# Consensus of Networks of Nonidentical Robots with Flexible Joints, Variable Time–Delays and Unmeasurable Velocities

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*Abstract*— The present paper proposes two controllers for solving a consensus problem to a given desired position of networks composed of a class of *under actuated* mechanical systems: flexible joints robots. One of the controllers makes use of joint (motor) velocity signals while the other only uses joint positions. The only assumption on the directed and weighted interconnection graph is that it is connected. Further, the interconnection may induce variable time-delays. The paper presents some experiments, using three 3-Degrees of Freedom manipulators, which show the performance of the proposed approaches.

#### I. INTRODUCTION

A wide range of applications in different areas are based on the consensus of networks of dynamic systems. The objective for the collective motion of the network is to reach some type of agreement between certain variables of interest of the interconnected systems. The literature that deals with the consensus of networks covers those composed of linear time invariant systems, which is relatively rich and large [1], [2], [3], [4], [5], [6], and those composed of nonlinear nodes, which is rapidly increasing [7], [8], [9], [10], [11].

Consensus of networks of Euler-Lagrange (EL) systems without time-delays has been considered in [12], [13] using simple proportional controllers together with filtered velocities. However, in both papers the authors assume that timedelays in the agents communications are negligible. The work of Nuño et al. [14] reports an adaptive controller for EL-systems that solves the consensus problem with constant time-delays. Further results are those by Liu and Chopra [15] and by Hatanaka et al. [16], which consider the consensus problem in Cartesian space with constant time-delays in the communications. Recently, in [17] it has been proved that networks composed by nonidentical EL-systems with variable time-delays can reach a consensus, using simple PD controllers, provided enough damping is injected. It should be underscored that, all these previous results deal with fully actuated EL-systems (fully actuated robots). However, in diverse applications, including space and surgical robots, the use of thin, lightweight and cable-driven manipulators is increasing. These systems exhibit joint or link flexibility and hence they are under actuated mechanical systems. It has been shown in [18] that the lumped (linear) dynamics of a flexible link is identical to the (linear) dynamics of a flexible joint.

On the other hand, the literature on the control of networks of *under actuated* EL-systems is scarce, with some nice exceptions [19] and, more recently, [20]. In [19] the Controlled–Lagrangian technique is employed to solve the consensus in networks without delays and in [20] the consensus problem is solved under the assumption that all the states are measurable, all the physical parameters are known and the time-delays are constant.

In the present work, inspired by [21] and [22], two different controllers that are capable of solving a consensus problem to a given desired position in a network composed by nonidentical flexible joint robots are proposed. One of the controller makes use of joint (motor) velocity measurements while the other only needs joint position measurements. The robots are interconnected with a directed and weighted network topology and the only assumption on the interconnection graph is that it is connected. Moreover, the interconnection can exhibit variable time-delays. It should be noted that since interconnection improves performance, as has been proved in [23], [24], the proposed controllers are, in principle, more robust to parameter uncertainties than the ones without the interconnection. Finally, using three 3-Degrees of Freedom (DOF) manipulators, the paper presents some experiments which show the performance of the controllers with and without the interconnection.

# A. Notation

 $\mathbb{R} := (-\infty, \infty), \mathbb{R}_{>0} := (0, \infty), \mathbb{R}_{\geq 0} := [0, \infty). \lambda_m \{\mathbf{A}\}$ and  $\lambda_M \{\mathbf{A}\}$  represent the minimum and maximum eigenvalues of matrix  $\mathbf{A}$ , respectively.  $||\mathbf{A}||$  denotes the matrixinduced 2-norm.  $|\mathbf{x}|$  stands for the standard Euclidean norm of vector  $\mathbf{x}$ .  $\mathbf{I}_k$  and  $\mathbf{0}_k$  represent the identity and all-zeros matrices of size  $k \times k$ .  $\mathbf{1}_k$  is a vector of all elements equal to one of size k. For any function  $\mathbf{f} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ , the  $\mathcal{L}_{\infty}$ norm is defined as  $\|\mathbf{f}\|_{\infty} := \sup_{\substack{t \geq 0 \\ t \geq 0}} |\mathbf{f}(t)|^2 dt$ . The  $\mathcal{L}_{\infty}$  and  $\mathcal{L}_2$  spaces are defined as the sets  $\{\mathbf{f} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n : \|\mathbf{f}\|_{\infty} < \infty\}$  and  $\{\mathbf{f} : \mathbb{R}_{>0} \to \mathbb{R}^n : \|\mathbf{f}\|_2 < \infty\}$ , respectively.

# II. MODELS OF THE ROBOT DYNAMICS AND THE NETWORK INTERCONNECTION

The complete system is composed by an interconnected network of different flexible–joint robots. This section presents the dynamics of the manipulators (nodes or agents) and the model of the interconnection of the network.

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#### A. Robots with Joint Flexibility

This work deals with networks composed of N nonidentical, flexible-joint manipulators with n-DOF. Revolute joints robots directly actuated are assumed. The nonlinear dynamics of these robots follow the Euler-Lagrange equations of motion and are of the form

$$\begin{split} \mathbf{M}_{i}(\mathbf{q}_{i})\ddot{\mathbf{q}}_{i} + \mathbf{C}_{i}(\mathbf{q}_{i},\dot{\mathbf{q}}_{i})\dot{\mathbf{q}}_{i} + \mathbf{g}_{i}(\mathbf{q}_{i}) &= \mathbf{S}_{i}[\boldsymbol{\theta}_{i} - \mathbf{q}_{i}] \\ \mathbf{J}_{i}\ddot{\boldsymbol{\theta}}_{i} + \mathbf{S}_{i}[\boldsymbol{\theta}_{i} - \mathbf{q}_{i}] &= \boldsymbol{\tau}_{i} \end{split}$$
(1b)

where  $\mathbf{q}_i \in \mathbb{R}^n$  are the link angular positions and  $oldsymbol{ heta}_i \in$  $\mathbb{R}^n$  are the joint (motor) angular positions.  $\mathbf{M}_i(\mathbf{q}_i) \in \mathbb{R}^{n imes n}$ are the inertia matrices,  $\mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i) \in \mathbb{R}^{n \times n}$  are the Coriolis and centrifugal effects matrices, defined via the Christoffel symbols of the first kind,  $\mathbf{J}_i \in \mathbb{R}^{n \times n}$  are symmetric and positive definite matrices corresponding to the motor inertia at the joints,  $\mathbf{S}_i \in \mathbb{R}^{n \times n}$  are symmetric and positive definite matrices that contain the joint stiffness and  $\boldsymbol{\tau}_i \in \mathbb{R}^n$  are the control inputs that will be defined later. The subindex  $i \in \overline{N} := \{1, ..., N\}.$ 

Dynamics (1) possess some important and well-known properties [25], [26]:

P1. 
$$\mathbf{M}_i(\mathbf{q}_i) = \mathbf{M}_i^{\top}(\mathbf{q}_i) \text{ and, } \forall \mathbf{q}_i \in \mathbb{R}^n,$$
  
 $0 < \lambda_{mi} \{ \mathbf{M}_i(\mathbf{q}_i) \} \mathbf{I} \le \mathbf{M}_i(\mathbf{q}_i) \le \lambda_{Mi} \{ \mathbf{M}_i(\mathbf{q}_i) \} \mathbf{I} < \infty$ 

P2. 
$$\forall \mathbf{x} \in \mathbb{R}^n, \, \mathbf{x}^{\top} [\mathbf{M}_i(\mathbf{q}_i) - 2\mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)] \mathbf{x} = 0.$$

- P3.  $\exists k_{ci} \in \mathbb{R}_{>0}$ :  $|\mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)\dot{\mathbf{q}}_i| \leq k_{ci}|\dot{\mathbf{q}}_i|^2$ . P4.  $\exists k_{gi} \in \mathbb{R}_{>0}$ :  $\left\|\frac{\partial \mathbf{g}_i(\mathbf{q}_i)}{\partial \mathbf{q}_i}\right\| \leq k_{gi}$ . Hence,  $\forall \mathbf{q}_{i1}, \mathbf{q}_{i2} \in \mathbb{R}^n, |\mathbf{g}_i(\mathbf{q}_{i1}) \mathbf{g}_i(\mathbf{q}_{i2})| \leq k_{gi}|\mathbf{q}_{i1} \mathbf{q}_{i2}|$ .
- P5. If  $\dot{\mathbf{q}}_i, \ddot{\mathbf{q}}_i \in \mathcal{L}_{\infty}$  then  $\frac{d}{dt} \mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)$  is a bounded operator.

## B. Network Model

The manipulators exchange information over a network described by a *directed* and *weighted* interconnection graph, where each manipulator (agent) is a node of the graph. The information exchange between the i-th and the j-th agent is subject to a variable time delay, denoted  $T_{ij}(t) \ge 0$ . With regards to the interconnection graph we assume the following A1. The interconnection graph is connected, that is, the

graph has a directed spanning tree.

The Laplacian matrix  $\mathbf{L} \in \mathbb{R}^{N \times N}$  that models the interconnection is given by

$$\ell_{ij} = \begin{cases} \sum_{j \in \mathcal{N}_i} \omega_{ij} & i = j \\ -\omega_{ij} & i \neq j \end{cases},$$
(2)

with  $\omega_{ij} > 0$  if  $j \in \mathcal{N}_i$ , and  $\omega_{ij} = 0$  otherwise, with  $\mathcal{N}_i$  the set of agents transmitting information to the *i*-th agent.  $\omega_{ii}$ represents the interconnection weights between the *i*-th and the *j*-th agent.

Note that, by construction,  $L1_N = 0$ . Moreover, Assumption A1 ensures that rank(L) = N - 1, that L has a single zero-eigenvalue, that the rest of the spectrum of L has positive real parts and that exists  $\beta = \operatorname{col}(\beta_i) \in \mathbb{R}^N$ such that  $\boldsymbol{\beta}^{\top} \mathbf{L} = \mathbf{0}$  with  $\beta_i > 0$ .

#### III. PROPOSED CONTROLLER

In the rest of the paper, it is assumed that the time-delays satisfy the following:

The time-delays do not grow or decrease faster than A2. time itself, hence  $-1 < \dot{T}_{ij}(t) < 1$ , and the higher order derivatives are bounded. Further, there exist  $\bar{\mu}_{ij}, \underline{\mu}_{ij} \in \mathbb{R}_{>0}$  such that  $-1 < -\underline{\mu}_{ij} \leq \dot{T}_{ij}(t) \leq \bar{\mu}_{ij} < 1$ .

Assumption A2 is a mild assumption that is widely used in the control of systems with variable delays [8], [27], [28].

Using the nice energy shaping idea of Tomei [21], latter generalized in [29], let us define

$$\boldsymbol{\theta}_{di} := \mathbf{q}_d + \mathbf{S}_i^{-1} \mathbf{g}_i(\mathbf{q}_d) \tag{3}$$

and note that  $\mathbf{S}_i(\boldsymbol{\theta}_{di} - \mathbf{q}_d) - \mathbf{g}_i(\mathbf{q}_d) = \mathbf{0}$ . Using this new desired position for the actuated state, model (1) can be written as

$$\mathbf{M}_{i}\ddot{\mathbf{q}}_{i} + \mathbf{C}_{i}\dot{\mathbf{q}}_{i} + \mathbf{S}_{i}(\mathbf{q}_{i} - \mathbf{q}_{d} - \boldsymbol{\theta}_{i} + \boldsymbol{\theta}_{di}) + \mathbf{g}_{i}(\mathbf{q}_{i}) = \mathbf{g}_{i}(\mathbf{q}_{d})$$
$$\mathbf{J}_{i}\ddot{\boldsymbol{\theta}}_{i} + \mathbf{S}_{i}(\boldsymbol{\theta}_{i} - \boldsymbol{\theta}_{di}) - \mathbf{S}_{i}(\mathbf{q}_{i} - \mathbf{q}_{d}) + \mathbf{g}_{i}(\mathbf{q}_{d}) = \boldsymbol{\tau}_{i}.$$
(4)

Let us now state the main control objective of this work.

# A. Control Objective

The control objective is to regulate the interconnected network at a desired link position. That is, find the controllers  $\boldsymbol{\tau}_i$  such that for any desired link position  $\mathbf{q}_d \in \mathbb{R}^n$  and  $\forall i \in \overline{N}$ ,  $\lim \mathbf{q}_i(t) = \mathbf{q}_d$  and  $\lim \dot{\mathbf{q}}_i(t) = \lim \dot{\boldsymbol{\theta}}_i(t) = \mathbf{0}$ .

The solution to this problem is given by two different controllers, one that assumes that joint velocities are available while the other assumes that only joint positions are measurable.

#### B. Case when Joint Velocities are Available

The controllers are given by the Tomei's scheme plus a velocity interconnection term, that is

$$\boldsymbol{\tau}_{i} = - \mathbf{B}_{i} \dot{\boldsymbol{\theta}}_{i} - \mathbf{K}_{i} (\boldsymbol{\theta}_{i} - \boldsymbol{\theta}_{di}) + \mathbf{g}_{i} (\mathbf{q}_{d})$$
(5)  
$$- a_{i} \sum_{j \in \mathcal{N}_{i}} \omega_{ij} \left( \dot{\boldsymbol{\theta}}_{i} - \gamma_{ij} \dot{\boldsymbol{\theta}}_{j} (t - T_{ij}(t)) \right)$$

where  $\mathbf{B}_i, \mathbf{K}_i \in \mathbb{R}^{n \times n}$  are diagonal and positive definite matrices,  $a_i, \omega_{ij} \in \mathbb{R}_{>0}$  and  $\gamma_{ij}^2 = 1 - T_{ij}(t)$ . If time delays increase or decrease, then the time-varying gains  $\gamma_{ij}$  dissipate the energy generated by the communications.

The equilibrium point  $(\hat{\theta}_i, \hat{\theta}_i) = (\mathbf{\ddot{q}}_i, \mathbf{\dot{q}}_i) = (\mathbf{0}, \mathbf{0})$  of (4), in closed-loop with (5), fulfils

$$\begin{aligned} \mathbf{S}_{i}(\mathbf{q}_{i} - \mathbf{q}_{d}) - \mathbf{S}_{i}(\boldsymbol{\theta}_{i} - \boldsymbol{\theta}_{di}) + \mathbf{g}_{i}(\mathbf{q}_{i}) - \mathbf{g}_{i}(\mathbf{q}_{d}) &= \mathbf{0} \\ (\mathbf{S}_{i} + \mathbf{K}_{i})(\boldsymbol{\theta}_{i} - \boldsymbol{\theta}_{di}) - \mathbf{S}_{i}(\mathbf{q}_{i} - \mathbf{q}_{d}) &= \mathbf{0} \end{aligned}$$

Defining  $\mathbf{x}_i := \operatorname{col}(\mathbf{q}_i, \boldsymbol{\theta}_i)$  and  $\mathbf{x}_{di} := \operatorname{col}(\mathbf{q}_d, \boldsymbol{\theta}_{di})$ , this equilibrium point can be further written as

$$\mathcal{K}_i(\mathbf{x}_i - \mathbf{x}_{di}) = \begin{bmatrix} \mathbf{g}_i(\mathbf{q}_d) - \mathbf{g}_i(\mathbf{q}_i) \\ \mathbf{0} \end{bmatrix}$$
  
where  $\mathcal{K}_i := \begin{bmatrix} \mathbf{S}_i & -\mathbf{S}_i \\ -\mathbf{S}_i & \mathbf{S}_i + \mathbf{K}_i \end{bmatrix}$ .

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Using the same procedure as in [21], it can be proved that, using Property P4 and setting  $\lambda_m \{\mathcal{K}_i\} > k_{gi}$ , the only equilibrium point is at  $\mathbf{x}_i \equiv \mathbf{x}_{di}$ . If it can be proved that this equilibrium is globally asymptotically stable (GAS) then the control objective is fulfilled.

Proposition 1: Consider a network of N flexible-joints manipulators of the form (1). Suppose that the motor velocities are measurable. Then, setting  $\lambda_m{\{\mathcal{K}_i\}} > k_{gi}$  ensures that the equilibrium point  $\dot{\mathbf{x}}_i \equiv \mathbf{0}$  and  $\mathbf{x}_i \equiv \mathbf{x}_{di}$  is GAS for any interconnection network and any time-delays for which Assumptions A1 and A2 hold, respectively.

*Proof:* Consider the following Lyapunov-Krasovskii functional

$$V_i = W_i + a_i \sum_{j \in \mathcal{N}_i} \omega_{ij} \int_{t-T_{ij}(t)}^t |\dot{\boldsymbol{\theta}}_j(\sigma)|^2 d\sigma$$

where

$$W_{i} := \frac{1}{2} \left[ \dot{\boldsymbol{\theta}}_{i}^{\top} \mathbf{J}_{i} \dot{\boldsymbol{\theta}}_{i} + \dot{\mathbf{q}}_{i}^{\top} \mathbf{M}_{i}(\mathbf{q}_{i}) \dot{\mathbf{q}}_{i} \right] + P_{i}(\mathbf{x}_{i}, \mathbf{x}_{di})$$
(7)

and

$$P_i := \frac{1}{2} (\mathbf{x}_i - \mathbf{x}_{di})^\top \mathcal{K}_i (\mathbf{x}_i - \mathbf{x}_{di}) + U_i (\mathbf{q}_i) - \mathbf{q}_i^\top \mathbf{g}_i (\mathbf{q}_d),$$
(8)

 $U_i(\mathbf{q}_i)$  is the potential energy due to the gravity and  $\mathbf{g}_i(\mathbf{q}_i) = \frac{\partial U_i(\mathbf{q}_i)}{\partial \mathbf{q}_i}$ . Using Property P5 and the fact that  $\lambda_m \{\mathcal{K}_i\} > k_{gi}$  ensures that  $P_i$  has only one global minimum at  $\mathbf{x}_i \equiv \mathbf{x}_{di}$  (see [21] for the proof). Hence  $V_i$  is positive semi-definite and radially unbounded with regards to  $\dot{\boldsymbol{\theta}}_i, \dot{\mathbf{q}}_i, \mathbf{x}_i - \mathbf{x}_{di}$ .

After some algebraic manipulations, the time derivative of  $V_i$  evaluated at (1) and (5) is

$$\dot{V}_{i} = - \dot{\boldsymbol{\theta}}_{i}^{\top} \left[ \mathbf{B}_{i} \dot{\boldsymbol{\theta}}_{i} + a_{i} \sum_{j \in \mathcal{N}_{i}} \omega_{ij} \left( \dot{\boldsymbol{\theta}}_{i} - \gamma_{ij} \dot{\boldsymbol{\theta}}_{j}(t - T_{ij}(t)) \right) \right] \\ + \frac{a_{i}}{2} \sum_{j \in \mathcal{N}_{i}} \omega_{ij} \left( |\dot{\boldsymbol{\theta}}_{j}|^{2} - \gamma_{ij}^{2} |\dot{\boldsymbol{\theta}}_{j}(t - T_{ij}(t))|^{2} \right).$$

where  $\gamma_{ij}^2 = 1 - \dot{T}_{ij}(t)$ . Gathering terms,  $\dot{V}_i$  can be reduced to

$$\dot{V}_{i} = - \dot{\boldsymbol{\theta}}_{i}^{\top} \mathbf{B}_{i} \dot{\boldsymbol{\theta}}_{i} - \frac{a_{i}}{2} \sum_{j \in \mathcal{N}_{i}} \omega_{ij} |\dot{\boldsymbol{\theta}}_{i} - \gamma_{ij} \dot{\boldsymbol{\theta}}_{j} (t - T_{ij})|^{2} - \frac{a_{i}}{2} \sum_{j \in \mathcal{N}_{i}} \omega_{ij} \left( |\dot{\boldsymbol{\theta}}_{i}|^{2} - |\dot{\boldsymbol{\theta}}_{j}|^{2} \right).$$

Further, note that  $\sum_{i=1}^{N} \beta_i \sum_{j \in \mathcal{N}_i} \left( |\dot{\boldsymbol{\theta}}_i|^2 - |\dot{\boldsymbol{\theta}}_j|^2 \right) = \boldsymbol{\beta}^\top \mathbf{L} \mathbf{F}$ , where  $\mathbf{F} := \operatorname{col}(|\dot{\boldsymbol{\theta}}_1|^2, \dots, |\dot{\boldsymbol{\theta}}_N|^2)$ . Hence, defining  $V = \sum_{i=1}^{N} \frac{\beta_i}{a_i} V_i$ , yields

$$\dot{V} = -\sum_{i=1}^{N} \frac{\beta_i}{a_i} \left( \mathbf{B}_i |\dot{\boldsymbol{\theta}}_i|^2 + \frac{a_i}{2} \sum_{j \in \mathcal{N}_i} \omega_{ij} |\dot{\boldsymbol{\theta}}_i - \gamma_{ij} \dot{\boldsymbol{\theta}}_j (t - T_{ij}(t))|^2 \right)$$

where the properties  $\boldsymbol{\beta}^{\top} \mathbf{L} = \mathbf{0}$  and  $\beta_i > 0$ , from Section II-B, have been used.

Since 
$$V \geq 0$$
 and  $\dot{V} \leq 0$ ,  $\dot{\boldsymbol{\theta}}_i, |\dot{\boldsymbol{\theta}}_i - \gamma_{ij}\dot{\boldsymbol{\theta}}_j(t - T_{ij}(t))| \in \mathcal{L}_2$ 

<sup>1</sup>Note that it is not possible to invoke set invariant arguments (LaSalle) because V is a time dependent Krasovskii functional. To prove convergence the rest of the proof follows Barbalăt's Lemma arguments.

and  $\dot{\boldsymbol{\theta}}_i, \dot{\mathbf{q}}_i, |\mathbf{x}_i - \mathbf{x}_{di}| \in \mathcal{L}_{\infty}$  for all  $i \in \overline{N}$  and  $j \in \mathcal{N}_i$ . This last implies, from (1) and (5), that  $\ddot{\boldsymbol{\theta}}_i \in \mathcal{L}_{\infty}$  which, together with  $\dot{\boldsymbol{\theta}} \in \mathcal{L}_2 \cap \mathcal{L}_{\infty}$ , in turn supports the asymptotic convergence to zero of  $\dot{\boldsymbol{\theta}}_i$ , i.e.,  $\lim_{t \to \infty} \dot{\boldsymbol{\theta}}_i(t) = \mathbf{0}$ . Moreover, since  $\dot{\mathbf{q}}_i, \dot{\boldsymbol{\theta}}_i, \ddot{\boldsymbol{\theta}}_i \in \mathcal{L}_{\infty}$ , it can be established that  $\frac{d}{dt} \ddot{\boldsymbol{\theta}}_i \in \mathcal{L}_{\infty}$ , which implies that  $\ddot{\boldsymbol{\theta}}_i$  is uniformly continuous. This, along with the existence and boundedness of the following limit  $\lim_{t\to\infty} \int_0^t \ddot{\boldsymbol{\theta}}_i(\sigma) d\sigma = -\dot{\boldsymbol{\theta}}_i(0) < \infty$ , proves that  $|\ddot{\boldsymbol{\theta}}_i| \to 0$  as  $t \to \infty$ .

On the other hand, boundedness of  $\dot{\mathbf{q}}_i$  and  $|\mathbf{x}_i - \mathbf{x}_{di}|$  imply, from (1a), that  $\ddot{\mathbf{q}}_i \in \mathcal{L}_{\infty}$ . Differentiating (1b), in closed–loop with (5), yields

$$\frac{d}{dt}\ddot{\boldsymbol{\theta}}_{i} = - \mathbf{J}_{i}^{-1} \left[ \mathbf{S}_{i}[\dot{\boldsymbol{\theta}}_{i} - \dot{\mathbf{q}}_{i}] + \mathbf{B}_{i}\ddot{\boldsymbol{\theta}}_{i} + \mathbf{K}_{i}\dot{\boldsymbol{\theta}}_{i} \right] - \\
- a_{i}\mathbf{J}_{i}^{-1} \sum_{j \in \mathcal{N}_{i}} \omega_{ij} \left( \ddot{\boldsymbol{\theta}}_{i} - \gamma_{ij}^{2}\ddot{\boldsymbol{\theta}}_{j}(t - T_{ij}(t)) \right) - \\
- a_{i}\mathbf{J}_{i}^{-1} \sum_{j \in \mathcal{N}_{i}} \omega_{ij} \frac{d\gamma_{ij}}{dt} \dot{\boldsymbol{\theta}}_{j}(t - T_{ij}(t)).$$

Since all the signals in the right hand side converge to zero, except  $\dot{\mathbf{q}}_i$ , the proof that  $|\dot{\mathbf{q}}_i| \to 0$  is established if it can be proved that  $|\frac{d}{dt}\ddot{\boldsymbol{\theta}}_i| \to 0$ . For, it suffices to prove that  $\frac{d^2}{dt^2}\ddot{\boldsymbol{\theta}}_i \in \mathcal{L}_{\infty}$ . Indeed, the fact that  $\ddot{\mathbf{q}}_i, \frac{d}{dt}\ddot{\boldsymbol{\theta}}_i, \ddot{\boldsymbol{\theta}}_i \in \mathcal{L}_{\infty}$  and Assumption A2 ensure that  $\frac{d^2}{dt^2}\ddot{\boldsymbol{\theta}}_i \in \mathcal{L}_{\infty}$ , as needed. Thus  $\lim_{t\to\infty} \dot{\mathbf{q}}_i(t) = \mathbf{0}$ .

Finally, to prove that  $|\mathbf{\ddot{q}}_i| \to 0$ , recall that with Properties P1, P3, P4, P5 and boundedness of  $\mathbf{\ddot{q}}_i, \mathbf{\dot{q}}_i, \mathbf{\dot{\theta}}_i$  and  $|\mathbf{x}_i - \mathbf{x}_{di}|$  it can be shown that  $\frac{d}{dt}\mathbf{\ddot{q}}_i \in \mathcal{L}_{\infty}$ . On the other hand,  $|\mathbf{\dot{q}}_i| \to 0$ implies that  $\lim_{t\to\infty} \int_0^t \mathbf{\ddot{q}}_i(\sigma)d\sigma$  exists and it is finite. Hence,  $|\mathbf{\ddot{q}}_i| \to 0$ , as required. This completes the proof.

C. Case when only Joint Positions are Available

$$\boldsymbol{\tau}_{i} = -\mathbf{K}_{i}(\boldsymbol{\theta}_{i} - \boldsymbol{\theta}_{di}) + \mathbf{g}_{i}(\mathbf{q}_{d}) - \mathbf{B}_{i}\dot{\mathbf{y}}_{i}$$
(9)

where  $\dot{\mathbf{y}}_i \in \mathbb{R}^n$  are joint velocity filters defined as

$$\dot{\mathbf{y}}_{i} = -\mathbf{D}_{i}\mathbf{y}_{i} + \mathbf{B}_{i}\boldsymbol{\theta}_{i} - a_{i}\sum_{j\in\mathcal{N}_{i}}\omega_{ij}\left(\mathbf{y}_{i} - \mathbf{y}_{j}(t - T_{ij}(t))\right)$$
(10)

where  $\mathbf{D}_i \in \mathbb{R}^{n \times n}$  are diagonal and positive definite matrices and  $a_i, \omega_{ij} \in \mathbb{R}_{>0}$ . In this case

$$\ddot{\mathbf{y}}_{i} = -\mathbf{D}_{i}\dot{\mathbf{y}}_{i} + \mathbf{B}_{i}\dot{\boldsymbol{\theta}}_{i} - a_{i}\sum_{j\in\mathcal{N}_{i}}\omega_{ij}\left(\dot{\mathbf{y}}_{i} - \gamma_{ij}^{2}\dot{\mathbf{y}}_{j}(t - T_{ij}(t))\right).$$
(11)

Note that, for the closed-loop system (1) and (9), the equilibrium point  $(\ddot{\boldsymbol{\theta}}_i, \dot{\boldsymbol{\theta}}_i) = (\ddot{\mathbf{q}}_i, \dot{\mathbf{q}}_i) = (\mathbf{0}, \mathbf{0})$  and  $\dot{\mathbf{y}}_i = \mathbf{0}$  coincides with (6).

The following proposition states the main result of this work and proves that the equilibrium point of the closed–loop system (1) and (9) is GAS.

Proposition 2: Consider a network of N flexible joint manipulators of the form (1). Assume that joint velocities are not available. Then, setting  $\lambda_m \{\mathcal{K}_i\} > k_{gi}$  and

$$\lambda_m \{ \mathbf{D}_i \} > \frac{a_i}{2} \sum_{j \in \mathcal{N}_i} \omega_{ij} \underline{\mu}_{ij}$$

ensures that the equilibrium point  $\dot{\mathbf{y}}_i = \mathbf{0}$ ,  $\dot{\mathbf{x}}_i \equiv \mathbf{0}$  and  $\mathbf{x}_i \equiv \mathbf{x}_{di}$  is GAS for any time-delays and any interconnection network for which Assumptions A1 and A2 hold, respectively.

*Proof:* Consider the function  $W_i$  defined in (7) that is positive definite and radially unbounded with regards to  $\dot{\boldsymbol{\theta}}_i, \dot{\mathbf{q}}_i, \mathbf{x}_i - \mathbf{x}_{di}$ . In this case,  $\dot{W}_i$  evaluated along (1) and (9) yields  $\dot{W}_i = -\dot{\boldsymbol{\theta}}_i^\top \mathbf{B}_i \dot{\mathbf{y}}_i$ .

Now, let us propose the following Lyapunov-Krasovskii functional

$$U_i = \frac{1}{2} |\dot{\mathbf{y}}_i|^2 + \frac{a_i}{2} \sum_{j \in \mathcal{N}_i} \omega_{ij} \int_{t-T_{ij}(t)}^t |\dot{\mathbf{y}}_j(\sigma)|^2 d\sigma$$

which is positive semi-definite and radially unbounded with regards to  $\dot{\mathbf{y}}_i$ . Its time-derivative along (10) and using (11) yields

$$\begin{split} \dot{U}_{i} &= - \dot{\mathbf{y}}_{i}^{\top} \left[ \mathbf{D}_{i} \dot{\mathbf{y}}_{i} - \mathbf{B}_{i} \dot{\boldsymbol{\theta}}_{i} \right] - \\ &- a_{i} \sum_{j \in \mathcal{N}_{i}} \omega_{ij} \left( |\dot{\mathbf{y}}_{i}|^{2} - \gamma_{ij}^{2} \dot{\mathbf{y}}_{i}^{\top} \dot{\mathbf{y}}_{j}(t - T_{ij}(t)) \right) + \\ &+ \frac{a_{i}}{2} \sum_{j \in \mathcal{N}_{i}} \omega_{ij} \left( |\dot{\mathbf{y}}_{j}|^{2} - \gamma_{ij}^{2} |\dot{\mathbf{y}}_{j}(t - T_{ij}(t))|^{2} \right) \\ &= - \dot{\mathbf{y}}_{i}^{\top} \left[ \mathbf{D}_{i} \dot{\mathbf{y}}_{i} - \mathbf{B}_{i} \dot{\boldsymbol{\theta}}_{i} \right] - \frac{a_{i}}{2} \sum_{j \in \mathcal{N}_{i}} \omega_{ij} (1 - \gamma_{ij}^{2}) |\dot{\mathbf{y}}_{i}|^{2} \\ &- \frac{a_{i}}{2} \sum_{j \in \mathcal{N}_{i}} \omega_{ij} \gamma_{ij}^{2} |\dot{\mathbf{y}}_{i} - \dot{\mathbf{y}}_{j}(t - T_{ij}(t))|^{2} - \\ &- \frac{a_{i}}{2} \sum_{j \in \mathcal{N}_{i}} \omega_{ij} \left( |\dot{\mathbf{y}}_{i}|^{2} - |\dot{\mathbf{y}}_{j}|^{2} \right) \end{split}$$

where to obtain the second equation the term  $\frac{a_i}{2} \sum_{j \in \mathcal{N}_i} \omega_{ij} \gamma_{ij}^2 |\mathbf{\dot{y}}_i|^2$  has been added and subtracted to complete the square.

Defining  $W = \sum_{i=1}^{N} \frac{\beta_i}{a_i} (W_i + U_i)$  and noting that  $(1 - \gamma_{ij}^2) = \dot{T}_{ij}(t)$  yields

$$\dot{W} = - \sum_{i=1}^{N} \frac{\beta_i}{a_i} \left( \dot{\mathbf{y}}_i^{\top} \mathbf{D}_i \dot{\mathbf{y}}_i + \frac{a_i}{2} \sum_{j \in \mathcal{N}_i} \omega_{ij} \dot{T}_{ij}(t) |\dot{\mathbf{y}}_i|^2 \right) - \sum_{i=1}^{N} \frac{\beta_i}{2} \sum_{j \in \mathcal{N}_i} \omega_{ij} \gamma_{ij}^2 |\dot{\mathbf{y}}_i - \dot{\mathbf{y}}_j(t - T_{ij}(t))|^2 - \sum_{i=1}^{N} \frac{\beta_i}{2} \sum_{j \in \mathcal{N}_i} \omega_{ij} \left( |\dot{\mathbf{y}}_i|^2 - |\dot{\mathbf{y}}_j|^2 \right).$$

Similar to the previous proof  $\sum_{i=1}^{N} \beta_i \sum_{j \in \mathcal{N}_i} \left( |\dot{\mathbf{y}}_i|^2 - |\dot{\mathbf{y}}_j|^2 \right) =$ 

 $\boldsymbol{\beta}^{\top} \mathbf{L} \mathbf{Y} = 0$ , for  $\mathbf{Y} := \operatorname{col}(|\dot{\mathbf{y}}_1|^2, \dots, |\dot{\mathbf{y}}_N|^2)$ . This last and Assumption A2, which ensures that  $\underline{\mu}_{ij} \leq \dot{T}_{ij}(t)$ , allow to write  $\dot{W}$  as

$$\dot{W} \leq -\sum_{i=1}^{N} \frac{\beta_i}{a_i} \left( \delta_i |\dot{\mathbf{y}}_i|^2 + \frac{a_i}{2} \sum_{j \in \mathcal{N}_i} \omega_{ij} \gamma_{ij}^2 |\dot{\mathbf{y}}_i - \dot{\mathbf{y}}_j(t - T_{ij}(t))|^2 \right).$$



Fig. 1. Experimental setup, composed by three PPHANTom Omni devices.

where  $\delta_i := \lambda_m \{ \mathbf{D}_i \} - \frac{a_i}{2} \sum_{j \in \mathcal{N}_i} \omega_{ij} \underline{\mu}_{ij}.$ 

Clearly, selecting  $\lambda_m \{ \mathbf{D}_i \}$  such that  $\delta_i > 0$  ensures that  $\dot{W} \leq 0$ . This last and the fact that  $W \geq 0$  ensure that  $\dot{\mathbf{y}}_i \in \mathcal{L}_2$  and that  $\dot{\boldsymbol{\theta}}_i, \dot{\mathbf{q}}_i, \mathbf{x}_i - \mathbf{x}_{di}, \dot{\mathbf{y}}_i \in \mathcal{L}_\infty$ . These bounded signals imply, from (11), that  $\ddot{\mathbf{y}}_i \in \mathcal{L}_\infty$ . Hence, Barbalǎt's Lemma supports that  $\lim_{t\to\infty} \dot{\mathbf{y}}_i = \mathbf{0}$ . Further,  $\dot{\mathbf{q}}_i, \mathbf{x}_i - \mathbf{x}_{di}, \dot{\mathbf{y}}_i \in \mathcal{L}_\infty$  ensures, from (1) and (9), that  $\ddot{\mathbf{q}}_i \in \mathcal{L}_\infty$ . This last, Assumption A2 and the fact that  $\ddot{\mathbf{y}}_i \in \mathcal{L}_\infty$  supports that  $\frac{d}{dt}\ddot{\mathbf{y}}_i \in \mathcal{L}_\infty$ . Moreover, since

$$\lim_{t \to \infty} \int_0^t \ddot{\mathbf{y}}_i(t) = \lim_{t \to \infty} \dot{\mathbf{y}}_i(t) - \dot{\mathbf{y}}_i(0) = -\dot{\mathbf{y}}_i(0)$$

it is proved that  $|\mathbf{\ddot{y}}_i| \to 0$  and, from (11), we also have that  $|\mathbf{\dot{\theta}}_i| \to 0$  as  $t \to \infty$ . The rest of the proof relies on the same arguments as in the proof of Proposition 1.

# **IV. EXPERIMENTS**

This section, shows the effectiveness of the proposed schemes through some experimental results. The setup is composed by 3-DOF manipulators. These devices are the PHANToM Omni, from Sensable Technologies (*http://sensable.com/*). Two devices (Agents 1 and 2) run in the same computer and the other (Agent 3) is connected through the Internet, as can be observed in Fig. 1. The controller and all software is implemented in Matlab Simulink. The computer-device communication, have been implemented using the PHANSIM Libraries and the communications over the Internet with the blocks *UDP send* and *UDP receive*, from the *Instrument Control Toolbox*. Since the PHANTom Omni devices have 3 fully actuated DOF, this paper emulates a flexible-joint behavior by using the scheme in Fig. 2.

The different network weights are  $\omega_{12} = 0.5, \omega_{13} = 0.6, \omega_{21} = 0.9, \omega_{23} = 0.4, \omega_{31} = 0.95$  and  $\omega_{32} = 0.8$ , and the Laplacian matrix of the experiments is

$$\mathbf{L} = \begin{bmatrix} 1.1 & -0.5 & -0.6\\ -0.9 & 1.3 & -0.4\\ -0.95 & -0.8 & 1.75 \end{bmatrix}$$



Fig. 2. Emulated flexible-joint manipulator.

The induced time-delays between Agents 1 and 2 are given by  $T_{12} = 0.02 \sin(10t) + 0.05 \sin(15t) + 0.35$  and  $T_{21} = 0.02 \sin(25t) + 0.05 \sin(5t) + 0.30$  (Fig. 3). The bound of their time-derivative is lower than one.



Fig. 3. Induced-time delay between Agents 1 and 2.

The corresponding gravity vector is given by,

$$\mathbf{g}_i(\mathbf{q}_i) = \begin{bmatrix} 0\\ \theta_1 g \sin(q_2 + q_3) + \theta_2 g \cos(q_2)\\ \theta_1 g \sin(q_2 + q_3) \end{bmatrix}$$

where, g = 9.81 is the acceleration of gravity constant,  $\theta_1 =$  $m_3l_2$  and  $\theta_2 = m_3l_2 + m_2l_1$ . These physical values have been experimentally estimated yielding  $\theta_1 = 0.0077$  and  $\theta_2 =$ 0.005. In this case, matrix  $J_i$  corresponds to the motor inertia 0.25 0 0 and it has been set to  $\mathbf{J}_i =$ 0 0.150 , matrix  $\mathbf{S}_i$ 0 0 0.1stands for the manipulator joint stiffness and it is given by  $\mathbf{S}_i = 0.9\mathbf{I}_3$ . The desired position is  $\mathbf{q}_* = \begin{bmatrix} 0 & 0.45 & 0.3 \end{bmatrix}$ rad.

In order to ensure the existence of a unique solution of the equilibria, the proportional gains  $\mathbf{K}_i$  have been set such that  $\lambda_m\{\mathcal{K}_i\} > \left\|\frac{\partial \mathbf{g}_i(\mathbf{q}_i)}{\partial \mathbf{q}_i}\right\|$ . In this case, for manipulators 1, 2 and 3,  $\left\|\frac{\partial \mathbf{g}_i(\mathbf{q}_i)}{\partial \mathbf{q}_i}\right\| \le 0.1795$ . Easy calculations show that the controllers gains satisfy  $\lambda_m\{\mathcal{K}_i\} > k_{gi}$  with  $\mathbf{K}_i = \mathbf{I}\mathbf{I}_3$ . The first set of experiments shows the results for controller (5) when velocities are measurable. In this case  $\mathbf{B}_1 = \mathbf{B}_2 = 0.3$  and  $\mathbf{B}_3 = 0.2$ . Fig. 4 shows the link and joint position performance when the interconnection gain  $a_i = 0$  and Fig. 5 for the case when  $a_i = 0.5$ . Despite that all robots start from different initial conditions, in both cases, the link positions asymptotically converge to the desired one. Note that by adding information exchange between the robots performance improves by reducing the convergence time.



Fig. 4. Link and joint positions for the case when velocities are measurable and with  $a_i = 0$  ( $\mathbf{q}_d$  in dotted line).



Fig. 5. Link and joint positions for the case when velocities are measurable and with  $a_i = 0.5$  ( $\mathbf{q}_d$  in dotted line).

The second set of experiments aims at demonstrating that consensus is also achieved when only joint positions are available for measurement. The controller is given by (9). The control gains have been set to  $\mathbf{B}_i = 2\mathbf{I}_3$ ,  $\mathbf{K}_i = 3\mathbf{I}_3$ and  $\mathbf{D}_i = 10\mathbf{I}_3$ . The gain  $\mathbf{D}_i$  has been set such that  $\lambda_m{\{\mathbf{D}_i\}} - \frac{a_i}{2}\sum_{j\in\mathcal{N}_i}\omega_{ij}\underline{\mu}_{ij} > 0$ . Fig. 6 depicts the link and joint positions behavior when  $a_i = 10$ . Despite that, velocities are not available, all the robots reach the desired consensus position. Moreover, compared with the previous set of simulations -specifically with those in Fig. 5- the performance of the network with controller (9) yields similar responses.



Fig. 6. Performance of the network controlled when only joint positions are available with  $a_i = 10$  ( $\mathbf{q}_d$  in dotted line).

# V. CONCLUSIONS

Two different controllers for networks composed of a class of under actuated mechanical systems have been reported in this work. One controller assumes that joint velocities are measurable while the other does not. The network is modeled as a directed and weighted interconnection graph and the only requirement for the network is to be connected. Further, it is assumed that the interconnection can induce variable time–delays. Both controllers solve the consensus problem to a given constant desired position. Experiments, using a network with three manipulators, confirm the effectiveness of the proposed schemes.

Two avenues guide the future work along this line, the case when the interconnection graph is time-varying and the case when the given desired position is also time-varying.

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