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An Output Feedback Attitude Tracking Controller Design for Quadrotor Unmanned Aerial Vehicles Using Quaternion

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***Abstract*— In this paper, a quaternion based nonlinear output feedback tracking controller is developed to address the attitude and altitude tracking problem of a quadrotor unmanned aerial vehicle (UAV) which is subject to structural uncertainties and unknown external disturbances. A set of filters are introduced to provide estimation for the unmeasurable quadrotor UAV's angular and translational velocity signals. The Lyapunov based stability analysis ensures that a semi-global asymptotic tracking result is achieved and all closed loop states remain bounded with a suitable choice of control gains.**

I. Introduction

As a special micro helicopter, the quadrotor UAV attracts great attention from military and civil applications in recent years. Due to its advantages such as vertical taking off and landing (VTOL), rapid maneuvering and precise hovering, the potential for the quadrotor UAV in applications as diverse as firefighting and environmental monitoring has been well established. Recently several literatures have proposed some new methods for the control of the quadrotor UAV's attitude, but design of nonlinear control mechanisms for quadrotor UAVs in presence of structural uncertainties and unknown external disturbances is still a challenging task.

Because the Euler angle representation always exhibits singularity, hence a lot of efforts have been directed toward the quaternion based controller design. In [1], a modified sliding mode quaternion feedback controller was proposed for the quadrotor UAV and an observer was introduced to estimate the time varying disturbance, which was treated as an unknown state and an uniform ultimate bounded tracking result was achieved. In [2], an output feedback controller based on PD control structure was applied to address the attitude tracking problem of a rigid body in quaternion coordinate space. A low-gain dynamic observer was introduced to provide estimation for the unmeasurable angular velocity and the Lyapunov stability analysis showed that this controller achieved an uniform practical asymptotic tracking result. Two quaternion based adaptive output feedback controllers were introduced in [3] for the attitude control of a spacecraft with uncertain dynamics. A nonlinear reduced-order observer was utilized to estimate the unmeasurable speed signals and a Chebyshev neural network (CNN) was introduced to approximate the spacecraft's motion. The approximation errors and external disturbances were compensated by a hyperbolic tangent function. Both robust adaptive controllers presented in [3] with CNN could guarantee an uniform ultimate bounded tracking result. In [4], the authors proposed a quaternion adaptive neural network controller to address the attitude tracking problem of manipulators, which yielded an uniform ultimate bounded tracking result. In [5], a quaternion hybrid control law was proposed. In the kinematic subsystem, a virtual optimal angular velocity was design. Then for the dynamic subsystem, a finite-time control law was employed to force the actual angular velocity to track the virtual optimal angular velocity. The hybrid controller yielded a global asymptotic set stability. A quaternion feedback controller was introduced in [6] which utilized a nested saturation method for prior input bound, and could force the closed loop trajectory into a prior fixed neighborhood of the origin in a finite time.

Motivated by the controllers presented in [7] and [8], we consider the quadrotor UAV as a rigid body having four degree of freedom (altitude and orientation) which is subject to structural uncertainties and unknown external disturbance. A set of filter signals are developed to estimate the unmeasurable angular and translational velocity signals. A nonlinear robust term was developed to eliminate the structured uncertainties and unknown time-varying disturbance. Compared with other attitude control design, this output feedback controller need less model knowledge and only the output states are available for the control development. The Lyapunov based stability analysis is used to prove that the proposed control mechanism can achieve a semi-global tracking result and all closed loop signals remain bounded.

The remainder of this paper is organized as follows. The dynamic model of a quadrotor UAV is presented in Section 2. Section 3 shows the attitude controller design and stability analysis. The altitude controller is developed and stability analysis is proposed in Section 4. Conclusion and future work are discussed in Section 5.

II. The Quadrotor UAV Model

In this paper, we consider the quadrotor UAV as a rigid body having four degree of freedom in the Cartesian space. Motivated by the need to obtain the dynamic and kinematic model of the quadrotor UAV, two right hand frames are utilized. Let $\mathcal{B}$ be a body-fixed frame located at the center of mass of the quadrotor UAV, denoted by $\mathcal{B} = \left\{ x_{\mathcal{B}} \quad y_{\mathcal{B}} \quad z_{\mathcal{B}} \right\}$. The other frame is the inertial

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frame fixed at some position of the earth, denoted by $\mathcal{I} = \left\{ \begin{array}{ccc} x_I & y_I & z_I \end{array} \right\}$ where z_I is the vertical direction to the earth. The quadrotor UAV system considered in the paper can be modeled via following state space representation, a combination of rotation and altitude motion [9]

$$\begin{array}{c} J\dot{\omega} = S(J\omega)\omega + N(\omega) + \tau + D_1 \\ m\ddot{z} = -k_z\dot{z} - mg + \cos\theta\cos\phi u + D_2 \end{array} \quad (1)$$

where $\omega(t) = [\begin{array}{ccc} \omega_1(t) & \omega_2(t) & \omega_3(t) \end{array}]^T \in \mathbb{R}^3$ denotes the angular velocity of the UAV with respect to the inertial frame $\mathcal{I}$ defined in the body fixed frame $\mathcal{B}$. The matrix $J \in \mathbb{R}^{3\times3}$ in (1) represents the unknown constant, diagonal, positive-definite, inertia matrix. The matrix $S(\cdot)$ in (1) denotes a general form of skew-symmetric matrix defined as follows

$$S(\xi) = \begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix} \quad (2)$$

for $\forall\xi = \left[\begin{array}{ccc} \xi_1 & \xi_2 & \xi_3 \end{array}\right]^T \in \mathbb{R}^3$. The nonlinear aerodynamic damping moment, denoted by $N(\omega) \in \mathbb{R}^3$ in (1), is defined as follows

$$N(\omega) = \begin{bmatrix} g_1 + g_2\left|\omega_1\right| & 0 & 0 \\ 0 & g_3 + g_4\left|\omega_2\right| & 0 \\ 0 & 0 & g_5 + g_6\left|\omega_3\right| \end{bmatrix}\omega \quad (3)$$

where $g_i \in \mathbb{R}$ for $i = 1\cdots6$ is unknown parameter. The input $u(t) \in \mathbb{R}$ in (1) denotes the vertical lift force and $\tau(t) = [\begin{array}{ccc} \tau_1(t) & \tau_2(t) & \tau_3(t) \end{array}]^T \in \mathbb{R}^3$ are three torques in roll, pitch and yaw directions. The external disturbance terms, denoted by $D_1(t) \in \mathbb{R}^3$ and $D_2(t) \in \mathbb{R}$, are defined in the following form

$$\begin{array}{c} D_1(t) = \left[\begin{array}{ccc} d_1(t) & d_2(t) & d_3(t) \end{array}\right]^T \\ D_2(t) = d_4(t) \end{array}. \quad (4)$$

In (1), the constant $m \in \mathbb{R}$ denotes the unknown mass of the quadrotor UAV. The altitude of the quadrotor UAV is denoted by $z(t) \in \mathbb{R}$, and $\phi(t)$, $\theta(t) \in \mathbb{R}$ in (1) represent roll and pitch angles, respectively. The constant $k_z \in \mathbb{R}$ is an unknown aerodynamic damping coefficient, and $g = 9.81m/s^2$ is the acceleration of gravity.

Remark 1: The disturbance signal $d_i(t)$ for $i = 1\cdots4$ in (4) is continuous differentiable and bounded up to its second derivative $d_i(t) \in \mathcal{C}^2$, i.e., $d_i(t)$, $\dot{d}_i(t)$, $\ddot{d}_i(t) \in \mathcal{L}_\infty$.

III. Rotation Subsystem Control Design

A. Quaternion Error Dynamics

According to the Euler's theorem [10], any rotation matrix can be uniquely represented by a rotation angle $\psi(t) \in \mathbb{R}$ about a suitable unit vector $k(t) \in \mathbb{R}^3$ [11] and [12]. Thus, utilizing the algorithm provided in [11], the angle $\psi(t)$ and an unit vector $k(t)$ can be calculated for any given rotation matrix. Given $(\psi, k) \in \mathbb{R}^4$, an alternative parametrization of the attitude is provided by a unit quaternion vector $q(t) = \left[\begin{array}{cc} q_o(t) & q_v^T(t) \end{array}\right]^T \in \mathbb{R}^4$ [10] and [12], which can be utilized to describe the orientation of the body fixed frame $\mathcal{B}$ with respect to the inertial frame $\mathcal{I}$. Specifically, the unit quaternion provides a method to describe the rigid body's attitude without singularity issue, and is defined via the angle-axis parameters as $q(t) = \left[\begin{array}{cc} \cos(\frac{1}{2}\psi(t)) & k^T(t)\sin(\frac{1}{2}\psi(t)) \end{array}\right]^T$. Note that the unit quaternion is subject to the constraint $q^Tq = 1$. It is important to mention that we obtain the following transformation matrix, $R(q) \in SO(3)$, for the Euler parameters [10], [13] and [14]

$$R(q) = (q_o^2 - q_v^Tq_v)I_3 + 2q_vq_v^T - 2q_oS(q_v). \quad (5)$$

Utilizing the fact $RR^T = 1$ and taking the time derivative of $R(q)$, we can obtain the following equation

$$\dot{R} = -S(w)R. \quad (6)$$

The unit quaternion $q(t) = \left[\begin{array}{cc} q_o(t) & q_v^T(t) \end{array}\right]^T \in \mathbb{R}^4$ can be directly derived form (5) and (6), and written as

$$\dot{q}_o = -\tfrac{1}{2}q_v^T\omega \quad \dot{q}_v = \tfrac{1}{2}(q_oI_3 + S(q_v))\omega \quad (7)$$

where I_3 denotes the 3×3 identity matrix. The expression in (7) can be modified as follows

$$\dot{q} = \frac{1}{2}B(q)\omega \quad (8)$$

where the auxiliary term $B(q) = \left[\begin{array}{cc} B_o^T & B_v^T \end{array}\right]^T \in \mathbb{R}^{4\times3}$ with $B_o = -q_v^T \in \mathbb{R}^{1\times3}$ and $B_v = q_oI_3 + S(q_v) \in \mathbb{R}^{3\times3}$.

The object of this section is to design the control input $\tau(t)$ to ensure the attitude tracking for the quadrotor UAV system shown in (1) without the measurement of the angular velocity signals $\omega(t)$. For this purpose, the desired attitude of the quadrotor UAV is represented by a desired body fixed, orthogonal coordinate frame $\mathcal{B}_d$. The corresponding rotation matrix is denoted by $R_d \in SO(3)$. The desired unit quaternion, $q_d(t) = \left[\begin{array}{cc} q_{od}(t) & q_{vd}^T(t) \end{array}\right]^T \in \mathbb{R}^4$, is utilized to described the orientation of $\mathcal{B}_d$. The desired angular velocity, denoted by $\omega_d(t)$, is the angular velocity of the desired body frame $\mathcal{B}_d$ with respect to the inertial frame $\mathcal{I}$ expressed in $\mathcal{B}_d$. The desired rotation matrix $R_d(t)$ can be expressed by $q_d(t)$ as follows

$$R_d(q_d) = (q_{od}^2 - q_{vd}^Tq_{vd})I_3 + 2q_{vd}q_{vd}^T - 2q_{od}S(q_{vd}). \quad (9)$$

The time derivative of $q_d(t)$ is related to the desired angular velocity $\omega_d(t)$ through the following kinematic equation

$$\dot{q}_d = \frac{1}{2}B_d(q_d)\omega_d \quad (10)$$

where the auxiliary term $B_d(q_d) = \left[\begin{array}{cc} B_{od}^T & B_{vd}^T \end{array}\right]^T \in \mathbb{R}^{4\times3}$ with $B_{od} = -q_{vd}^T \in \mathbb{R}^{1\times3}$ and $B_{vd} = q_{od}I_3 + S(q_{vd}) \in \mathbb{R}^{3\times3}$ similar to (8). To quantify the mismatch between the current and desired orientation of the quadrotor UAV, the quaternion tracking error, denoted by $e_q(t) = \left[\begin{array}{cc} e_o(t) & e_v^T(t) \end{array}\right]^T \in \mathbb{R}^4$, is defined as follows

$$\begin{array}{c} e_o = q_oq_{od} + q_v^Tq_{vd} \\ e_v = q_{od}q_v - q_oq_{vd} + S(q_v)q_{vd} \end{array} \quad (11)$$

which also satisfies the constraint

$$e_q^Te_q = 1. \quad (12)$$

We define the rotation matrix, denoted by $\tilde{R} \in SO(3)$, that brings $\mathcal{B}_d$ into $\mathcal{B}$ as follows

$$\tilde{R} = RR_d^T = (e_o^2 - e_v^T e_v)I_3 + 2e_v e_v^T - 2e_o S(e_v). \quad (13)$$

Utilizing the rotation matrix $\tilde{R}$, the desired unit quaternion describing the orientation of the desired body fixed frame $\mathcal{B}_d$ can be expressed in frame $\mathcal{B}$ as follows

$$\begin{aligned} \dot{q}_{od} &= -\tfrac{1}{2} q_{vd}^T \tilde{R}\omega_d \\ \dot{q}_{vd} &= \tfrac{1}{2}(q_{od} I_3 + S(q_{vd}))\tilde{R}\omega_d \end{aligned} . \quad (14)$$

The angular velocity of $\mathcal{B}$ with respect to $\mathcal{B}_d$ expressed in $\mathcal{B}$, denoted by $\tilde{\omega} \in \mathbb{R}^3$, can be written as follows [11]

$$\tilde{\omega} = \omega - \tilde{R}\omega_d. \quad (15)$$

Based on the previous definitions, the quadrotor UAV's attitude tracking objective can be stated as follows

$$\lim_{t\to\infty} e_v(t) = 0 \quad \lim_{t\to\infty} \tilde{R}(t) = I_{3\times 3} \ . \quad (16)$$

From (11) and (13), it can be obtained that if $\lim_{t\to\infty} e_v(t) = 0$ then $\lim_{t\to\infty} \tilde{R}(t) = I_{3\times 3}$.

Remark 2: The desired unit quaternion $q_d(t)$ is selected such that $q_d^{(i)} \in \mathcal{L}_\infty$, for $i = 0,\ 1,\ 2$. It ensures that $\omega_d^{(i)} \in \mathcal{L}_\infty$, for $i = 0,\ 1$.

Remark 3: For the rotation matrix R, R_d and $\tilde{R}$, their time derivative can be calculated utilizing $\dot{R} = -S(\omega)R$, $\dot{R}_d = -S(\omega_d)R_d$ and $\dot{\tilde{R}} = -S(\tilde{\omega})\tilde{R}$, respectively.

To design the attitude tracking controller $\tau(t)$, we will take the time derivative of $e_q(t)$. Refer to [7], the time derivative of $e_o(t)$ and $e_v(t)$ can be expressed as follows

$$\dot{e}_o = -\tfrac{1}{2} e_v^T \tilde{\omega} \quad \dot{e}_v = \tfrac{1}{2}\left[S(e_v) + e_o I_{3\times 3}\right]\tilde{\omega} \quad (17)$$

where the angular velocity $\tilde{\omega}(t)$ has been introduced in (15). The relation between angular velocity $\tilde{\omega}(t)$ and quaternion error $e_q(t)$ can be rewritten via the following kinematic equation

$$\dot{e}_q = \frac{1}{2} B_e(e_q)\tilde{\omega} \quad (18)$$

where $B_e(e_q) = \begin{bmatrix} B_{eo}^T & B_{ev}^T \end{bmatrix}^T \in \mathbb{R}^{4\times 3}$ with $B_{eo} = -e_v^T \in \mathbb{R}^{1\times 3}$ and $B_{ev} = e_o I_3 + S(e_v) \in \mathbb{R}^{3\times 3}$. After taking the time derivative of (15) and multiplying the resulting equation with J, it can be obtained

$$\begin{aligned} J\dot{\tilde{\omega}} &= N(\omega) + \tau + D_1 + J\left[S(\tilde{\omega})\tilde{R}\omega_d - \tilde{R}\dot{\omega}_d\right] \quad (19) \\ &\quad - S(\tilde{\omega} + \tilde{R}\omega_d)J(\tilde{\omega} + \tilde{R}\omega_d) \end{aligned}$$

where (1), (15) and Remark 3 have been utilized. Based on (18), it can be obtained that

$$\dot{e}_v = \frac{1}{2} B_{ev}\tilde{\omega}. \quad (20)$$

To facilitate the development of the tracking controller, two auxiliary matrices, denoted by $J_{ev}(t) \in \mathbb{R}^{3\times 3}$ and $P(t) \in \mathbb{R}^{3\times 3}$, are defined as

$$J_{ev} = B_{ev}^{-T} J B_{ev}^{-1} \quad P = B_{ev}^{-1} \ . \quad (21)$$

Due to the fact that $\det(B_{ev}) = e_o(t)$, the initial value of $e_o(t)$ is not zero and the subsequent control law will be designed to guarantee that $e_o(t) \neq 0$. By taking the time derivative of (20), and multiplying the resulting equation by J_{ev}, the following equation can be obtained

$$J_{ev}\ddot{e}_v = \frac{1}{2} J_{ev}\dot{B}_{ev}\tilde{\omega} + \frac{1}{2} B_{ev}^{-T} J\dot{\tilde{\omega}}. \quad (22)$$

After substituting (19) into (22), the following equation can be obtained

$$J_{ev}\ddot{e}_v + C^*\dot{e}_v + N^* = \frac{1}{2} B_d^{-T} D_1 + \frac{1}{2} B_d^{-T} N(\omega_d) + \tau_{eq} \quad (23)$$

where the auxiliary function $C^*(t) \in \mathbb{R}^{3\times 3}$, $N^*(t) \in \mathbb{R}^3$ and $\tau_{eq}(t) \in \mathbb{R}^3$ are defined as

$$C^* = -J_{ev}\dot{P}^{-1}P - 2P^T S(JP\dot{e}_v)P, \quad (24)$$

$$\begin{aligned} N^* &= -\frac{1}{2} P^T J\left[S(2P\dot{e}_v)\tilde{R}\omega_d - \tilde{R}\dot{\omega}_d\right] + P^T S(P\dot{e}_v)J\tilde{R}\omega_d \\ &+ \frac{1}{2} P^T S(\tilde{R}\omega_d)J\tilde{R}\omega_d + P^T S(\tilde{R}\omega_d)JP\dot{e}_v - \frac{1}{2} P^T D_1 \\ &+ \frac{1}{2} B_d^{-T} D_1 - \frac{1}{2} P^T N(\omega) + \frac{1}{2} B_d^{-T} N(\omega_d), \end{aligned} \quad (25)$$

$$\tau_{eq} = \frac{1}{2} B_{ev}^{-T}\tau = \frac{1}{2} P^T \tau. \quad (26)$$

The two following properties of the dynamics in (23) will be employed in the subsequent controller design and stability analysis [7].

Remark 4: The inertia matrix $J_{ev}(t)$ and centripetal-coriolis matrix $C^*(t)$ satisfy the following skew-symmetric equation

$$\alpha^T(\tfrac{1}{2}\dot{J}_{ev} - C^*)\alpha = 0 \quad \forall \alpha \in \mathbb{R}^3 \ . \quad (27)$$

Remark 5: The inertia matrix $J_{ev}(t)$ is symmetric and positive definite, and satisfies the following inequalities

$$j_1 \left\|\alpha\right\|^2 \leq \alpha^T J_{ev}\alpha \leq j_2 \left\|\alpha\right\|^2 \quad (28)$$

where j_1 and j_2 are some positive constants.

B. Output Feedback Controller Development

In this section, control torque input $\tau(t)$ is designed based on the restriction that the quadrotor UAV's angular velocity $\omega(t)$ is not measurable and the dynamic model parameters in (1) are unknown. The following filters are employed to provide estimation for the unmeasurable angular velocity signals [15]

$$\begin{aligned} \eta &= e_v + \dot{e}_v + e_f \\ e_f &= p - k_2 e_v \\ \dot{p} &= -(k_2 + 1)p + k_2^2 e_v + \tfrac{e_v}{(1 - e_v^T e_v)^2} \end{aligned} \quad (29)$$

where $e_f(t)$, $r_f(t) \in \mathbb{R}^3$ are the outputs of the filters, $p(t) \in \mathbb{R}^3$ is an auxiliary variable used in the filter implementation, $k_2 \in \mathbb{R}$ is a positive constant. The initial value of $p(t)$ is set to be $p(0) = k_2 e_v(0)$. Based on the signals in (29) and subsequent analysis, the output feedback tracking controller $\tau_{eq}(t)$ can be designed as

$$\tau_{eq} = -K_1 sgn(e_v + e_f) + k_2 e_f - \frac{e_v}{(1 - e_v^T e_v)^2} \quad (30)$$

where $K_1 \in \mathbb{R}^{3\times3}$ is a constant, diagonal, positive definite matrix, and the function $sgn(\bullet)$ is defined as

$$sgn(\alpha) = \begin{bmatrix} sgn(\alpha_1) & sgn(\alpha_2) & sgn(\alpha_3) \end{bmatrix}^T \quad (31)$$

$\forall\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}^T$. The actual control torque input $\tau(t)$ can be computed via (26). By taking the time derivative of (29), multiplying the resulting equation by $J_{ev}(t)$, and substituting (23), the following expression can be obtained

$$\begin{aligned} J_{ev}\dot{\eta} &= -C^*\eta - k_2 J_{ev}\eta + \frac{1}{2}B_d D_1 + N_{ev} + \frac{1}{2}B_d N(\omega_d) \\ &+ \tau_{eq} + J\dot{\omega}_d + \frac{1}{2}S(\omega_d)J\omega_d \end{aligned} \quad (32)$$

where the auxiliary function $N_{ev}(t) \in \mathbb{R}^3$ is defined as

$$\begin{aligned} N_{ev} &= C^*(e_f + e_v) + J_{ev}(\eta - e_v + \frac{e_v}{(1 - e_v^T e_v)^2}) \quad (33) \\ &- 2J_{ev}e_f - N^*. \end{aligned}$$

After substitution (30) into (32), the following closed loop system can be obtained

$$\begin{aligned} J_{ev}\dot{\eta} &= -C^*\eta - k_2 J_{ev}\eta + D_1' + N_{ev} + k_2 e_f \quad (34) \\ &- K_1 sgn(e_v + e_f) - \frac{e_v}{(1 - e_v^T e_v)^2} \end{aligned}$$

where the auxiliary function $D_1'(t) \in \mathbb{R}^3$ is defined as

$$D_1' = \frac{1}{2}B_d N(\omega_d) + \frac{1}{2}B_d D_1 + J\dot{\omega}_d + \frac{1}{2}S(\omega_d)J\omega_d. \quad (35)$$

It is not difficult to check that $D_1'(t)$ and $\frac{dD_1'(t)}{dt}$ are bounded.

Remark 6: Based on Remark 1, the fact that the rotation matrixes R, R_d, $\tilde{R}$ are always bounded, and the structure of (29) and (33), it can be shown that [7]

$$\|N_{ev}\| \le \rho(\|z_q\|)\|z_q\| \quad (36)$$

where $z_q(t) \in \mathbb{R}^9$ is defined as

$$z_q = \begin{bmatrix} e_f^T & \eta^T & \frac{e_v^T}{\sqrt{1 - e_v^T e_v}} \end{bmatrix}^T \quad (37)$$

and the positive function $\rho(\|z_q\|)$ is a non-decreasing in $\|z_q\|$.

C. Stability Analysis

Before presenting the main result of the section, a lemma which will be invoked later is stated firstly.

Lemma 1: Let the auxiliary function $L(t) \in \mathbb{R}$ be defined as follows

$$L = \eta^T \left[D_1' - K_1 sgn(e_v + e_f)\right]. \quad (38)$$

If the control gain matrix K_1, introduced in (30), is selected to satisfy the following sufficient condition

$$K_{1i} > \|D_1'(t)\|_\infty + \left\|\frac{d}{d(t)}[D_1'(t)]\right\|_\infty \quad (39)$$

where K_{1i} is the element of matrix K_1, and $\|\bullet\|_\infty$ denotes the infinity norm, then

$$\int_0^t L(\tau)d\tau \le \zeta_b \quad (40)$$

where the positive constant ζ_b is defined as

$$\zeta_b = \textstyle\sum_{i=1}^3 K_{1i}|e_{vi}(0)| - e_v^T(0)D_1'(0). \quad (41)$$

Proof: Please refer to [15]. ■

Theorem 1: The control law proposed in (30) ensures that all the closed loop signals are bounded and semi-global asymptotic tracking in the sense of (16). This result holds provided that K_1 is selected to satisfy (39), and k_2 is selected to satisfy the following condition

$$k_2 > \frac{(k_n + 1)}{j_1} \quad (42)$$

and

$$k_n > \frac{1}{4}\rho^2(\sqrt{\frac{\lambda_2}{\lambda_1}}\|y(0)\|) \quad (43)$$

where λ_1 and λ_2 are defined as

$$\lambda_1 = \tfrac{1}{2}\min(1, j_1) \quad \lambda_2 = \max(\tfrac{1}{2}j_2, 1)\ . \quad (44)$$

The positive constants j_1 and j_2 have been defined in (28). The initial value of auxiliary variable $y(t)$ is set as

$$\|y(0)\| = \sqrt{\begin{array}{c} \left\|\frac{e_v(0)}{(1 - e_v^T(0)e_v(0))^2}\right\| + \|\eta(0)\| \\ + \sum_{i=1}^3 K_{1i}|e_{vi}(0)| - \frac{1}{2}e_v^T(0)B_d^{-T}D_1'(0) \end{array}}. \quad (45)$$

Proof: Let the auxiliary function $Q(t) \in \mathbb{R}$ be defined as follows

$$Q = \zeta_b - \int_0^t L(\tau)d\tau \quad (46)$$

where the ζ_b and $L(t)$ have been introduced in Lemma 1. It is not difficult to check that $Q \ge 0$. To prove the above theorem, an nonnegative function $V(t) \in \mathbb{R}$ is defined as follows

$$V = \frac{1}{2}\eta^T J_{ev}\eta + \frac{1}{2}e_f^T e_f + \frac{1}{2}\frac{e_v^T e_v}{(1 - e_v^T e_v)^2} + Q. \quad (47)$$

Note the function $V(t)$ can be bounded as

$$\lambda_1\|y\|^2 \le V \le \lambda_2\|y\|^2 \quad (48)$$

where $y = \begin{bmatrix} z_q^T & \sqrt{Q} \end{bmatrix}^T \in \mathbb{R}^{10}$. After taking time derivative of (47), and substituting (29), (32) and (46) into the resulting equation, the following expression can be obtained

$$\begin{aligned} \dot{V} &= -e_f^T e_f - \frac{e_v^T e_v}{(1 - e_v^T e_v)^2} - k_2\eta^T J_{ev}\eta + \eta^T N_{ev} \quad (49) \\ &= -\|z_q\|^2 + \eta^T\eta - k_2\eta^T J_{ev}\eta + \eta^T N_{ev} \end{aligned}$$

upon the use of the definition of $z_q(t)$. After applying (28) and (36) to (49), it can be obtained

$$\begin{aligned} \dot{V} &\le -\|z_q\|^2 + \left[\|\eta\|\|z_q\|\|\rho(\|z_q\|)\| - k_n\|\eta\|^2\right] \quad (50) \\ &\le -\left(1 - \frac{\rho^2(\|z_q\|)}{4k_n}\right)\|z_q\|^2 \end{aligned}$$

where the constant k_n satisfies the following condition

$$k_n < k_2 j_1 - 1. \quad (51)$$

From previous equations, it can be obtained that

$$\dot{V} \leq -\gamma \left\| z_q \right\|^2 \quad k_n > \tfrac{1}{4}\rho^2(\|z_q\|) \tag{52}$$

where γ is some positive constant. It should be noted that even though the selction of k_n in (52) is related with the state $z_q(t)$, but it can be transfter to the condition (43) which is only dependent on the initial value of the system states by following the similar steps in [15]. By utilizing (52) and following the similar steps in [15], we can conclude that all the closed loop signals remain bounded and the attitude tracking is achieved in the sense (16), provided the control gains being selected to satisfy (41), (42), (43), and (44). ■

IV. Altitude Direction Controller Development

A. Altitude Error System

To facilitate the controller $u(t)$ development, the dynamics of altitude subsystem in (1) can be rewritten as

$$m\ddot{z} = -k_z\dot{z} - mg + u^* + d_4 \tag{53}$$

where the auxiliary function $u^*(t) \in \mathbb{R}$ is defined as

$$u^* = \cos\theta\cos\phi u. \tag{54}$$

The Euler angles can be computed from the quaternion via

$$\begin{array}{c} \theta = \arcsin(2(q_0q_2 - q_1q_3)) \\ \phi = [\arctan \frac{2(q_0q_1+q_2q_3)}{1-2(q_1^2+q_2^2)}] \end{array} . \tag{55}$$

The altitude tracking error, denoted by $e_z(t) \in \mathbb{R}$, is defined as

$$e_z = z_d - z \tag{56}$$

where $z_d(t) \in \mathbb{R}$ denotes the reference trajectory of altitude. Our objective is to design the controller $u(t)$ such that $e_z(t) \to 0$ as $t \to \infty$, under the restriction that the velocity signal $\dot{z}(t)$ is not measurable. Inspired by [15], we propose the following filters to solve the above stated problem

$$\dot{e}_{fz} = -e_{fz} + r_{fz} \quad e_{fz}(0) = 0 \tag{57}$$

$$r_{fz} = p_z - (k_{2z}+1)e_z \tag{58}$$

$$\begin{array}{c} \dot{p}_z = -r_{fz} - (k_{2z}+1)(e_z + r_{fz}) + e_z - e_{fz} \\ p_z(0) = (k_{2z}+1)e_z(0) \end{array} \tag{59}$$

where $e_{fz}(t)$, $r_{fz}(t) \in \mathbb{R}$ are the outputs of the filters, $p_z(t) \in \mathbb{R}$ is an auxiliary variable used in the filter implementation, $k_{2z} \in \mathbb{R}$ is a positive constant. An auxiliary term, denoted by $\eta_z(t) \in \mathbb{R}$, is defined as follows

$$\eta_z = \dot{e}_z + e_z + r_{fz}. \tag{60}$$

After taking the time derivative of (58), we obtain the following equation

$$\dot{r}_{fz} = -r_{fz} - (k_{2z}+1)\eta_z + e_z - e_{fz}. \tag{61}$$

After taking the time derivative of (60) and substituting (56) and (61) into the resulting equation, the following expression can be obtained

$$\dot{\eta}_z = \ddot{z}_d - \ddot{z} - 2r_{fz} - e_{fz} - k_{2z}\eta_z. \tag{62}$$

After multiplying (62) by m and substituting (53) into the resulting equation, we have

$$m\dot{\eta}_z = -k_{2z}m\eta_z + N_z - m(2r_{fz} + e_{fz}) - u^* \tag{63}$$

where the auxiliary function $N_z(t) \in \mathbb{R}$ is defined as

$$N_z = k\dot{z} + mg + m\ddot{z}_d - d_4. \tag{64}$$

Let $N_{zd}(t) = N_z(\begin{array}{cc} z_d & \dot{z}_d \end{array})$, due to the fact that $z_d(t)$ and $\dot{z}_d(t) \in \mathcal{L}_\infty$, $N_{zd}(t)$ and $\dot{N}_{zd}(t) \in \mathcal{L}_\infty$. The expression in (63) can be rewritten as

$$m\dot{\eta}_z = -k_{2z}m\eta_z + N_{zd} + \tilde{N}_z - u^* \tag{65}$$

where the function $\tilde{N}_z(t) \in \mathbb{R}$ is defined as

$$\tilde{N}_z = N_z - N_{zd} - m(2r_{fz} + e_{fz}). \tag{66}$$

Remark 7: Based on the definitions of $N_z(t)$ and $N_{zd}(t)$, we can show $\tilde{N}_z(t)$ can be upper bounded as following

$$\left\|\tilde{N}_z\right\| \leq \rho_z(\|z_z\|)\,\|z_z\| \tag{67}$$

where $z_z = \left[\begin{array}{cccc} e_z & e_{fz} & r_{fz} & \eta_z \end{array}\right]^T \in \mathbb{R}^4$, and the positive function $\rho_z(\|z_z\|)$ is non-decreasing in $\|z_z\|$.

B. Output Feedback Controller Development

To achieve the control objective, the altitude tracking controller $u^*(t) \in \mathbb{R}$ is designed as

$$u^* = k_{1z}sgn(e_z + e_{fz}) - (k_{2z}+1)r_{fz} + e_z \tag{68}$$

where k_{1z} and k_{2z} are some positive control gains, $sgn(\bullet)$ is a standard sign function. The actual lift force input $u(t)$ can be computed via (54). After substituting (68) into (65), the closed loop dynamics of $\eta_z(t)$ can be obtained as

$$\begin{array}{l} m\dot{\eta}_z = -k_{2z}m\eta_z - k_{1z}sgn(e_z + e_{fz}) + \tilde{N}_z \\ \quad + N_{zd} + (k_{2z}+1)r_{fz} - e_z. \end{array} \tag{69}$$

C. Stability Analysis

Before presenting the main result, the following lemma to be invoked later is stated as follows.

Lemma 2: Let the auxiliary function $L_z(t) \in \mathbb{R}$ be defined as follows

$$L_z = \eta_z(N_{zd} - k_{1z}sgn(e_z + e_{fz})). \tag{70}$$

If the control gain $k_{1z} \in \mathbb{R}$ is selected to satisfy the following sufficient condition

$$k_{1z} > \|N_{dz}(t)\|_\infty + \left\|\dot{N}_{dz}(t)\right\|_\infty \tag{71}$$

where $\|\bullet\|_\infty$ denotes the infinity norm, then

$$\int_\infty^t L_z(\tau)d\tau \leq \zeta_{bz} \tag{72}$$

where the positive constants $\zeta_{bz} \in \mathbb{R}$ is defined as

$$\zeta_{bz} = k_{1z}\,|e_z(0)| - e_z(0)N_{zd}(0). \tag{73}$$

Proof: Please refer to [15]. ■

Theorem 2: The control law in (68) ensures that all closed-loop signals are bounded and $e_z(t)$ and $\dot{e}_z(t) \to 0$

as $t \to \infty$ provided the control gain k_{1z} satisfies (71), k_{2z} satisfies the following condition

$$k_{2z} > \frac{k_{nz}+1}{m} \tag{74}$$

and the positive constant k_{nz} is selected to satisfy

$$k_{nz} > \frac{1}{4}\rho_z^2\left(\sqrt{\frac{\lambda_{1z}}{\lambda_{2z}}}\left\|y_z(0)\right\|\right) \tag{75}$$

where λ_{1z}, $\lambda_{2z} \in \mathbb{R}$ are defined as follows

$$\lambda_{1z} = \tfrac{1}{2}\min(1,m) \quad \lambda_{2z} = \max(\tfrac{1}{2}m, 1) \tag{76}$$

and

$$\begin{aligned} &|y_z(0)| \\ &= \sqrt{|e_z^2(0)| + |\eta_z^2(0)| + k_{1z}\left|e_z(0)\right| - e_z(0)N_{zd}(0)}. \end{aligned} \tag{77}$$

Proof: Let the auxiliary function $Q_z(t) \in \mathbb{R}$ be defined as follows

$$Q_z = \zeta_{bz} - \int_0^t L_z(\tau)d\tau \tag{78}$$

where the ζ_{bz} and $L_z(t)$ have been introduced in Lemma 2. To prove the above theorem, an nonnegative function $V_z(t) \in \mathbb{R}$ is defined as follows

$$V = \frac{1}{2}m\eta_z^2 + \frac{1}{2}e_{fz}^T + \frac{1}{2}e_z^2 + \frac{1}{2}r_{fz}^2 + Q_z. \tag{79}$$

Note the function $V_z(t)$ can be bounded as

$$\lambda_{1z}\left\|y_z\right\|^2 \le V_z \le \lambda_{2z}\left\|y_z\right\|^2 \tag{80}$$

where $y_z = \begin{bmatrix} z_z^T & \sqrt{Q_z} \end{bmatrix}^T \in \mathbb{R}^5$, and λ_{1z}, λ_{2z} have been defined in (76). After taking time derivative of (79), and substituting (57), (60), (61) (69) and (78) into the resulting equation, the following expression can be obtained

$$\begin{aligned} \dot{V}_z &= -e_{fz}^2 - e_z^2 - r_{fz}^2 - k_{2z}m\eta_z^2 + \eta_z\tilde{N}_z \\ &= -\left\|z_z\right\|^2 + (1-k_{2z}m)\eta_z^2 + \eta_z\tilde{N}_z \end{aligned} \tag{81}$$

upon the use of the definition of $z_z(t)$. After applying (67) to (81), it can be obtained

$$\begin{aligned} \dot{V} &\le -\left\|z_z\right\|^2 + \left[\left\|\eta_z\right\|\rho_z(\left\|z_z\right\|) - k_{nz}\left\|\eta_z\right\|^2\right] \\ &\le -\left(1 - \frac{\rho_z^2(\left\|z_z\right\|)}{4k_{nz}}\right)\left\|z_z\right\|^2 \end{aligned} \tag{82}$$

where $k_{nz} \in \mathbb{R}$ is a constant and satisfies the following condition

$$k_{nz} < k_{2z}m - 1. \tag{83}$$

From previous equation, it can be obtained that

$$\dot{V}_z \le -\gamma_z\left\|z_z\right\|^2 \quad k_{nz} > \tfrac{1}{4}\rho_z^2(\left\|z_z\right\|) \tag{84}$$

where γ_z is some positive constant. It should be noted that even though the selction of k_{nz} in (84) is related with the state $z_z(t)$, but it can be transfter to the condition (75) which is only dependent on the initial value of the system states by following the similar steps in [15]. By utilizing (84) and following the similar steps in [15], it can be proved that all the closed loop signals remain bounded and the attitude tracking is achieved, provided the control gains being selected to satisfy (73), (74), (75) and (76). ■

V. CONCLUSIONS

In this paper, a quaternion based output feedback tracking controller for a quadrotor UAV system with structural uncertainties and unknown external disturbances is developed. Through a Lyapunov based stability analysis, we have demonstrated that a semi-global asymptotic altitude and orientation tracking result is achieved and all closed loop signals remain bounded. Future work will focuses on the implementation of the proposed control algorithm on a real-time quadrotor flying testbed.

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