dot{S} &= S[x_1y_2 - x_2y_1 + \frac{1}{2k}(x_1u_2 - x_2u_1)] \\ &= S[x_1y_2 - x_2y_1 + \frac{1}{2k}(-fS) - x_1y_2 + x_2y_1] \\ &= -\frac{f}{2k}S^2 \leq 0 \end{aligned} \tag{35}$$

In addition, when using Eq. (34), $\dot{S}$ is reduced to

$$\dot{S}(t) = -\frac{f}{2k}S(t) \tag{36}$$

Note however that u_2 in Eq. (34) may diverge as $x_1(t) \to 0$. Using the above equation and the representation of the solution to $x_1(t)$ in Eq. (20) with the input of Eq. (34), if $x_1(0) \neq 0$ and $x_1(0) + y_1(0)/2k \neq 0$, then it can be said that

$$\begin{aligned}\lim_{t\to\infty} \frac{S(\boldsymbol{x})}{x_1(t)} &= \lim_{t\to\infty} \frac{S(\boldsymbol{x}(0))e^{-\frac{f}{2k}t}}{x_1(0) - \frac{1}{2k}y_1(0)(e^{-2kt}-1)} \\ &= \frac{0}{x_1(0) + \frac{y_1(0)}{2k}} = 0 \end{aligned} \tag{37}$$

Otherwise, if $x_1(0) \neq 0$ and $x_1(0) + y_1(0)/2k = 0$, then it results in

$$\begin{aligned}\left|\frac{S(\boldsymbol{x})}{x_1(t)}\right| &= \left|\frac{S(\boldsymbol{x}(0))e^{-\frac{f}{2k}t}}{-\frac{1}{2k}y_1(0)e^{-2kt}}\right| \\ &\leq \left|\frac{2kS(\boldsymbol{x}(0))}{y_1(0)}\right| e^{-(\frac{f}{2k}-2k)t} \end{aligned} \tag{38}$$

so that $S(\boldsymbol{x})/x_1$ decreases exponentially with the exponential rate $(f/2k - 2k)$ if $f/2k > 2k$, consequently it can be prevented from the divergence of u_2, because $2kS(\boldsymbol{x}(0))/y_1(0)$ is finite even if $x_1(0) \neq 0$, $x_2(0) = 0$, $x_3(0) = 0$, and $y_2(0) = 0$, as long as $y_1(0) \neq 0$ (in fact, $y_1(0) = -2kx_1(0) \neq 0$ because of the assumption that $x_1(0) \neq 0$ and $x_1(0) + y_1(0)/2k = 0$). In general, it is known [10], [12] that, in a differential equation $\dot{x}(t) = -ax(t) + g(x,t)$, when $|g(x,t)|$ decreases with an exponential rate greater than $a > 0$, $x(t)$ decreases with the exponential rate a. From this fact, it is seen that, if $f/2k - 2k \geq 2k$, i.e., $f \geq 8k^2$, and $2kS(\boldsymbol{x}(0))/y_1(0)$ is finite then y_2 converges exponentially to zero with the exponential rate $2k$.

VII. Quasi-continuous Exponential Stabilizing Control for Extended System

Now, consider a quasi-continuous exponential stabilizing control with the control input consisting of Eqs. (32) and (34) such as

$$\begin{aligned} u_1 &= -k^2x_1 - 2ky_1 \\ u_2 &= -fS(\boldsymbol{x})/x_1 - k^2x_2 - 2ky_2 \end{aligned} \tag{39}$$

Then, it is found that

$$\begin{aligned}\dot{S} &= x_1y_2 - x_2y_1 + \frac{1}{2k}(x_1u_2 - x_2u_1) \\ &= x_1y_2 - x_2y_1 + \frac{1}{2k}(-fS - k^2x_1x_2 - 2kx_1y_2 \\ &\quad + k^2x_1x_2 + 2kx_2y_1) \\ &= -\frac{f}{2k}S \end{aligned} \tag{40}$$

and the corresponding time response is given by

$$S(t) = S(\boldsymbol{x}(0))e^{-\frac{f}{2k}t} \tag{41}$$

so that $S(t) \to 0$ as $t \to \infty$.

In addition, to keep the input of u_2 finite, it must be that

$$\left|\frac{S}{x_1}\right| < \infty \tag{42}$$

In order to examine this condition, it is found from Eq. (26) that for the case of using Eq. (39) the response of x_1 can be reduced to

$$\begin{aligned} x_1(t) &= x_1(0)[e^{-kt} + kte^{-kt}] + y_1(0)te^{-kt} \\ &= e^{-kt}[x_1(0)(1+kt) + y_1(0)t] \\ &= \frac{x_1(0)(1+kt) + y_1(0)t}{e^{kt}} \end{aligned} \tag{43}$$

Since the limiting value of $x_1(t)$ is just an indefinite form such as $\lim_{t\to\infty} x_1(t) = \infty/\infty$, using the L'hospital's theorem gives

$$\begin{aligned}\lim_{t\to\infty} x_1(t) &= \lim_{t\to\infty} \frac{[x_1(0)(1+kt) + y_1(0)t]'}{(e^{kt})'} \\ &= \lim_{t\to\infty} \frac{x_1(0)k + y_1(0)}{ke^{kt}} = 0 \end{aligned} \tag{44}$$

Similarly, it can be proved that $y_1(t)$ is asymptotically stable. Therefore, if $f/2k - k > 0$, then it follows that

$$\begin{aligned}\lim_{t\to\infty} \frac{S(t)}{x_1(t)} &= \lim_{t\to\infty} \frac{S(\boldsymbol{x}(0))e^{-\frac{f}{2k}t}}{e^{-kt}[x_1(0)(1+kt) + y_1(0)t]} \\ &= \lim_{t\to\infty} \frac{kS(\boldsymbol{x}(0))e^{-(\frac{f}{2k}-k)t}}{x_1(0)k + y_1(0)} = 0 \end{aligned} \tag{45}$$

and furthermore since it is found from this relation that

$$\left|\frac{S(t)}{x_1(t)}\right| \leq \left|\frac{kS(\boldsymbol{x}(0))}{x_1(0)k + y_1(0)}\right| e^{-(\frac{f}{2k}-k)t} \tag{46}$$

$S/x_1(t)$ decreases with the exponential rate $(\frac{f}{2k} - k)$.

On the other hand, since when using u_2 in Eq. (39), the differential equations for x_2 and y_2 in Eq. (20) are given by

$$\begin{aligned} \dot{x}_2 &= y_2 \\ \dot{y}_2 &= -k^2x_2 - 2ky_2 - fS(\boldsymbol{x})/x_1 \end{aligned} \tag{47}$$

it is seen that

$$\frac{d}{dt}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k^2 & -2k \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{fS(\boldsymbol{x})}{x_1} \end{bmatrix} \tag{48}$$

Assuming that

$$A_s = \begin{bmatrix} 0 & 1 \\ -k^2 & -2k \end{bmatrix}, \quad d_1(t) = -\frac{fS(\boldsymbol{x})}{x_1} \tag{49}$$

it is obtained that

$$e^{A_st} = \begin{bmatrix} e^{-kt} + kte^{-kt} & te^{-kt} \\ -k^2te^{-kt} & e^{-kt} - kte^{-kt} \end{bmatrix} \tag{50}$$

so that the following solutions are obtained:

$$\begin{aligned} x_2(t) &= x_2(0)[e^{-kt} + kte^{-kt}] + y_2(0)te^{-kt} \\ &\quad + \int_0^t (t-\tau)e^{-k(t-\tau)}d_1(\tau)d\tau \\ y_2(t) &= x_2(0)[-k^2te^{-kt}] + y_2(0)[e^{-kt} - kte^{-kt}] \\ &\quad + \int_0^t [e^{-k(t-\tau)} - k(t-\tau)e^{-k(t-\tau)}]d_1(\tau)d\tau \end{aligned} \tag{51}$$

Since $S(\boldsymbol{x})/x_1$ has been shown to decrease with the exponential rate $(f/2k - k)$ and the free response term of Eq. (51) can be shown to converge to zero with the exponential rate k as the same as the previous $x_1(t)$ and $y_1(t)$, it is seen easily that $x_2(t)$ and $y_2(t)$ in Eq. (51) converge to zero with the exponential rate k, if $f/2k - k \geq k$, i.e., $f \geq 4k^2$.

It is concluded from these results that if $S(\boldsymbol{x})$, $x_1(t)$, $x_2(t)$, $y_1(t)$, and $y_2(t)$ converge to zero as $t \to \infty$, then $x_3(t)$, from Eq. (28), also converges to zero.

VIII. Simulation Examples

A. The case of kinematic model

Consider the kinematic model for the following mobile robot with two-independent driving wheels:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \tag{52}$$

where (x, y) denotes the geometrical center, θ is the azimuth, v is the translational velocity of the robot, and ω denotes the azimuth velocity. Then, taking a transformation from (x, y, θ) to the intermediate variables (z_1, z_2, z_3)

$$\begin{aligned} z_1 &= \theta \\ z_2 &= x\cos\theta + y\sin\theta \\ z_3 &= x\sin\theta - y\cos\theta \end{aligned} \tag{53}$$

and moreover taking a transformation from (z_1, z_2, z_3) to (x_1, x_2, x_3) [13]

$$\begin{aligned} x_1 &= z_1 \\ x_2 &= z_2 \\ x_3 &= -2z_3 + z_1 z_2 \end{aligned} \tag{54}$$

the double integrator model in Eq. (1) is obtained, where

$$u_1 = \omega, \quad u_2 = z_3\omega + v \tag{55}$$

Therefore, for the case of using the kinematic model, the states x_1, x_2 and x_3 are generated by using the transformation equations of Eq. (53) and Eq. (54) with the actual measurements x, y and θ, any quasi-continuous exponential stabilizing control law that was presented in section IV and based on the kinematic model is calculated, and finally the actual inputs v and ω are generated through the inverse transformation of Eq. (55).

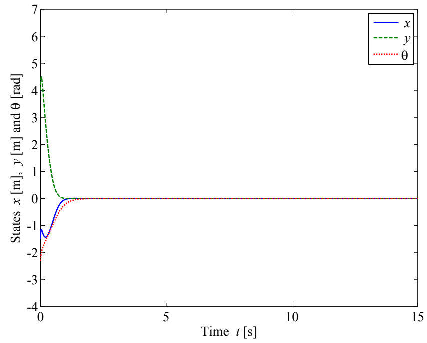
Fig. 1 shows the time histories of the states and inputs for the case of implementing a quasi-continuous exponential stabilizing control law with W, under the conditions that $x(0) = -1.5$ [m], $y(0) = 4$ [m], and $\theta(0) = -2.3$ [rad], where it was assumed that the sampling width was $\Delta t = 0.01$ [s] and the control gains were set to $k = 4$ and $f = 10$.

It is found from these figures that all the states in the kinematics converge to the origin very quickly.

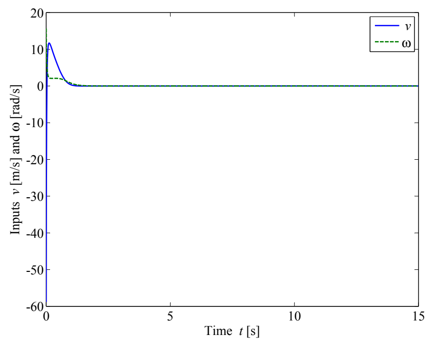
B. The case of dynamical model

Combining the kinematic model given in Eq. (52) and the so-called steering model for such a mobile robot with two-independent driving wheels, the following dynamical model is obtained by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{v} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} v\cos\theta \\ v\sin\theta \\ \omega \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M} \\ 0 \end{bmatrix} F + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{I} \end{bmatrix} \tau \tag{56}$$



(a) States



(b) Inputs

Fig. 1. Time histories of the states and control inputs for the quasi-continuous control law (15)

where M denotes the mass of the robot, I is the moment of inertia for the robot, F is the force, and τ is the torque, where using the left and right torques (τ_1, τ_2) it is obtained that

$$F = \frac{1}{R}(\tau_1 + \tau_2), \quad \tau = \frac{L}{R}(\tau_1 - \tau_2) \tag{57}$$

in which R denotes the radius of the wheel and $2L$ is the tread. Note here that (v, ω) denote the translational velocity of the steering axis and its rotational speed.

Then, taking a transformation from $(x, y, \theta, v, \omega)$ to $(z_1, z_2, z_3, z_4, z_5)$

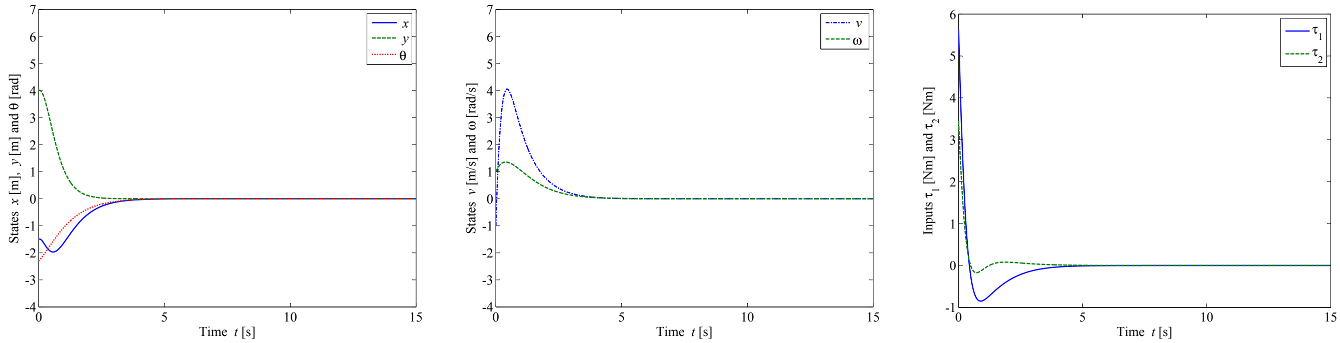
$$\begin{aligned} z_1 &= \theta \\ z_2 &= x\cos\theta + y\sin\theta \\ z_3 &= x\sin\theta - y\cos\theta \\ z_4 &= \omega \\ z_5 &= v - (x\sin\theta - y\cos\theta)\omega \end{aligned} \tag{58}$$

and further taking a transformation from $(z_1, z_2, z_3, z_4, z_5)$ to $(x_1, x_2, x_3, y_1, y_2)$ [13]

$$\begin{aligned} x_1 &= z_1 \\ x_2 &= z_2 \\ x_3 &= -2z_3 + z_1 z_2 \\ y_1 &= z_4 \\ y_2 &= z_5 \end{aligned} \tag{59}$$

the extended double integrator model in Eq. (20) is obtained, where u_1 and u_2 are given by

$$\begin{aligned} u_1 &= \frac{\tau}{I} \\ u_2 &= -z_4^2 z_2 - \frac{\tau}{I} z_3 + \frac{F}{M} \end{aligned} \tag{60}$$



(a) States 1, 2 and 3 (b) States 4 and 5 (c) Inputs

Fig. 2. Time histories of the states and inputs for the quasi-continuous control law (39)

Therefore, for this control of using a dynamical model, the states x_1, x_2, x_3, y_1, and y_2 are generated by using the transformation equations of Eq. (58) and Eq. (59) with the actual measurements x, y, θ, v, and ω, the quasi-continuous exponential stabilizing control law that was presented in section VII and based on the dynamical model is calculated, and the inputs F and τ are solved through an inverse transformation of Eq. (60), and finally the left and right torque inputs τ_1 and τ_2 are generated for the actual robot through the inverse transformation of Eq. (57).

Fig. 2 shows the time histories of the states and inputs for the case of implementing a quasi-continuous exponential stabilizing control law, under the conditions that $x(0) = -1.5$ [m], $y(0) = 4$ [m], $\theta(0) = -2.3$ [rad], $v(0) = -1$ [m], and $\omega(0) = 1$ [rad/s] and using the physical parameters such as $M = 10$ [kg], $I = 2$ [kgm^2], $R = 0.03$ [m], and $L = 0.06$ [m], where it was assumed that the sampling width was $\Delta t = 0.01$ [s] and the control gains were set to $k = 1.5$ and $f = 9$.

Observe from this experiment that for the case of using this dynamical controller a satisfied convergence is obtained very quickly, as well as the case of using the kinematic controller. Especially it is worthy to note that the controlled system due to the present approach is very simpler compared to the conventional dynamical approach, e.g., due to Fierro and Lewis [14], [15]. Because in their approach the controlled system is very complicated due to the combination of three parts: any kinematic controller, a backstepping controller, and a partial linearization of the degenerate state-space model.

IX. Conclusions

In this paper, a new underactuated control method has been proposed for nonholonomic mobile robots by applying a double integrator model and the invariant manifold theory. Especially, the quasi-continuous exponential stabilizing controller was able to be applied to both a kinematics-based model and a dynamics-based model, where the latter model should be interpreted as an extended double integrator model, whereas the former one is a conventional double integrator model in nonholonomic systems. Although a stabilizing control problem was only considered at present, there remain other path following control problem and trajectory tracking control problem as future work.

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