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Geometric Properties of Zero-Torsion Parallel Kinematics Machines

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***Abstract*— The advantages of tilt-and-torsion angles in analysis of *zero-torsion* parallel kinematics machines (PKM) have been reported by several literatures. However, geometric properties of tilt-and-torsion angles are not completely understood and fully utilized in synthesis of novel zero-torsion PKMs. In this paper, we study geometric properties of the so called zero-torsion motion types via differential geometry of Lie groups. We show that zero-torsion motion types admit simple representations under *canonical coordinates of the first kind* of the special Euclidean group $SE(3)$. Using the proposed representation, we give a classification of zero-torsion PKMs. The synthesis condition for several well known zero-torsion PKMs are correctly identified. We will conduct type synthesis of zero-torsion PKMs in a separate paper.**

I. INTRODUCTION

Recently, type synthesis of *parallel kinematics machines* (PKM) with mixed translational and rotational *degree of freedoms* (D) becomes a popular area of research [1], [2], [3], [4], [5]. For convenience, they are often referred to as $mTnR$ PKMs. Such notation is however not accurate, since it does not give information about the way of composition of each translational and rotational degree of freedom. For this reason, we prefer the more precise notion of *motion type* [6].

If the set of rigid motions (attainable by the end-effector of a PKM) relative to a reference configuration agrees with a subgroup G (or a submanifold N) of the special Euclidean group $SE(3)$ in a neighborhood of the identity, we say that the mechanism has the motion type of G (or N). In this paper, our notations for subgroups and submanifolds largely follow [6].

Take $1T2R$ PKMs for example, reference [4]-Fig.4 shows a PKM having the same motion type as that of a prismatic joint $\mathcal{T}(z)$ (along z-axis) followed by a universal joint $\mathcal{U}(o,x,y)$, which is a product of one 1-D translational subgroup $T(z)$ and two 1-D rotational subgroups $R(o,x)$ and $R(o,y)$:

$$T(z)U(o,x,y) = T(z)R(o,x)R(o,y) \tag{1}$$

or a *product of exponential*[1] (PoE) [7], [6]. We refer to such motion types as *universal motion types*, since the $2R$ DoF is represented by $U(o,x,y)$, the motion type of a universal joint.

Another example of $1T2R$ PKMs is the symmetric 3-$\mathcal{TTS}$ (or planar-spherical bond, [8]) PKM family[2] [3], whose motion type is the product of $T(z)$ with a complex $2R$ motion type. Reference [9] used Rodrigues formula to show that its rotation matrix is generated by a horizontal rotation vector. Reference [10] used tilt-and-torsion angle [11] to parameterize its rotation matrix, and showed that its torsion angle is constantly zero. Such PKMs are referred to as *zero-torsion* PKMs. Zero-torsion PKMs have been extensively studied by [12], [13], [14].

Since tilt-and-torsion angles do not depend on the choice of rotation axes as is the case of $U(o,x,y)$, it is ideal for modeling $mT2R$ PKMs, such as $2R$ orientation devices, $3T2R$ five-axis machine tools and their $1T2R$ modules. However, tilt angle and zero-torsion PKMs are less referred to in PKM synthesis literatures. This is probably due to lack of both a proper definition of *zero-torsion motion type*s, and knowledge of their geometric properties. To the authors' knowledge, most PKM synthesis literatures focus on $mT2R$ universal motion types [15], [4], [16], [17].

In fact, the majority of PKM synthesis literatures are concerned with subchains and PKMs with PoEs [18], [19], [20], [6]; while zero-torsion motion types has no PoE representation. Another reason for unpopularity of zero-torsion motion type is that, there is a lack of motivation for type synthesis of PKMs with parasitic motions. While PoE usually avoid the presence of parasitic motions, zero-torsion motion types usually have parasitic translations.

This paper is multi-purpose. First, it fills the gap between analysis and synthesis of $mT2R$ zero-torsion PKMs; second, it brings more sense into $mT2R$ motion types with parasitic motions, and gives a unified picture of PKMs with or without parasitic motions, and also PKMs with universal or zero-torsion motion types; third, it throws some light on innovation of PKM synthesis with new motion types.

This paper is organized as follows: in Section II, we first review the basic knowledge of Lie group theory, which is necessary for understanding the rest of the paper, then we utilize these geometric tools in definition and analysis of zero-torsion motion types. We derive several geometric properties of zero-torsion motion types, which are crucial to a unified classification and synthesis of zero-torsion PKMs. Finally, we give our conclusion and foresee our future work.

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[1]Here the reason for the name "exponential" shall be clear in Section II.

[2]Here $\mathcal{S}$ denotes a spherical joint.

The synthesis of zero-torsion PKMs is treated in a separate paper.

II. Geometric Properties of Zero-Torsion Motion Types

In this section, we first give a minimal review of mathematics necessary for understanding the rest of the material. Then we use canonical coordinates of the first kind to give a precise definition for zero-torsion motion types, based on which we give a classification of zero-torsion motion types.

A. Basic Lie group theory

We give a brief review of Lie group theory in a similar fashion to [7], [6]. A formal treatment of the material can be found in [21], [22].

By attaching a copy of the reference frame to a rigid body, a rigid displacement can be identified with an element $g = (p, R) \in SE(3)$ which, in homogeneous representation, has the following form:

$$SE(3) \triangleq \left\{ \left[\begin{array}{cc} R & p \\ 0 & 1 \end{array} \right] \middle| p \in \mathbb{R}^3, R \in SO(3) \right\} \qquad (2)$$

where $SO(3)$ denotes the set of 3×3 orthogonal matrices of determinant 1. Given a change of reference frame $g_0 \in SE(3)$, a rigid displacement g in the new reference frame is given by its *conjugation*, denoted $I_{g_0}(g)$:

$$I_{g_0}(g) \triangleq g_0 g g_0^{-1} \qquad (3)$$

The *Lie algebra* of $SE(3)$, denoted $se(3)$, is a 6-D vector space consisting of all 4×4 matrices of the form:

$$\begin{aligned} se(3) &\triangleq \left\{ \hat{\xi} \in \mathbb{R}^{4\times 4} | e^{\hat{\xi} t} \in SE(3), \forall t \in \mathbb{R} \right\} \\ &= \left\{ \left[\begin{array}{cc} \hat{\omega} & v \\ 0 & 0 \end{array} \right] \middle| \hat{\omega}^T = -\hat{\omega} \in \mathbb{R}^{3\times 3}, v \in \mathbb{R}^3 \right\} \end{aligned} \qquad (4)$$

An element of $se(3)$ is called a *twist*, usually denoted by $\hat{\xi}, \hat{\eta}$, etc. $se(3)$ is isomorphic to $\mathbb{R}^6$ by:

$$\wedge : \mathbb{R}^6 \mapsto se(3), \left[\begin{array}{c} v \\ \omega \end{array} \right]^{\wedge} = \left[\begin{array}{cc} \hat{\omega} & v \\ 0 & 0 \end{array} \right] \qquad (5)$$

where

$$\hat{\omega} = \left[\begin{array}{c} \omega_1 \\ \omega_2 \\ \omega_3 \end{array} \right]^{\wedge} = \left[\begin{array}{ccc} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{array} \right] \qquad (6)$$

The $\wedge$ operator satisfies:

$$\begin{aligned} \hat{\omega} v &= \omega \times v, v \in \mathbb{R}^3 \\ \hat{\omega}^2 &= \omega\omega^T - \|\omega\|^2 I \end{aligned} \qquad (7)$$

The *exponential map*:

$$\exp : se(3) \mapsto SE(3), \exp(\hat{\xi}) = e^{\hat{\xi}} \qquad (8)$$

defines a local diffeomorphism taking the zero vector of $se(3)$ to the identity of $SE(3)$. It can be computed that $(e^{\hat{\xi}})^{-1} = e^{-\hat{\xi}}$.

Given a twist $\xi = (v, \omega)^T$, $e^{\hat{\xi}}$ is given by:

$$e^{\hat{\xi}} = \left[\begin{array}{cc} e^{\hat{\omega}} & [(I - e^{\hat{\omega}})\hat{\omega} v + \omega\omega^T v]\frac{1}{\|\omega\|^2} \\ 0 & 1 \end{array} \right], \|\omega \neq 0 \qquad (9)$$

where $e^{\hat{\omega}}$ is given by Rodrigues formula:

$$e^{\hat{\omega}} = I + \frac{\hat{\omega}}{\|\omega\|} \sin\|\omega\| + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos\|\omega\|) \qquad (10)$$

Its inverse is given by:

$$\hat{\xi} = \log \left[\begin{array}{cc} R & p \\ 0 & 1 \end{array} \right] = \left[\begin{array}{cc} \hat{\omega} & Ap \\ 0 & 1 \end{array} \right] \qquad (11)$$

where $\hat{\omega} = \log R$ is given by:

$$2\cos\|\omega\| + 1 = \mathrm{trace}(R), \hat{\omega} = \frac{1}{2\sin\|\omega\|}(R - R^T) \qquad (12)$$

and

$$A = I - \frac{1}{2}\hat{\omega} + \frac{2\sin\|\omega\| - \|\omega\|(1 + \cos\|\omega\|)}{2\|\omega\|^2 \sin\|\omega\|}\hat{\omega}^2 \qquad (13)$$

Given an arbitrary basis $(\hat{v}_1, \ldots, \hat{v}_6)$ of $se(3)$, there are two ways to locally parameterize $SE(3)$:

- *Canonical coordinate of the first kind*:
$$(\theta_1, \ldots, \theta_6) \mapsto e^{\sum_{i=1}^{6} \hat{v}_i \theta_i} \qquad (14)$$
- *Canonical coordinate of the second kind*:
$$(\theta_1, \ldots, \theta_6) \mapsto e^{\hat{v}_1 \theta_1} \cdots e^{\hat{v}_6 \theta_6} \qquad (15)$$
also known as *product of exponential* (POE) [7].

In general, $e^{\sum_{i=1}^{6} \hat{v}_i \theta_i}$ does not equal $e^{\hat{v}_1 \theta_1} \cdots e^{\hat{v}_6 \theta_6}$.

A *Lie subgroup* G of $SE(3)$ is a closed subset of $SE(3)$ such that:

$$\forall g_1, g_2 \in G \subset SE(3) \Rightarrow g_1 g_2^{-1} \in G \qquad (16)$$

One can check by (16) that $I_{g_0}(G) \triangleq \{I_{g_0}(g) | g \in G\}$ is also a subgroup of $SE(3)$, called a *conjugate subgroup*.

The Lie algebra $\mathfrak{g}$ of a subgroup G is given by:

$$\mathfrak{g} \triangleq \left\{ \hat{\xi} \in se(3) | e^{\hat{\xi} t} \in G, \forall t \in \mathbb{R} \right\} \qquad (17)$$

which is a subspace of $se(3)$ and is closed under the *Lie bracket*:

$$[,] : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}, [\hat{\xi}, \hat{\eta}] \triangleq \hat{\xi}\hat{\eta} - \hat{\eta}\hat{\xi} \qquad (18)$$

$\mathfrak{g}$ is called a *Lie subalgebra* of $se(3)$.

The Lie subalgebra of a conjugate subgroup $I_{g_0} G$ is given by:

$$Ad_{g_0} \mathfrak{g} \triangleq \{ g_0 \hat{\xi} g_0^{-1} | \hat{\xi} \in \mathfrak{g} \} \qquad (19)$$

The *Adjoint transformation* $Ad_{g_0}(\hat{\xi}) = g_0 \hat{\xi} g_0^{-1}$ induces a linear map on $\mathbb{R}^6 \cong se(3)$:

$$Ad_{g_0} \xi = \left[\begin{array}{cc} R_0 & \hat{p}_0 R_0 \\ 0 & R_0 \end{array} \right] \xi \qquad (20)$$

for some $g_0 = (p_0, R_0) \in SE(3)$. Ad_{g_0} relates to I_{g_0} via:

$$I_{g_0}(e^{\hat{\xi}}) = e^{Ad_{g_0} \hat{\xi}} \qquad (21)$$

Lie subgroups of $SE(3)$ provide model spaces for rigid displacements generated by say, the six lower pairs (revolute,

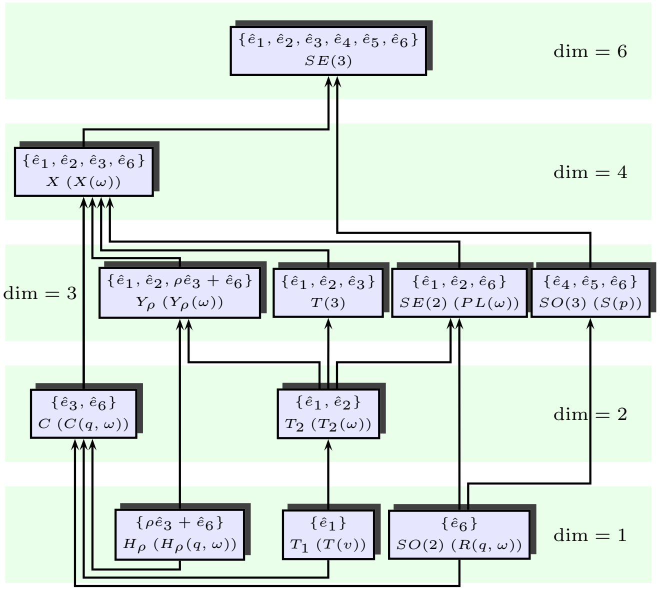


Fig. 1. A classification of Lie subgroups of $SE(3)$. The upper part of each box denotes the Lie subalgebra of the corresponding Lie subgroup in its normal form, and the lower part denotes the conjugacy class. Enclosed in the parenthesis is a generic member of the conjugacy class.

prismatic, helical, cylindrical, spherical and planar joints). Classification of Lie subgroups of $SE(3)$, up to a conjugation, have been studied in [23], [24], [25], and the results are displayed in Fig.1. In the figure, $\{e_i\}_{i=1}^6$ is the canonical basis of $\mathbb{R}^6$, $\{\hat{e}_{i_1}, \ldots, \hat{e}_{i_n}\}$ is the Lie subalgebra spanned by $\hat{e}_{i_1}, \ldots, \hat{e}_{i_n}$. In other words, the preimage of Lie subgroups of $SE(3)$ under $\exp$ are Lie subalgebras of $se(3)$.

Rigid displacements generated by a parallelogram joint $\mathcal{P}_a$ or a universal joint are not closed under the group operation, and so is that of a five-axis machine as there simply exists no Lie subgroup of dimension 5 at all. For this reason, (regular) submanifolds of SE(3) are introduced in [6], [8], [4] to provide additional modeling spaces for mechanism motions. The preimage of a submanifold M of $SE(3)$ under $\exp$ is no longer a Lie subalgebra, but a hypersurface of $se(3)$ in a neighborhood of $0 \in se(3)$. Thus for convenience, we use the word *hypersurface* to mean its corresponding submanifold under $\exp$.

Two special families of regular submanifolds are defined: (i) *Category I submanifolds* of the form $N_1 \cdot N_2$, where N_1 is a submanifold of $T(3)$ and N_2 a submanifold of $S(o)$, and (ii) *Category II submanifolds* of the form $H_1 \cdot H_2$, where H_1 and H_2 are Lie subgroups of $SE(3)$ with nontrivial intersection (also referred to as dependent products). Category I submanifolds are used to model desired task motions, and 25 dependent products of Category II submanifolds are enumerated in Table III of [6], some of which are used in [8], [19], [6] for subchain (or limb) synthesis.

B. Canonical representation of zero-torsion motion types

The tilt-and-torsion angle (or modified Euler angles) is proposed in [11] for orientation workspace analysis, and is well explained in [11], [10]. Here we use a slightly different convention: we parameterize the tilt axis, denoted ω, by its angle between x-axis and ω:

$$\omega = x\cos\theta + y\sin\theta \tag{22}$$

an denote the tilt angle by α, then the corresponding torsion-less (or zero-torsion) rotation matrix R is given by:

$$R = e^{\hat{\omega}\alpha} = e^{(\hat{x}\cos\theta + \hat{y}\sin\theta)\alpha} \tag{23}$$

Using the following coordinate transformation:

$$\begin{cases} \theta_1 = \alpha\cos\theta \\ \theta_2 = \alpha\sin\theta \end{cases} \tag{24}$$

the zero-torsion rotation, in its homogeneous representation, is represented by:

$$\begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} = e^{\hat{e}_4\theta_1 + \hat{e}_5\theta_2} \tag{25}$$

It is not difficult to see that the zero-torsion rotation is exactly represented by canonical coordinates of the first kind (14). Thus,

$$\begin{aligned} &\exp\{\hat{e}_4, \hat{e}_5\} \triangleq \{e^{\hat{\xi}} | \hat{\xi} \in \{\hat{e}_4, \hat{e}_5\}\} \\ &= \{e^{(\hat{e}_4\cos\theta + \hat{e}_5\sin\theta)\alpha} | \theta \in [0, 2\pi], \alpha \in [-\pi, \pi]\} \end{aligned} \tag{26}$$

defines a 2-D zero-torsion submanifold of $SO(3)$.

$\exp\{\hat{e}_4, \hat{e}_5\}$ has the following geometric properties:

1) **Invariance by change of basis:** By (26), given any other basis $(\hat{v}_1, \hat{v}_2)$ of $\{\hat{e}_4, \hat{e}_5\}$, $\exp\{\hat{v}_1, \hat{v}_2\} = \exp\{\hat{e}_4, \hat{e}_5\}$ on the domain of intersection. This is a property not shared by the PoE $R(o, x)R(o, y)$;
2) **Invariance by inverse:** Since $(e^{\hat{\xi}})^{-1} = e^{-\hat{\xi}}$, and for any $\hat{\xi} \in \{\hat{e}_4, \hat{e}_5\}$, we also have $-\hat{\xi} \in \{\hat{e}_4, \hat{e}_5\}$. $(\exp\{\hat{e}_4, \hat{e}_5\})^{-1} = \exp\{\hat{e}_4, \hat{e}_5\}$. This is a property not shared by $U(o, x, y) = R(o, x)R(o, y)$;
3) **z-axial symmetry:** This is equivalent to saying that $I_{g_0}(\exp\{\hat{e}_4, \hat{e}_5\}) = \exp\{\hat{e}_4, \hat{e}_5\}$ for all $g_0 \in R(o, z)$. From (21), we have:

$$I_{g_0}(\exp\{\hat{e}_4, \hat{e}_5\}) = \exp\{Ad_{g_0}\hat{e}_4, Ad_{g_0}\hat{e}_5\} \tag{27}$$

where $(Ad_{g_0}\hat{e}_4, Ad_{g_0}\hat{e}_5)$ is just a change of basis for $\{\hat{e}_4, \hat{e}_5\}$. To see this, let $g_0 = R_z(\psi)$, then:

$$\begin{cases} Ad_{g_0}\hat{e}_4 = \hat{e}_4\cos\psi + \hat{e}_5\sin\psi \\ Ad_{g_0}\hat{e}_5 = -\hat{e}_4\sin\psi + \hat{e}_5\cos\psi \end{cases} \tag{28}$$

The symmetry follows from 1);
4) $\exp\{\hat{e}_4, \hat{e}_5\}$ **is omni-directional**, i.e. for any $\omega \perp z$, the 1-D subgroup $R(o, \omega)$ is contained in $\exp\{\hat{e}_4, \hat{e}_5\}$. This is a great advantage of $\exp\{\hat{e}_4, \hat{e}_5\}$ over $U(o, x, y) = \exp\{\hat{e}_4\}\exp\{\hat{e}_5\}$, which only contains 2 1-D subgroups of $SO(3)$, namely $R(o, x) = \exp\{\hat{e}_4\}$ and $R(o, y) = \exp\{\hat{e}_5\}$.

As compared to $U(o, x, y)$, $\exp\{\hat{e}_4, \hat{e}_5\}$ is more like a Lie subgroup of $SE(3)$, in the sense that it corresponds to a 2-D subspace (a hyperplane) $\{\hat{e}_4, \hat{e}_5\}$ of $se(3)$, only that it is no longer a Lie subalgebra (since $[\hat{e}_4, \hat{e}_5] = \hat{e}_6 \notin \{\hat{e}_4, \hat{e}_5\}$, it is not closed under Lie bracket).

Following this lead, we look at a $1T2R$ zero-torsion submanifold, $\exp\{\hat{e}_3, \hat{e}_4, \hat{e}_5\}$. It shares all geometric properties of $\exp\{\hat{e}_4, \hat{e}_5\}$. Moreover, $\exp\{\hat{e}_3, \hat{e}_4, \hat{e}_5\}$ is not only z-axial symmetric but also invariant under any horizontal translation. To see this, let $p = (p_x, p_y, 0)^T \in T_2(z)$ and note that:

$$I_p \exp\{\hat{e}_3, \hat{e}_4, \hat{e}_5\} = \exp\{Ad_p\hat{e}_3, Ad_p\hat{e}_4, Ad_p\hat{e}_5\} \tag{29}$$

A straightforward calculation shows that:

$$\begin{cases} Ad_p\hat{e}_3 = \hat{e}_3 \\ Ad_p\hat{e}_4 = \hat{e}_4 - p_y\hat{e}_3 \\ Ad_p\hat{e}_5 = \hat{e}_5 + p_x\hat{e}_3 \end{cases} \tag{30}$$

and thus,

$$\{Ad_p\hat{e}_3, Ad_p\hat{e}_4, Ad_p\hat{e}_5\} = \{\hat{e}_3, \hat{e}_4, \hat{e}_5\} \tag{31}$$

(31) and (29) prove the invariance. We also say its symmetry is $T_2(z)R(o, z) = PL(z)$, the planar Euclidean subgroup. $\exp\{\hat{e}_3, \hat{e}_4, \hat{e}_5\}$ contains $T(z)$ and all 1-D rotation subgroups with rotation axis lying on the xy-plane:

$$\begin{aligned} &\forall \omega \perp z, p = (p_x, p_y, 0)^T \in T_2(z) \Rightarrow \\ &R(p, \omega) \subset \exp\{\hat{e}_3, \hat{e}_4, \hat{e}_5\} \end{aligned} \tag{32}$$

Example 1 (omni-wrist): The omni-wrist (or omni-wrist III) [26], as shown in Fig.3, is a $2R$ zero-torsion PKM. Its motion type, denoted $O(o, dz)$ for the omni-wrist situated at the origin and with dimension d (see the Schematic in Fig.3(b)), is given by:

$$\begin{aligned} O(o, dz) = \Bigg\{ \begin{bmatrix} e^{2\hat{\omega}\alpha} & -\frac{d}{2}(e^{\hat{\omega}\alpha} - I)^2 z \\ 0 & 1 \end{bmatrix} \Bigg| \\ \omega = x\cos\theta + y\sin\theta, \theta \in [0, 2\pi], \alpha \in (-\varepsilon, \varepsilon) \Bigg\} \end{aligned} \tag{33}$$

where ε is the tilt limit. We use (11)-(13) to derive the preimage of $O(o, dz)$ in $se(3)$. Bear in mind that the ω in (13) is actually $2\alpha\omega$ in this example where ω is normalized.

$$A = I - \alpha\hat{\omega} + \frac{\sin 2\alpha - \alpha(1 + \cos 2\alpha)}{\sin 2\alpha}\hat{\omega}^2 \tag{34}$$

Then by (11),

$$v = -\frac{d}{2}A(e^{\hat{\omega}\alpha} - I)^2 z = (d\alpha \tan\frac{\alpha}{2})z \tag{35}$$

and

$$\hat{\xi} = \hat{e}_4(2\alpha\cos\theta) + \hat{e}_5(2\alpha\sin\theta) + \hat{e}_3(d\alpha\tan\frac{\alpha}{2}) \tag{36}$$

Then the coordinate transformation:

$$\begin{cases} \theta_1 = 2\alpha\cos\theta \\ \theta_2 = 2\alpha\sin\theta \\ \theta_3 = d\alpha\tan\dfrac{\alpha}{2} = \theta_3(\theta_1, \theta_2) \\ \quad = \dfrac{d}{2}(\theta_1^2 + \theta_2^2)^{\frac{1}{2}} \tan\dfrac{(\theta_1^2 + \theta_2^2)^{\frac{1}{2}}}{4} \end{cases} \tag{37}$$

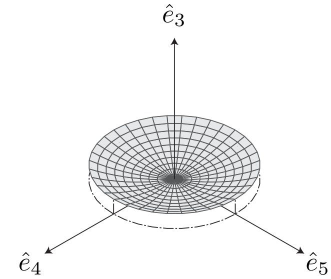
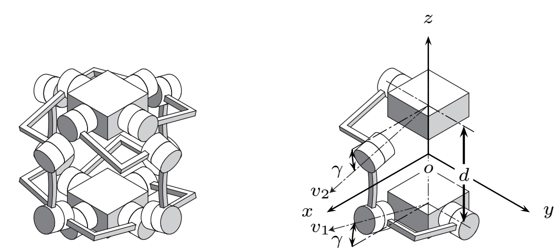


Fig. 2. 2-D hypersurface of the omni-wrist.



(a) (b)

Fig. 3. (a): omni-wrist ($\mathcal{O}$), a PKM with four symmetric $\mathcal{UU}$ subchains; (b): Schematic of a single leg of the omni-wrist.

shows that

$$\begin{aligned} &O(o, dz) = \{e^{\hat{\xi}} | \hat{\xi} = \hat{e}_4\theta_1 + \hat{e}_5\theta_2 + \hat{e}_3\theta_3(\theta_1, \theta_2), \\ &\theta_1 \in (-\varepsilon_1, \varepsilon_1), \theta_2 \in (-\varepsilon_2, \varepsilon_2)\} \end{aligned} \tag{38}$$

That is to say, $O(o, dz)$ corresponds to a 2-D hypersurface of the 3-D vector space $\{\hat{e}_3, \hat{e}_4, \hat{e}_5\}$, as shown in Fig.2.

It is interesting to note that this 2-D surface is still z-axial symmetric, and $\{\hat{e}_4, \hat{e}_5\}$ is its tangent space at $0 \in se(3)$. The former is because $Ad_{R_z(\psi)}\hat{e}_3 = R_z(\psi)\hat{e}_3 \equiv \hat{e}_3$ for any angle ψ, and that the coefficient θ_3 of $\hat{e}_3$ does not depend on the tilt axis parameter θ. ♢

Example 2 (Reflected tripod): The reflected tripod [27], [10], as shown in Fig.4, is a $1T2R$ zero-torsion PKM. Its motion type is given by:

$$\begin{aligned} \Bigg\{ \begin{bmatrix} e^{2\hat{\omega}\alpha} & -\frac{d}{2}(e^{\hat{\omega}\alpha} - I)^2 z + e^{\hat{\omega}\alpha} z\theta_3 \\ 0 & 1 \end{bmatrix} \Bigg| \omega = x\cos\theta \\ + y\sin\theta, \theta \in [0, 2\pi], \alpha \in (-\varepsilon, \varepsilon), \theta_3 \in (-\delta, \delta) \Bigg\} \end{aligned} \tag{39}$$

A similar computation as in Example 1 shows that:

$$v = (d\alpha\tan\frac{\alpha}{2} + \frac{\alpha}{\sin\alpha}\theta_3)z \tag{40}$$

Then the coordinate transformation:

$$\begin{cases} \theta_1 = 2\alpha\cos\theta \\ \theta_2 = 2\alpha\sin\theta \\ \theta_3' = d\alpha\tan\dfrac{\alpha}{2} + \dfrac{\alpha}{\sin\alpha}\theta_3 \end{cases} \tag{41}$$

shows that (39) can be parameterized by canonical coordinate of the second kind:

$$(\theta_1, \theta_2, \theta_3') \mapsto e^{\hat{e}_4\theta_1 + \hat{e}_5\theta_2 + \hat{e}_3\theta_3'} \tag{42}$$

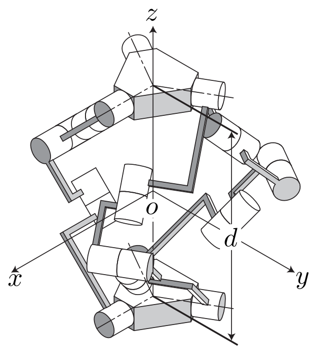


Fig. 4. The reflected tripod, with three $\mathcal{RSR}$ subchain.

In other words, the motion type of the reflected tripod is exactly $\exp\{\hat{e}_3, \hat{e}_4, \hat{e}_5\}$, which contains $O(o, dz)$ as a 2-D submanifold. $\diamondsuit$

Generalizing from the above examples, we see that zero-torsion motion types can be defined in the following way.

Definition 1 (zero-torsion motion type):

A submanifold M of $SE(3)$ is said to be a *zero-torsion motion type* if the preimage $V \triangleq \exp^{-1}(M)$ projects one-to-one onto $\{\hat{e}_4, \hat{e}_5\}$, where the projection is defined by:

$$\begin{aligned} &\pi : se(3) \mapsto so(3) \triangleq \{\hat{e}_4, \hat{e}_5, \hat{e}_6\}, \\ &\pi(v, w)^T = (0, w)^T \end{aligned} \tag{43}$$

$\diamondsuit$

We could make use of the quotient space $se(3)/t(3)$ and its natural projection for a more elegant definition, if bigger space were allowed.

C. *Classification of zero-torsion motion types*

Now we are ready to give a classification of zero-torsion motion types. Starting from the simplest case, subspaces (hyperplanes) of $se(3)$ that contain $\{\hat{e}_4, \hat{e}_5\}$ but not $\hat{e}_6$ correspond to zero-torsion motion types. Such subspaces include: the 3-D $\{\hat{e}_i, \hat{e}_4, \hat{e}_5\}$ for $i = 1, 2, 3$, the 4-D $\{\hat{e}_i, \hat{e}_j, \hat{e}_4, \hat{e}_5\}$ for $i \neq j \in \{1, 2, 3\}$, and the 5-D $\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4, \hat{e}_5\}$. Subspaces that contain $\{\hat{e}_3, \hat{e}_4, \hat{e}_5\}$ are of particular interest for their abundant geometric properties. General zero-torsion motion types are hypersurfaces of the aforementioned subspaces. Fig.5 shows a classification of the zero-torsion subspaces, together with related $3R$ motion types.

It needs to be pointed out that the classification of general zero-torsion motion types relies on the notion of quotient space, which does not fit the scope of this paper. Synthesis of zero-torsion PKMs based on the classification has to be separated from this paper for the same reason.

III. Geometric Properties of Zero-Torsion Parallel Kinematics Machine

The symmetry properties of hypersurfaces of $se(3)$ are closely related to geometric properties of zero-torsion PKMs. Take the omni-wrist, as shown in Fig.3, for example. Recall that its motion type (38) is a 2-D hypersurface of the 3-D subspace $\{\hat{e}_3, \hat{e}_4, \hat{e}_5\}$, with a symmetry of $R(o, z)$. Each $\mathcal{UU}$ subchain of the omni-wrist, a 4-D hypersurface of

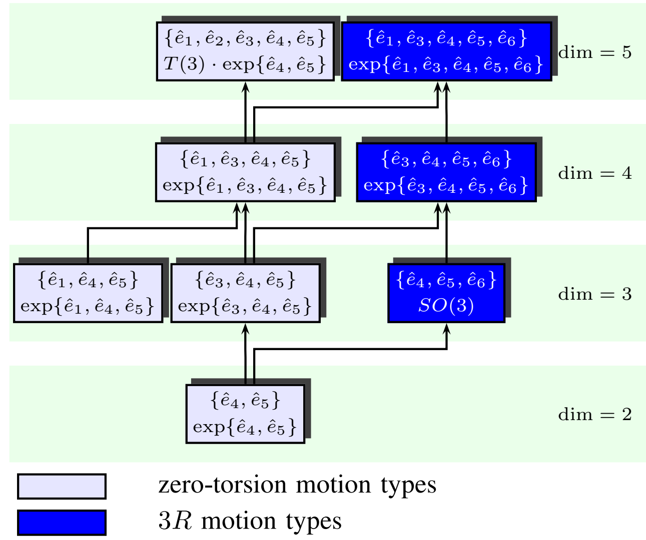


Fig. 5. A classification of zero-torsion motion types

$se(3)$, though not axial symmetric itself, contains the 2-D hypersurface of omni-wrist. Thus the omni-wrist's subchains can be arbitrarily rotated about z-axis without affecting the PKM's motion type, so long as the *Force Matching Condition* (FMC, [6]-Prop.6) is satisfied. This can be illustrated by the 3-subchained version of omni-wrist [28], whose neighboring subchains are conjugate by $120°$ rotation about z-axis.

A. *Geometric properties of 3-$\mathcal{RSR}$ PKM*

The 3-$\mathcal{RSR}$ PKM has a motion type of $\exp\{\hat{e}_3, \hat{e}_4, \hat{e}_5\}$. According to the geometric properties of $\exp\{\hat{e}_3, \hat{e}_4, \hat{e}_5\}$, each $\mathcal{RSR}$ subchain of the PKM can be displaced by a planar transformation in $PL(z)$ without changing the fact that it contains the 3-D hyperplane $\{\hat{e}_3, \hat{e}_4, \hat{e}_5\}$. This shows that subchain symmetry of the reflected tripod is not an essential condition for its synthesis. Moreover, its subchain can be arbitrarily translated not only radially, but also tangentially.

B. *Geometric properties of 3-planar-spherical-bond PKM family*

The symmetric 3-$\mathcal{TRS}$ PKM, as shown in Fig.6, is a member of the 3-planar-spherical-bond PKM family. It is shown to have a $1T2R$ zero-torsion motion type by [10]. Its motion type can be computed:

$$\left\{ \left[\begin{array}{c|c} e^{(\hat{x}\cos\theta + \hat{y}\sin\theta)\alpha} & p \\ \hline 0 & 1 \end{array} \right] \middle| \, \theta \in [0, 2\pi], \alpha \in (-\varepsilon_1, \varepsilon_1), \right.$$
$$\left. \theta_3 \in (-\varepsilon_2, \varepsilon_2), p = \begin{bmatrix} \frac{h}{2}\cos(-2\theta)(1-\cos\alpha) \\ \frac{h}{2}\sin(-2\theta)(1-\cos\alpha) \\ \theta_3 \end{bmatrix} \in \mathbb{R}^3 \right\} \tag{44}$$

A straightforward computation using (11)-(13) shows that it does not correspond to the 3-D hyperplane $\{\hat{e}_3, \hat{e}_4, \hat{e}_5\}$, but rather a 3-D hypersurface of $\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4, \hat{e}_5\}$.

Since the zero-torsion matrix $e^{(\hat{x}\cos\theta+\hat{y}\sin\theta)\alpha}$ is z-axial symmetric, we only need to analyze symmetry of the parasitic motion. Observe (44) that its parasitic translation in $T_2(z)$ has a magnitude of $\frac{h}{2}(1-\cos\alpha)$, which is independent of the tilt axis parameter θ; its direction is given by -2θ. Thus, the PKM's motion type remains unchanged only when parasitic translations of all three subchains coincide for an arbitrary tilt axis θ and tilt angle α. The conjugation $R_z(\psi)$ between neighboring subchains is given by:

$$\psi - 2(\theta - \psi) = -2\theta + 2k\pi, \forall\theta \in [0, 2\pi] \quad (45)$$

from which we get $\psi = \frac{2k\pi}{3}, k = 1, 2$. The PKM remains zero-torsion only when neighboring subchains are conjugate by $R_z(\frac{2\pi}{3})$. The 3-$\mathcal{TST}$ PKM [10] can be analyzed in a similar fashion. Its subchains must remain symmetric in order to generate a zero-torsion motion type.

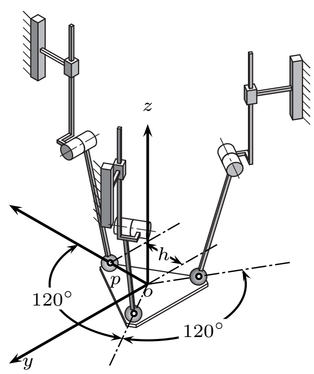


Fig. 6. Schematic of the symmetric 3-$\mathcal{TRS}$ PKM.

IV. Conclusion

In this paper, we have proposed a geometric study on zero-torsion parallel kinematics machines (PKMs). The notion of zero-torsion motion types is defined and studied in a Lie group theory framework. In particular, we give a clear representation of zero-torsion rotations by canonical coordinates of the first kind of $SE(3)$. All zero-torsion motion types are represented by hyperplanes or hypersurfaces of $\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4, \hat{e}_5\}$. Using such representations, the problem of PKM synthesis becomes the intersection problem of different hyperplanes or hypersurfaces of $se(3)$, whose geometric properties are easily extracted and utilized in analysis and synthesis of zero-torsion PKMs. The synthesis condition of several well known zero-torsion PKMs are easily identified without exhaustive computation of loop constraints.

In a separate paper, we will dig deeper into geometric properties of zero-torsion motion types for the synthesis of novel zero-torsion parallel or hybrid kinematics machines.

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