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# The unconstrained and inequality constrained moving horizon approach to robot localization

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*Abstract*— **We present a moving horizon approach for estimating the state of a nonlinear dynamic system that may be subject to inequality constraints. The method takes advantage of a recent smoothing algorithm proposed in the literature based on interior point techniques. The approach exploits the same decomposition used for unconstrained Kalman-Bucy smoothers. Hence, the number of operations required by the algorithm scales linearly with the length of the horizon, making it suitable for online applications. We apply this method to the robot localization problem, showing that it is able to produce much more accurate results than the iterated Kalman filter with little additional computational effort.**

## I. Introduction

Kalman filters (KF) are widely used to estimate states of dynamic stochastic systems in many different fields such as biomedicine, economy and robotics (see [13], [22] for detailed expositions of their properties). When the system under study is nonlinear, the simplest implementation is the extended or iterated Kalman filter (IKF) [9]. However, the estimates obtained by IKF may be quite different from the minimum variance estimate. In addition, this filter may be quite sensitive to unknown initial conditions and local minima. Particle filters (PF) are an important alternative where optimization is replaced by propagation of a posterior density in sampled form and Monte Carlo integration [21], [24], [25], [11]. One problem with these techniques is the need for delicate tuning of proposal densities to improve convergence rates. In addition, robust statistical convergence criteria are still missing. We also notice that in many applications additional knowledge on the system state may be available, e.g. in the form of inequality constraints. Including this information may be important to in order to improve the estimation process. However, while linear and nonlinear equality constraints on the state vector may be handled by augmenting the measurement model (see e.g. [7], [10], [20], [26]), imposing affine or nonlinear inequality constraints is more difficult. This can further complicate the implementation of IKF and PF [4], [17], [20]. To deal with the above issues, in this paper we propose a new moving horizon approach (see e.g. [6], [16], [17]), i.e. a filter that is able to processes a moving window of past measurements, and to efficiently handle inequality constraints on the state. In particular, we use a recently proposed optimization algorithm that relies upon interior point methods [3]. The key feature of this approach is that it takes advantage of the same decomposition used for unconstrained Kalman smoothers [2], [18]. In this way, the required operations scale linearly with the horizon length, making it it feasible to simultaneously optimize with respect to the state value at all time indices in the moving window. This leads to a filter which is much less sensitive to unknown initial conditions and local minima. In order to demonstrate the effectiveness of our technique, the estimator is applied to a robot localization problem. Due to its practical importance, this is one of the most widely studied topics in mobile robotics. Our focus is on mobile robots moving in planar environments where the goal is to estimate the position in the plane and the *yaw* orientation of the robot. More precisely, in the experimental section presented in this paper the agent moves in an environment conditioned with landmarks located at known locations, and is equipped with a sensor identifying the landmarks and estimating their distance. The goal is to estimate the triple $(x, y, \theta)$, indicated as *pose* in the following. In this setting the inequality constraints can represent the known region where the robot is moving. The reader is referred to the book by Thrun et al. [23] for a thorough description of the most commonly used probabilistic techniques used to solve this problem. Most of the published literature in the field resorts to IKF for position tracking (where the initial pose of the robot is known) [12] and PF for global localization problems [11], [14], [21], [24], [25]. In this latter problem, the initial pose is unknown and it is crucial to track non Gaussian distributions. Remarkably, the moving horizon approach proposed in this paper will prove to be robust also for solving this kind of estimation task, with little additional computational effort in comparison with IKF. The paper is organized as follows. Section II formally defines the problem and presents the theory supporting the presented moving horizon approach. Algorithmic details are presented in section III. The experimental setup and results are presented in Section IV. Finally, conclusions are offered in Section V.

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## II. Statement of the problem

We are given the noisy output measurements $z_k \in \mathbf{R}^m$ coming from the following dynamic system

$$x_k = g_k(x_{k-1}) + w_k \quad , \quad z_k = h_k(x_k) + v_k \tag{1}$$

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where $x_k \in \mathbf{R}^n$. The noise vectors $w_k \in \mathbf{R}^n$ and $v_k \in \mathbf{R}^m$ are all mutually independent and

$$w_k \sim \mathbf{N}(0, Q_k) \quad , \quad v_k \sim \mathbf{N}(0, R_k) \tag{2}$$

where $Q_k \in \mathbf{R}^{n \times n}$ and $R_k \in \mathbf{R}^{m \times m}$ are the known autocovariance matrices.We also use $f_k : \mathbf{R}^n \rightarrow \mathbf{R}^\ell$ to denote known functions that model inequality constraints on the state:

$$f_k(x_k) \leq 0 \quad \text{for} \quad k = 1, \ldots, N \ . \tag{3}$$

Let

$$\begin{aligned} S_k(x_k, x_{k-1}) &= \tfrac{1}{2}[z_k - h_k(x_k)]^{\mathrm{T}} R_k^{-1} [z_k - h_k(x_k)] \\ &+ \tfrac{1}{2}[x_k - g_k(x_{k-1})]^{\mathrm{T}} Q_k^{-1} [x_k - g_k(x_{k-1})] \end{aligned}$$

denote the residual sum of squares at time index $k$.

In the spirit of moving horizon methods, let $K$ be the moving index and $M$ the length of the horizon. In addition, define

$$\begin{aligned} & x^K = \{x_q, \ldots, x_K\} \qquad z^K = \{z_q, \ldots, z_K\} \\ & q = \max\{1, K - M + 1\} \ . \end{aligned}$$

Letting

$$S^K(x^K) = \sum_{k=q}^{K} S_k(x_k, x_{k-1})$$

the corresponding moving horizon problem is

$$\begin{array}{ll} \text{minimize} & S^K(x^K, y^K) \qquad \text{w.r.t.} \quad x^K \\ & \text{s.t.} \quad f_k(x_k) \leq 0, \qquad k = q, q+1, \ldots, K \ . \end{array} \tag{4}$$

In the above equations, $x_{q-1}$ is treated as a known parameter and the initial conditions are given by the following special inizializitations

$$g_q(x_{q-1}) = x_{q-1}, \qquad w_q \sim \mathbf{N}(0, V_q)$$

that model the state vector $x_q$ as Gaussian with mean $x_{q-1}$ and autocovariance $V_q$. Notice that no matter how large $N$ is, problem (4) is bounded in size by $M$.

The following result is easily obtained.

*Theorem 1:* Assume that $x_q \sim \mathbf{N}(x_{q-1}, V_q)$ where $x_{q-1}$ and $V_q$ are known quantities. Then, the maximum a posteriori estimate of the process $x^K$, conditional on $z^K$ and the constraints (3) acting at instants $k = q, q+1, \ldots, K$, is the solution of the problem 4.

In the scenario of robot localization, the constraints (3) may e.g. represent the region where the robot is moving. Notice that another interpretation of our model is that the event $f(x_k) \leq 0$ accounts for the physical constraints of the environment by making the distribution of $x_k$ conditional on $x_{k-1}$, a truncated Gaussian.

## III. The Numerical Algorithm

### A. The Quadratic Programming Sub-problem

The algorithm presented in [3] is now briefly recalled and easily adapted to our framework. In particular, we will show that for any moving horizon index $K$, obtaining the estimate of $x^K$ requires only $O(Mn^3)$ operations. In particular, each moving horizon problem is solved by introducing affine approximations that are first-order accurate for a state sequence $y^K = \{y_q, \ldots, y_K\}$ near a fixed state sequence $x^K$. Define the affine approximations $\tilde{f}_k$, $\tilde{g}_k$, and $\tilde{h}_k$ by

$$\begin{aligned} \tilde{f}_k(x_k; y_k) &= f_k(x_k) + f_k^{(1)}(x_k)(y_k - x_k) \\ \tilde{g}_k(x_k; y_k) &= g_k(x_k) + g_k^{(1)}(x_k)(y_k - x_k) \\ \tilde{h}_k(x_k; y_k) &= h_k(x_k) + h_k^{(1)}(x_k)(y_k - x_k) \ . \end{aligned}$$

Then, the residual sum of squares function associated with the $K$-th moving horizon problem and with the above affine approximations is denoted by

$$\tilde{S}^K(x^K; y^K) = \sum_{k=q}^{K} \tilde{S}_k(x_k, x_{k-1}; y_k, y_{k-1}) \ , \tag{5}$$

where

$$\begin{aligned} & \tilde{S}_k(x_k, x_{k-1}; y_k, y_{k-1}) = \\ & \quad (1/2)[y_k - \tilde{g}_k(x_{k-1}; y_{k-1})]^{\mathrm{T}} Q_k^{-1} [y_k - \tilde{g}_k(x_{k-1}; y_{k-1})] \\ & \quad +(1/2)[z_k - \tilde{h}_k(x_k; y_k)]^{\mathrm{T}} R_k^{-1} [z_k - \tilde{h}_k(x_k; y_k)] \ . \end{aligned}$$

Following [3], the nonlinear problem (4) is solved by solving quadratic programming (QP) subproblems given by:

$$\begin{array}{lll} \text{minimize} & \tilde{S}^K(x^K; y^K) & \text{w.r.t.} \quad y^K \\ \text{subject to} & \tilde{f}_k(x_k; y_k) \leq 0, & k = q, q+1, \ldots, K \ . \end{array} \tag{6}$$

Define $A_k \in \mathbf{R}^{n \times n}$ and $C_k \in \mathbf{R}^{n \times n}$ by

$$\begin{aligned} & A_k = -M_k^{-1} g_k^{(1)}(x_{k-1}) \\ & C_k = \left\{ \begin{array}{l} M_k^{-1} + h_k^{(1)}(x_k)^{\mathrm{T}} R_k^{-1} h_k^{(1)}(x_k) \\ \quad + g_{k+1}^{(1)}(x_k)^{\mathrm{T}} M_{k+1}^{-1} g_{k+1}^{(1)}(x_k) \end{array} \right\} \\ & M_k = \left\{ \begin{array}{ll} V_k & \text{if} \quad k = q \\ Q_k & \text{otherwise} \ . \end{array} \right. \end{aligned} \tag{7}$$

Then, the matrix $C^K \in \mathbf{R}^{nM \times nM}$ given by

$$C^K = \begin{pmatrix} C_q & A_{q+1}^{\mathrm{T}} & 0 & \\ A_{q+1} & C_{q+1} & A_{q+2}^{\mathrm{T}} & 0 \\ 0 & \ddots & \ddots & \ddots \\ & 0 & A_K & C_K \end{pmatrix} \tag{8}$$

is precisely the Hessian of the objective $\tilde{S}^K$ in (6) with respect to $y^K$, see [3] for details. Now, define the vector $a_k \in \mathbf{R}^n$ by

$$\begin{aligned} a_k \quad = \quad & M_k^{-1}[x_k - g_k(x_{k-1})] - h_k^{(1)}(x_k)^{\mathrm{T}} R_k^{-1} [z_k - h_k(x_k)] \\ & - g_{k+1}^{(1)}(x_k)^{\mathrm{T}} M_{k+1}^{-1} [x_{k+1} - g_{k+1}(x_k)] \end{aligned}$$

Let $a^K \in \mathbf{R}^{nM}$ be the column vector representing $\{a_q, \ldots, a_K\}$. Then, notice that $a^K$ represents the gradient of $\tilde{S}^K(x^K, y^K)$ with respect to $y^K$ at $y_k = x_k$, $k = q, \ldots, K$. Now, let

$$B^K = \begin{pmatrix} f_q^{(1)}(x_q) & 0 & \\ 0 & \ddots & 0 \\ & 0 & f_K^{(1)}(x_K) \end{pmatrix} \tag{9}$$

and notice that the affine approximation to the constraints (3) for the $K$-th moving horizon problem is given by $b^K + B^K y^K \leq 0$. Thus, the QP subproblem (6) becomes

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}(y^K)^{\mathrm{T}} C^K y^K + (d^K)^{\mathrm{T}} y^K \quad \text{w.r.t. } y^K \in \mathbf{R}^{nM} \\ \text{subject to} & b^K + B^K y^K \leq 0 \end{array} \tag{10}$$

where $d^K = a^K - C^K x^K$. The QP subproblem (10) can now be solved using the interior point approach presented in [3] with $O(Mn^3)$ operations. Here, we just recall that the interior point approaches apply a damped Newton's method to a relaxation of the Karush-Kuhn-Tucker (KKT) conditions. In our case, the relaxed subproblem (that contains a log barrier) is given by:

$$\begin{array}{ll} \text{minimize} & (1/2)(y^K)^{\mathrm{T}} C^K y^K + (d^K)^{\mathrm{T}} y^K \\ & \qquad -\mu \sum_{i=1}^{\ell M} \log(s_i) \\ \text{w.r.t} & (y^K, s) \in \mathbf{R}^{nM} \times \mathbf{R}_+^{\ell M} \\ \text{s.t.} & s + b^K + B^K y^K = 0 \,, \end{array} \tag{11}$$

where $\mu$ is the relaxation parameter and $s$ is the vector containing the slack variables.

### *B. The Nonlinear Algorithm for solving the moving horizon problems*

The termination criteria for the $K$-th nonlinear moving horizon problem rely upon the KKT conditions for problem (4). Given the current value for $K$, we use $p$ to denote the iteration counter of our optimization scheme. Let $x^K(p) \in \mathbf{R}^{nM}$ where $x^K(p)$ contains the blocks $x_k^K(p) \in \mathbf{R}^n$, $k = q, \ldots, K$ while $u^K(p) \in \mathbf{R}_+^{\ell M}$ is the Lagrange multiplier vector containing the blocks $u_k^K(p) \in \mathbf{R}^{\ell}$, $k = q, \ldots, K$. Then, for each $K$, the algorithm terminates at a primal vector $x^K(p)$ and Lagrange multiplier vector $u^K(p) \in \mathbf{R}_+^{\ell M}$ such that for $k = q, \ldots, K$

$$\begin{array}{l} f_k(x_k^K(p)) \leq \varepsilon \,, \ \|u_k^K(p) \cdot f_k(x_k^K(p))\|_\infty \leq \varepsilon \text{ and} \\ \|(u_k^K(p))^{\mathrm{T}} f_k^{(1)}(x_k^K(p)) + \partial_{x_k^K(p)} S^K(x^K(p))\|_\infty \leq \varepsilon \end{array} \tag{12}$$

where $\varepsilon$ is a termination tolerance. Given a vector $w \in \mathbf{R}^m$, $\max(0, w) \in \mathbf{R}^m$ denotes the vector with $i$-th component equal to $\max(0, w_i)$. Given a $x^K \in \mathbf{R}^{nM}$, the $\boldsymbol{\ell}_1$ distance from the constraint function values $\{f_k(x_k^K)\}_{k=q}^K$ to the constraint set is

$$\phi(x^K) \;=\; \sum_{k=q}^{K} \sum_{i=1}^{\ell} \max([f_k(x_k^K)]_i, 0)$$

with its approximation given by

$$\tilde{\phi}(x^K; y^K) \;=\; \sum_{k=q}^{K} \sum_{i=1}^{\ell} \max([\tilde{f}_k(x_k^K; y_k^K)]_i, 0) \,.$$

We are now in a position to describe the algorithm that solves in an online-manner the sequence of nonlinear moving horizons problems (4). Needless to say, the unconstrained version becomes just a special case of the scheme described below.

*Algorithm 2:* Moving Horizon Version of the Inequality Constrained Nonlinear Smoother

1) Initialization: Set $K = 0$
2) Set $K = K + 1$, $q = \max\{1, K - M + 1\}$, $x_{q-1}$ to its estimate and $V_q$ to a numerical approximation of its posterior autocovariance[1]. In addition, set the iteration counter $p = 0$ and the initial penalty parameter $\alpha_0 = 0$.
3) Affine approximation: Substitute $x^K(p)$ for $x^K$ in equations (8), (9) and let $a(p)$, $b(p)$, $B(p)$, $C(p)$, and $d(p)$ be the corresponding values for $a^K$, $b^K$, $B^K$, $C^K$, and $d^K$ in QP (10).
4) Solve this QP using Algorithm 4 described in [3] with inputs $\delta = \varepsilon \times 10^{-2}$ and let $y(p)$ and $u(p)$ be the resulting solution.
5) If the convergence criteria (12) are satisfied, return $x(p)$, $u(p)$ as the solution for the $K$-th moving horizon problem and go to Step 2.
6) If $\alpha_p > 0$, set $\hat{\alpha}_p = \alpha_p$; otherwise, $\hat{\alpha}_p = \|u(p)\|_\infty$. Define the value

$$\begin{array}{rcl} \zeta_p &=& (y(p) - x(p))^{\mathrm{T}} C(p)(y(p) - x(p)) \\ &+& (a(p))^{\mathrm{T}}(y(p) - x(p)) \end{array}$$

If $\zeta_p \leq \hat{\alpha}_p \phi(x(p))$, set $\alpha_{p+1} = \hat{\alpha}_p$; otherwise, $\alpha_{p+1} = \max[\zeta_p / \phi(x(p)) \;,\; 2\hat{\alpha}_p]$.
7) Compute the line search step size $\lambda_p$ as follows:

$$\begin{array}{l} \eta_p = (a(p))^{\mathrm{T}}(y(p) - x(p)) \\ \qquad +\alpha_{p+1}[\tilde{\phi}(x(p); y(p)) - \phi(x(p))] \\ H_p(\lambda) = S^K[x(p) + \lambda(y(p) - x(p))] \\ \qquad +\alpha_{p+1}\phi[x(p) + \lambda(y(p) - x(p))] \\ \lambda_p = \max\{\; 2^{-r} \mid r \in \mathbf{Z}_+ \text{ and} \\ \qquad H_p(2^{-r}) - H_p(0) \leq 2^{-r}\eta_p/10 \;\} \,. \end{array}$$

8) Set $x^{p+1} = x(p) + \lambda_p(y(p) - x(p))$, then set $p = p+1$ and go to step 3.

The following convergence results is taken from [3], see also [5].

*Theorem 3:* Suppose $\varepsilon = 0$, for each $K$ all the quadratic subproblems in step 3 have feasible solutions, the corresponding sequence $\{y(p)\}$ is bounded, and every cluster point of $\{x(p)\}$ satisfies the Mangasarian-Fromowitz Constraint Qualification. Then the sequence $\{x(p)\}$ is bounded and each of its cluster points is a KKT point for problem (4), i.e. satisfies convergence criteria (12) for some vector of Lagrange multipliers.

## IV. Experimental results

In this section we offer an extensive set of experimental results aimed to outline the properties of the proposed algorithm (MH – moving horizon), and we also contrast it with the iterative Kalman filter. The chosen benchmark problem is a classical localization problem in a given map with landmarks located at known positions. All the code and data needed to replicate the results presented in this section are available for download on `http://robotics.ucmerced.edu`.

[1] In the numerical implementation, this autocovariance has been obtained by the last affine approximation providing the solution of the nonlinear problem (4) without constraints.

### A. System model

We consider a differential drive robot moving on a flat terrain populated with $m$ landmarks placed at known locations. Let the location of the $i$-th landmark be $(p_x^i, p_y^i)$. As usual, the pose of the robot is indicated as $(x, y, \vartheta) \in \mathbf{R}^3$, and we hypothesize two inputs control the system, namely the translational speed $u_v$ and the rotational speed $u_w$. Consistently with the formerly depicted framework, we model the system with discrete time equations describing how the state evolves over time. When $u_w$ is different from 0, the following relationships[2] hold (see [23], chapter 5):

$$
\begin{aligned}
x_k &= x_{k-1} - \frac{u_v}{u_w}\sin\theta_{k-1} + \frac{u_v}{u_w}\sin(\theta_{k-1} + u_w) \\
y_k &= y_{k-1} + \frac{u_v}{u_w}\cos\theta_{k-1} - \frac{u_v}{u_w}\cos(\theta_{k-1} + u_w) \\
\theta_k &= \theta_{k-1} + u_w
\end{aligned}
$$

When $u_w$ is equal to 0 the robot simply moves forward, so these relationships simplify in a straightforward way with $\theta_k$ remaining equal to $\theta_{k-1}$, and $x_k, y_k$ changing according to the heading. Following the hypotheses presented while introducing the problem, we assume that state evolution is affected by Gaussian noise $w_k \sim \mathbf{N}(0, Q_k)$. $Q_k$ is a known $3 \times 3$ diagonal covariance matrix whose values on the diagonal are not all necessarily equal. It is furthermore assumed that $Q_k$ is the same for each $k$, though this is not necessary. The robot is supposed to be equipped with a single sensor returning the distance from the landmarks, provided that they are closer than a known constant threshold $T$. That is, at step $k$ the sensor returns a vector $z_k \in \mathbf{R}^m$ where the $i$-th entry is either

$$d_k^i = \sqrt{(x_k - p_x^i)^2 + (y_k - p_y^i)^2}$$

if $d_k^i < T$, or 0 otherwise. In all examples presented in this section, $T = 7m$. Entries larger than 0, i.e. entries corresponding to actual readings, are corrupted by Gaussian noise $w_k \sim \mathbf{N}(0, R_k)$. $R_k$ is assumed to be an $m \times m$ diagonal matrix with identical values on the main diagonal, although it is possible to consider situations where the values are not the same. This extension will not be considered in this paper, but the price to pay is just a slightly more articulated implementation. The reader should note that the sensor does not return any information about the heading of the robot, whereas the sensor values are exclusively dependent from the $x, y$ components. This aspect will be important to consider while evaluating the impact of the proposed technique, and also when comparing it with the IKF. Also, we implicitly assumed that the landmark correspondence problem does not occur, i.e. at every time step the sensor knows the identity of the landmarks being seen. This hypothesis is realistic if the landmarks are properly designed (see e.g. [8]), and when this is not the case the problem can be addressed using a maximum likelihood approach, as evidenced in [23].

[2]In this section $x_k$ indicates the $x$ component of the pose at time $k$, while in Section II $x_k$ indicated the whole state at time $k$. This slight abuse of notation is accepted to give an immediate physical meaning to the individual components of the state.

In this section we consider the three classical problems contemplated in localization literature, i.e.

- tracking: the robot starts from a known location;
- global localization: the robot starts from an unknown location;
- kidnapped robot: the localization algorithm starts with a strong confidence about the position of the robot, but the position is wrong.

Global localization and the kidnapped robot are considered even if they are problematic for the IKF estimator in order to outline that the proposed method can handle those as well. Moreover, we consider different scenarios with varying degrees of noise.

### B. Sensitivity to horizon length

The first experiment aims to verify the impact of the moving horizon length on the localization error. Figure 1 displays the root mean square error (RMSE) error obtained in 100 runs with the moving horizon length varying from 2 to 10. The three subplots display the errors for the $x$, $y$, and $\vartheta$ components of the state. The specific problem was the kidnapped robot problem.

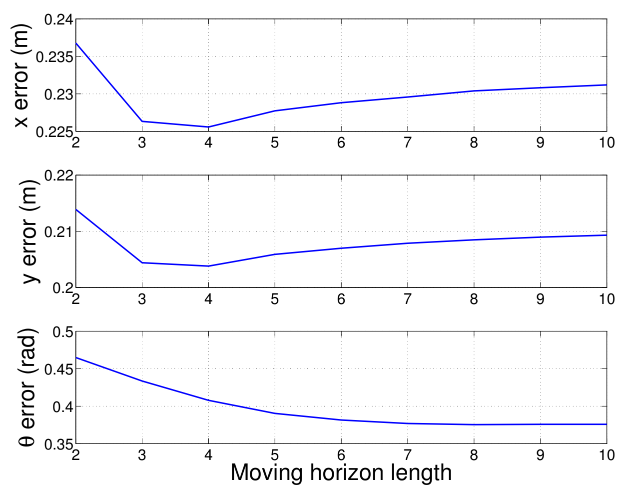


Fig. 1. Error trend for the $x, y, \vartheta$ components for different values of the moving horizon length.

Similar trends were obtained when considering the tracking and global localization problems. The chart shows that only modest improvements are obtained by increasing the size of the horizon. Consequently, in order to keep processing times at the minimum, in all subsequent tests presented in this section we used an horizon of length 2.

### C. Comparison with the IKF for the unconstrained case

In this subsection we contrast the performance of the proposed estimator with an iterated Kalman filter performing 10 iterations at each estimation step. Both estimators are

| | | IKF | | | MH | | |
|---|---|---|---|---|---|---|---|
| | | $x$ | $y$ | $\vartheta$ | $x$ | $y$ | $\vartheta$ |
| Tracking | Case 1 | 0.0874 | 0.0865 | 0.1331 | 0.0552 | 0.0554 | 0.1394 |
| | Case 2 | 0.1013 | 0.1023 | 0.1393 | 0.0680 | 0.0688 | 0.1443 |
| | Case 3 | 0.1385 | 0.1384 | 0.3998 | 0.0771 | 0.0783 | 0.4012 |
| | Case 4 | 0.1515 | 0.1516 | 0.4060 | 0.0998 | 0.1021 | 0.4062 |
| Global | Case 1 | 0.2399 | 0.1171 | 0.2901 | 0.0579 | 0.0573 | 0.2208 |
| | Case 2 | 0.2542 | 0.1517 | 0.3450 | 0.0717 | 0.0725 | 0.2445 |
| | Case 3 | 0.2538 | 0.1544 | 0.4881 | 0.0774 | 0.0786 | 0.4572 |
| | Case 4 | 0.2639 | 0.1667 | 0.5010 | 0.1005 | 0.1029 | 0.4704 |
| Kidnapped | Case 1 | 0.3667 | 0.1654 | 0.4832 | 0.2321 | 0.2090 | 0.4624 |
| | Case 2 | 0.4797 | 0.2094 | 0.5090 | 0.2677 | 0.2471 | 0.4963 |
| | Case 3 | 0.2613 | 0.1563 | 0.5509 | 0.1601 | 0.1503 | 0.6014 |
| | Case 4 | 0.2836 | 0.1728 | 0.6389 | 0.1944 | 0.1817 | 0.6077 |

TABLE I

COMPARISON BETWEEN THE PERFORMANCE OF IKF AND THE PROPOSED ESTIMATOR (MH — MOVING HORIZON). EACH ROW DISPLAYS THE RMSE AVERAGED OVER 100 INDEPENDENT TRIALS. ERRORS FOR $\vartheta$ ARE EXPRESSED IN RADIANS.

fed with exactly the same data, and a priori information about the matrices $Q_k$ and $R_k$. For each of the three estimation problems considered, i.e. tracking, global localization and kidnapped robot problem, we consider four different situations characterized by different matrices $Q_k$ and $R_k$ (see the four cases in Table I for every scenario). Table I shows the overall results. Again, we display RMSE obtained averaging 100 runs. For cases where the estimator starts with an erroneous value for $x_0$, the same wrong value is used for both estimators. In order to get an idea about the relative importance of the plotted numbers the reader is referred to figure 2 for the dimensions of the experimental environment. It is clear from Table I that for the $x$ and $y$ components of the state MH largely outperforms IKF in every instance of the tracking and global localization problems. The situation is less drastic in the the kidnapped case, where IKF prevails in 3 out of 8 cases. However, one should keep in mind that MH has been run with the shortest possible horizon length, i.e. 2. By increasing this parameter a more accurate estimate is expected, whereas for the problems at hand we observed that with 10 iterations the IKF seems to converge to a point where adding more iterations would not help. For the $\vartheta$ component of the state the situation is less crisp — there is no clear winner. This is not surprising, since the chosen sensor bears no information about the orientation of the robot. This problem equally affects both estimators. In this scenario IKF mostly relies on prediction, while MH may get stuck in local minima during its search for the minimum. In the next stage of this research we will experiment with a different sensor model returning not only the distance from the landmarks, but also the relative orientation, i.e. a range-bearing sensor. This information would then be used to infer the orientation. Based on the preliminary results seen for the $x$ and $y$ components, we also expect MH to outperform IKF for this state variable.

### D. Pose estimation with constraints

The aim of the last experiments is to show the utility of the proposed estimation algorithm when inequality constraints are imposed on the state. In many scenarios dealing with robots moving on the plane, it may be known upfront that the robot is bound to remain within the convex hull defined by the position of the landmarks. This is for example the case when the robot is moving indoors, and it is known that some landmarks are located on the walls surrounding the working area. Discarding or using this valuable a priori information may make a difference in the estimation process. For example, this information can be exploited in order to filter away pose estimates placing the robot outside the area of interest. Figure 2 illustrates the environment used throughout the tests presented in this section, with the blue segments bounding the working area.

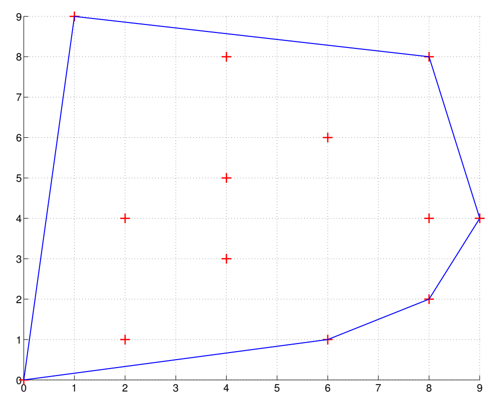

Fig. 2. The experimental environment used in all experiments described in this section. Distances are expressed in meters, red crosses indicate landmarks locations, and blue segments delimit the convex hull.

In the last batch of experiments we generated 100 robot paths that often get close to the boundaries of the area. Next, we used both MH and IKF to track the robot pose. Figure 3 shows a zoomed version of the prototypical results produced by IKF (green) and MH (red). The figure also shows the ground truth (black) and the boundaries (blue). The data has been generated in a situation where significant noise affects the evolution of the state, as it can be guessed from the jagged black path. The figure clearly shows that while IKF estimates a trajectory that exists the boundary of the

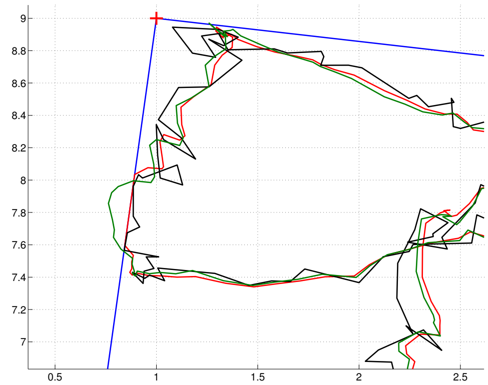


Fig. 3. A detail of a situation where the IKF estimate exits from the boundaries while MH respects the constraints.

environment, MH enforces the constraint, to the point that for a certain number of time steps its estimated trajectory coincides exactly with the imposed boundaries (see the part where the red path perfectly overlaps with the blue segment). Throughout the 100 runs, MH produced estimates that never left the boundaries. IKF, instead, being uninformed of the constraints, repeatedly violated them. To be precise, with each run being 400 steps long, on average IKF violated the constraints 15.32 times, with individual runs violating up to 68 times the constraints. To put these numbers into the right context, one should consider that in each run the trajectory stays close to the boundaries less than half of the time (i.e. less than 200 steps). In fact, almost every time the state trajectory approached the boundary, IKF generated an estimate violating them. MH, instead, seamlessly enforced these constraints.

## V. Conclusions

The estimates obtained by the IKF for nonlinear dynamic systems may be quite far from optimal. Particle filters provide an alternative approach, but these techniques require delicate tuning of proposal densities in order to improve their convergence rates, and detection of convergence is often uncertain. These design problems can be further complicated when inequality constraints have to be included in the model to improve the state estimate.

In this paper, we have shown that moving horizon (MH) approaches may provide a good alternative. In particular, we have used MH to recast a recently proposed inequality constrained smoother to this online situation. The key feature of this approach is that it exploits the same decomposition used for unconstrained Kalman-Bucy smoothers. Thus, the required operations scale linearly with the horizon length. Results show that when the horizon length is set just to 2 or 3, the quality of the estimates improves significantly and with little additional computational effort in comparison with IKF. In particular, the moving horizon approach makes the filter much less sensitive to unknown initial conditions and local minima.

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